# 6+∞ new expressions for the Euler–Mascheroni constant

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**Abstract**

In the first part we review some formulae for the Euler–Mascheroni constant $\gamma$. For the four formulae we present comparison of the computer determinations of these expressions with the actual value of $\gamma$. Next we give new formulae expressing the $\gamma$ constant in terms of the Ramanujan–Soldner constant $\mu$. Employing the cosine integral we obtain another infinity of formulae for $\gamma$. Finally we express $\gamma$ in terms of $\pi$.

# 1 Introduction

The Euler–Mascheroni constant is defined by the following limit:

$$\gamma = \lim_{k\to\infty}\left(\sum_{n=1}^{k}\frac{1}{n} - \log(k)\right) = 0.57721566490153286\ldots. \tag{1}$$

see e.g. [19], [21]. It is not known whether $\gamma$ is irrational, see [31], [21]. It is known that if $\gamma$ is rational and equal to a simple fraction $p/q$ than $q > 10^{242080}$, see [19, p.97].

The Euler–Mascheroni constant $\gamma$ is a first element of the sequence of the Stieltjes constants $\gamma_n$ defined by

$$\gamma_n = \lim_{m\to\infty}\left[\left(\sum_{k=1}^{m}\frac{(\log k)^n}{k}\right) - \frac{(\log m)^{n+1}}{n+1}\right]. \tag{2}$$

When $n = 0$ (what corresponds to Euler–Mascheroni constant $\gamma = \gamma_0$) the numerator of the fraction in the first summand in (2) is formally $0^0$ which is taken to be 1. These constants are coefficients of the Laurent series for the Riemann's $\zeta(s)$ function:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n\,(s-1)^n \tag{3}$$

The limit in (1) is very slowly convergent (like $n^{-1}$) and in [14] it was shown that slight modification of (1):

$$\gamma = \lim_{k\to\infty}\left(\sum_{n=1}^{k}\frac{1}{n} - \log(k+\frac{1}{2})\right)$$

improves convergence to $1/n^2$. Presently sequences converging to $\gamma$ much faster are known, see [23] where sequence which converge to $\gamma$ like $n^{-6}$ is presented. There are numerous formulae expressing $\gamma$ as limits, series, integrals or products, see [19] and e.g. [17], [10], [21]. We highlight here the infinity of formulae for $\gamma$ [17, p.4]:

$$\gamma = \sum_{k=1}^{n}\frac{1}{k} - \log(n) - \sum_{k=2}^{\infty}\frac{\zeta(k,n+1)}{k}, \qquad n = 2,3,\ldots\ , \tag{4}$$

where the Hurwitz zeta function:

$$\zeta(s,k)=\sum_{n=0}^{\infty}\frac{1}{(n+k)^s}, \qquad \Re(s)>1 \quad k\neq -1,-2,-3\ldots. \tag{5}$$

A second infinite set of formulaae for $\gamma$ is found in [7, eq.(9.3.10)]:

$$\gamma=\sum_{k=1}^{n}\frac{1}{k}-\log n-\int_n^{\infty}\frac{\{x\}}{x^2}dx, \qquad n=1,2,3,\ldots\ . \tag{6}$$

where $\{x\}$ is the fractional part of $x$.

The third example includes uncountable many formulae for $\gamma$, see e.g. [8]: for real $r>0$

$$\gamma=\lim_{n\to\infty}\frac{\sum_{k=0}^{\infty}\{(\frac{n^k}{k!})^r(\sum_{j=1}^{k}\frac{1}{j}-\log(k))\}}{\sum_{k=0}^{\infty}(\frac{n^k}{k!})^r}. \tag{7}$$

That this formula gives $\gamma$ for each $r>0$ means that the derivative of rhs with respect to $r$ is zero. There are also doubly uncountable formulaae for $\gamma$. As the fourth example, we present the formula (3.13) from [10]:

$$\gamma=r\int_0^{\infty}\left(\frac{1}{1+x^q}-\exp(-x^r)\right)\frac{dx}{x}, \qquad q>0,\ r>0. \tag{8}$$

The numerical value of the Euler–Mascheroni constant has been calculated many times to ever increasing decimal places, see e.g. [8]. The current world record (as of 14 June 2023) is $700,000,000,000$ decimal digits of $\gamma$ and belongs to Jordan Ranous and Kevin O'Brien , see [1].

The Euler–Mascheroni constant appears in numerous places in number theory including the theory of the Riemann zeta function, such as in the Nicolas' and Robin's criterions for the Riemann Hypothesis, see e.g. [9, vol.1, chapters 5 and 7]. One of the most amazing appearances of the $\gamma$ constant is in F. Mertens's two products over primes [18, p.351], one of which involves constants $\pi,\ e,\ \gamma$ ("holy trinity"):

$$\lim_{n\to\infty}\frac{1}{\log(n)}\prod_{p<n}\left(1+\frac{1}{p}\right)=\frac{6e^\gamma}{\pi^2} \tag{9}$$

from which we obtain

$$\gamma=\log\Big(\frac{\pi^2}{6}\lim_{n\to\infty}\frac{1}{\log(n)}\prod_{p<n}\Big(1+\frac{1}{p}\Big)\Big). \tag{10}$$

With today's computers we can check the accuracy of the above relation. In the Table 1 we present numerical test of (9) along with the values of the $\gamma(n)$ computed from finite products over primes:

$$\gamma(n)=\log\Big(\frac{\pi^2}{6}\frac{1}{\log(n)}\prod_{p<n}\Big(1+\frac{1}{p}\Big)\Big). \tag{11}$$

We present these numerical calculations as an illustration of (9) not the way to compute $\gamma$, as there are more efficient algorithms known, see e.g. [8].

Another appearance of $\gamma$ is found in the formula for the average value of the divisor function $d(n)$, which counts the number of divisors of $n$ including 1 and $n$, is given by the theorem proved by Dirichlet, see e.g. [18, Th.320]:

$$\frac{1}{n}\sum_{k=1}^{n}d(k)=\log n+2\gamma-1+\mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big). \tag{12}$$

Values of $\gamma$ obtained from the formula above for $n=2^{15},\ldots,2^{23}$ are presented in Table 2.

TABLE 1 The values of the product in (9) up to $n = 1000, 10000, \ldots, 10^{13}$ (second column) and values following from the Mertens's formula (third column), their ratio in fourth column and finite approximations to $\gamma$ in the last column. The fluctuations in the last digits of the values obtained from the computer are possibly caused by fluctuations in the prime distrubution or by cumulation of floating–point errors.

| $n$ | $\prod_{p<n}(1+1/p_n)$ | $6e^{\gamma}\log(n)/\pi^2$ | ratio | $\gamma(n)$ |
|---|---|---|---|---|
| $10^3$ | 7.5094464 | 7.4891425 | 1.0027111 | 0.57992110 |
| $10^4$ | 9.9849904 | 9.9733461 | 1.0011675 | 0.57838053 |
| $10^5$ | 12.4756558 | 12.4721158 | 1.0002838 | 0.57749746 |
| $10^6$ | 14.9651229 | 14.9643917 | 1.0000489 | 0.57726252 |
| $10^7$ | 17.4570890 | 17.4568441 | 1.0000140 | 0.57722769 |
| $10^8$ | 19.9494269 | 19.9493052 | 1.0000061 | 0.57721977 |
| $10^9$ | 22.4418428 | 22.4417674 | 1.0000034 | 0.57721703 |
| $10^{10}$ | 24.9342956 | 24.9342295 | 1.0000027 | 0.57721631 |
| $10^{11}$ | 27.4267504 | 27.4266917 | 1.0000021 | 0.57721581 |
| $10^{12}$ | 29.9192150 | 29.9191539 | 1.0000020 | 0.57721571 |
| $10^{13}$ | 32.4116846 | 32.4116161 | 1.0000021 | 0.57721578 |

The relation (12) suggests that there is a connection between $\gamma$ and the distribution of primes. Further example of this relation we found in [11, Corrolary 1]:

$$\gamma = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1-\Lambda(n)}{n}\right), \tag{13}$$

where the von Mangoldt function$\Lambda(n)$ is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

TABLE 2 The values of $\gamma$ obtained from (12) for $n = 2^{20}, 2^{22}, \ldots, 2^{32}$.

| $n$ | $\sum_{k=1}^{n} d(k)$ | $\gamma$ from (12) |
|---|---|---|
| $2^{20} = 1048576$ | 14698342 | 0.57724382397818362 |
| $2^{22} = 4194304$ | 64607782 | 0.57722880551235453 |
| $2^{24} = 16777216$ | 281689074 | 0.57722242972001786 |
| $2^{26} = 67108864$ | 1219788256 | 0.57721736522986625 |
| $2^{28} = 268435456$ | 5251282902 | 0.57721609681052878 |
| $2^{30} = 1073741824$ | 22493653324 | 0.57721585470866853 |
| $2^{32} = 4294967296$ | 95928700948 | 0.57721570434208188 |

The rhs of (13) is a sum of two diverging series: just harmonic series and decimated harmonic series where instead of ones in the nominator are values of $\Lambda(n) > 1$, see Table 3.

TABLE 3 The values of $\gamma$ obtained from (13) for $n = 10, \ldots, 19^7$ compared with the actual value of the Euler–Mascheroni constant.

| $n$ | eq. (13) | ratio |
|---|---|---|
| 10000 | 0.576533359060 | 1.001183462901 |
| 100000 | 0.576946987410 | 1.000465688352 |
| 1000000 | 0.577417595935 | 0.999650285972 |
| 10000000 | 0.577287376546 | 0.999875778257 |

A surprising appearance of $\gamma$ is in the "harmonic" sum of reciprocals of non–trivial zeros $\rho$ of the Riemann's zeta function [15, p.67 and p. 159], [13, pp.80–82]:

$$\sum_\rho \frac{1}{\rho} \;=\; 1+\frac{1}{2}\gamma-\frac{1}{2}\log(4\pi)=0.023095708966\ldots\;. \tag{15}$$

Above again three constants $\pi, e, \gamma$ appear: $e$ is hidden in the natural logarithm. The sum (15) is real and convergent when zeros $\rho$ and complex conjugate $\overline{\rho}$ are paired together and summed according to increasing absolute values of the imaginary parts of $\rho$. Several years ago using the L-function calculator written by Michael Rubinstein (see [27]) we calculated 100,000,000 imaginary parts of zeros of $\zeta(s)$; the last obtained zero has the value $\rho_{100000000} = \frac{1}{2} + \imath 42653549.7609515$. In Table 4 we present approximations to $\gamma$ obtained from (15) after summing over 1000, 10,000, ..., 100,000,000 zeros of zeta function.

Another connection with the Riemann's zeta function $\zeta(s)$ is given by the astonishing fact: the Riemann's Hypothesis is true iff the following relation holds, see [30]:

$$\frac{1}{\pi}\int_0^\infty \frac{2t\arg(\zeta(1/2+it))}{(1/4+t^2)^2}dt=\gamma-3 \tag{16}$$

The largest known prime numbers are of the form $2^p-1$ where $p$ is also a prime and are called Mersenne primes, see eg. https://www.mersenne.org/. In [25, p.101], [33, p.388] (see also [29, §3.5]) the Lenstra–Pomerance–Wagstaff conjecture was formulated: if $\mathcal{M}_n$ denotes the $n$-th Mersenne prime, then $\mathcal{M}_n$ grows doubly exponentially with $n$:

$$\log_2\log_2\mathcal{M}_n \sim ne^{-\gamma}, \tag{17}$$

The presence of $\gamma$ here comes from Mertens's result (9). In Fig. 1 we compare the Lenstra–Pomerance – Wagstaff conjecture with all 51 presently known Mersenne primes [2]. The fit by least square method gives the line with the slope 0.54 what leads to rather poor value 0.61 for $\gamma$.

Fast converging to $\gamma$ formulae were presented in [8]; they are the most commonly utilized formulae for numerical calculations of gamma to high precison.

In this paper we present some new formulae for $\gamma$ obtained by using special values for the argument of the logarithmic and cosine integrals. Similar idea appeared in [32], where the series for the exponential integral was used to calculate $\gamma$ up to 3566 decimal places. A few of these new expressions present the Euler–Mascheroni constant in the form of the difference of two numbers, one of which is transcendental. It gives hopes for the proof of the irrationality and perhaps the transcendentality of $\gamma$.

The astonishing formula was found by A. Kawalec [20]. He expressed $\gamma$ in terms of the imaginary parts $t_l$ of the nontrivial zeros of the Riemann's zeta function on the critical line $\zeta(\frac{1}{2}+it_n)=0$:

$$\gamma=\lim_{k\to\infty}\left(2\sum_{n=1}^{k}\sum_{m=n+1}^{k}\frac{(-1)^m(-1)^{n+1}}{\sqrt{mn}}\cos(t_l\log(m/n))-\log(k)\right)\qquad m=1,2,3,\ldots. \tag{18}$$

It is infinitude of formulae as there are infinitely many nontrivial zeta zeros. The above formula does not depend on the Riemann's Hypothesis: if there are any zeros off critical line they do not enter (18).

## 2 Logarithmic integral

The logarithmic integral is defined for all positive real numbers $x \neq 1$ by the definite integral

$$\mathrm{li}(x)\equiv\begin{cases} p.v.\displaystyle\int_0^x\frac{du}{\log(u)}, & \text{for } x>1;\\ \displaystyle\int_0^x\frac{du}{\log(u)}, & \text{for } 0<x<1,\end{cases} \tag{19}$$

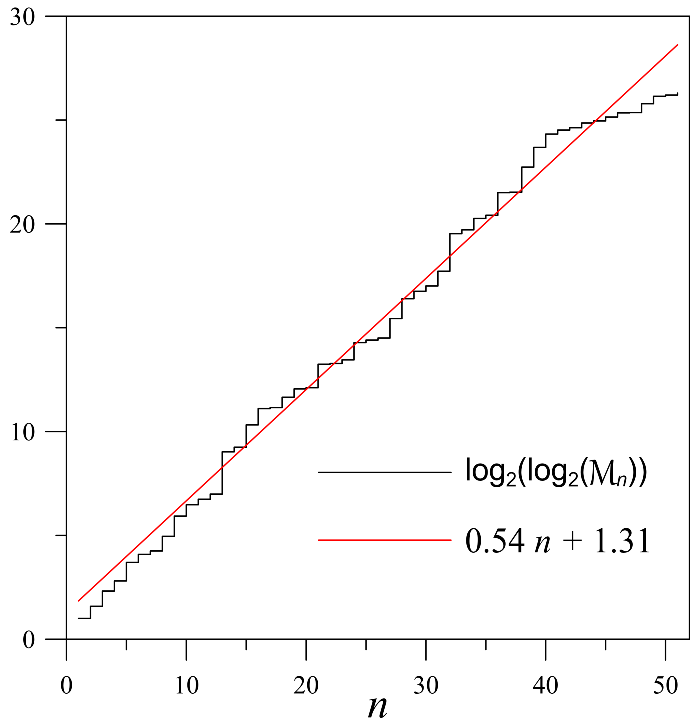


Fig.1 The plot illustrating the Lenstra–Pomerance–Wagstaff conjecture. The least–square fit was done to all known $\mathcal{M}_n$ and it is $0.54n + 1.31$, which gives rather bad value for $\gamma$ of 0.61.

where $p.v.$ stands for Cauchy principal value around $u = 1$:

$$p.v. \int_0^x \frac{du}{\log(u)} = \lim_{\epsilon \to 0} \left( \int_0^{1-\epsilon} \frac{du}{\log(u)} + \int_{1+\epsilon}^x \frac{du}{\log(u)} \right). \tag{20}$$

There is a series giving logarithmic integral $\mathrm{li}(x)$ for all $x > 1$ (see [3, formulae 5.1.3 and 5.1.10])

$$\mathrm{li}(x) \;=\; \gamma + \log\log x + \sum_{n=1}^{\infty} \frac{\log^n}{n!} \quad \text{for } x > 1. \tag{21}$$

This series is quickly convergent because it has $nn!$ in denominator which eventually overwhelms the $\log^n(x)$ term in the numerator. The above expansion was known to C.F. Gauss and F.W. Bessel, see remarks by R. Dedekind after the famous paper "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" by B. Riemann in [26, p. 168].

TABLE 4 The value of $\gamma$ obtained from (15) after summing over $n = 1000, 10000, \ldots, 100000000$ zeros of $\zeta(s)$.

| $n$ | $\gamma$ |
|---|---|
| 1000 | 0.5757765 |
| 10000 | 0.5769463 |
| 100000 | 0.5771715 |
| 1000000 | 0.5772091 |
| 10000000 | 0.5772147 |
| 100000000 | 0.5772155 |

After a change of variables, a variant of the above series is given by:

$$\int_x^\infty \frac{e^{-t}}{t}dt \;=\; -\gamma - \log x + \sum_{n=1}^\infty \frac{(-1)^{n-1}x^n}{n\cdot n!}. \tag{22}$$

This was used in [32] for large $x>0$, when the lhs of above equation is practically zero (in fact it is $\mathcal{O}(e^{-x}/x)$), to compute 3566 digits of $\gamma$, see also [8].

The logarithmic integral takes a value 0 at only one real number which is denoted by $\mu$ and is called the Ramanujan–Soldner constant

$$\int_0^\mu \frac{du}{\log u} = 0, \tag{23}$$

see e.g. [6, entry 14, p.126, eq.(11.3)] and its numerical value is:

$$\mu = 1.45136923488338105028396848589202745\ldots\ .$$

Thus for $x>\mu$ we have:

$$\mathrm{li}(x) = \int_\mu^x \frac{du}{\log(u)}. \tag{24}$$

Inserting in (21) $x=\mu>1$ we obtain the first formula expressing the Euler–Mascheroni constant via the Ramanujan–Soldner constant:

$$\gamma \;=\; -\log\log\mu - \sum_{n=1}^\infty \frac{\log^n\mu}{n\cdot n!}. \tag{25}$$

Appearing here constant $\log\mu = 0.37250741078136663446\ldots$ is the zero of the exponential integral $\mathrm{Ei}(x)$, see (45). The series in (25) is very quickly convergent. Using PARI [24] we checked that summing in (25) to only $n=20$ reproduces 31 digits of $\gamma$. In the **Appendix** we give the script to check (25) to any desired number of digits. In constrast, calculating (1) at $k=1000000$ gives only 5 digits of $\gamma$.

Even faster converging series was discovered by Ramanujan [6, p.130]:

$$\int_\mu^x \frac{du}{\log u} = \gamma + \log\log x + \sqrt{x}\sum_{n=1}^\infty \frac{(-1)^{n-1}(\log x)^n}{n!\,2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \frac{1}{2k+1} \qquad \text{for } x>1. \tag{26}$$

Putting here $x=\mu$ we obtain second formula for the Euler—Mascheroni constant:

$$\gamma \;=\; -\log\log\mu + \sqrt{\mu}\sum_{n=1}^\infty \frac{(-1)^{n}(\log \mu)^n}{n!\,2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \frac{1}{2k+1}. \tag{27}$$

We checked using PARI that summing above to $n=20$ reproduces correctly 37 digits of $\gamma$.

Using $x=e$ in (21) greatly simplifies the series leading to the third expression for the Euler—Mascheroni constant:

$$\gamma \;=\; \int_\mu^e \frac{du}{\log u} - \sum_{n=1}^\infty \frac{1}{n\cdot n!} := \alpha - \beta, \tag{28}$$

where the numbers

$$\alpha := \int_\mu^e \frac{du}{\log u} = 1.89511781635593675546652\ldots\ , \tag{29}$$

$$\beta := \sum_{n=1}^\infty \frac{1}{n\cdot n!} = 1.3179021514544038948 6\ldots\ . \tag{30}$$

The number $\beta$ is irrational by the same reasoning which proves the irrationality of $e = \sum_{n=0}^{\infty} 1/n!$ (see e.g. [28, p.65]) which can be repeated here *mutatis mutandis*. In fact from the Siegel–Shidlovsky theorem [16, see eq.5.2 for $k = 1$] it follows that $\beta$ (30) is transcendental.

Putting $x = e$ in (26) yields the fourth expression for the Euler–Mascheroni constant

$$\gamma \ = \ \int_{\mu}^{e} \frac{du}{\log u} + \sqrt{e} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\, 2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}. \tag{31}$$

Finally, let us notice that in [19] at several places (e.g. pp. 52, 104), we find that Euler had hoped that $\gamma$ is the logarithm of some important number. Above we have given two series for $\gamma$ in terms of the logarithm of the Ramanujan–Soldner constant $\mu$.

## 3 Cosine integral

Many special functions involve in theirs expansions the Euler-Mascheroni constant. The cosine integral $\mathrm{Ci}(x)$ function for $x > 0$ has a series expansion also containing $\gamma$ (see e.g. [3, §5.2, formula 5.2.16]):

$$\begin{aligned} \mathrm{Ci}(x) \ &= \ -\int_{x}^{\infty} \frac{\cos u}{u} du \ = \ \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{2n(2n)!} \\ &= \ \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{2^{n+1} n n! (2n-1)!!}, \end{aligned} \tag{32}$$

because $(2n)! = 2^n n!(2n-1)!!$, where odd factorial $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$. Using $x = 1$ we obtain the fifth expression for the Euler—Mascheroni constant:

$$\gamma \ = \ -\int_{1}^{\infty} \frac{\cos u}{u} du + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n)!}, \tag{33}$$

where:

$$\int_{1}^{\infty} \frac{\cos u}{u} du = -0.3374039229009681346626\ldots \tag{34}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n)!} \ = \ 0.2398117420005647259439\ldots \ . \tag{35}$$

Again, $\gamma$ can be expressed simply in terms of two other numbers. The cosine integral $\mathrm{Ci}(x)$ has infinitely many zeros that do not have their own names and are non–periodic. They are usually denoted by $c_k$, see [22, eq.(9)]. The first zeros are $c_0 = 0.61650548562\ldots$, $c_1 = 3.38418042255\ldots$, $c_2 = 6.42704774405\ldots$, .... In [22], A.J. MacLeod gives the asymptotic expansion for these zeros:

$$c_k \approx k\pi + \frac{1}{k\pi} - \frac{16}{3(k\pi)^3} + \frac{1673}{15(k\pi)^5} - \frac{507746}{105(k\pi)^7} + \frac{111566353}{315(k\pi)^9} \ldots . \tag{36}$$

Putting any specific zero $c_k$ into (32) we obtain an infinity of expressions for $\gamma$

$$\gamma \ = \ -\sum_{n=1}^{\infty} \frac{(-c_k^2)^n}{2n(2n)!} - \log c_k, \qquad k = 0, 1, 2, \ldots \ . \tag{37}$$

In Table 5 we present values for $\gamma$ obtained from this formula when $c_k$ are calculated from (36). In the last column the differences between values in third column and $\gamma$ are given.

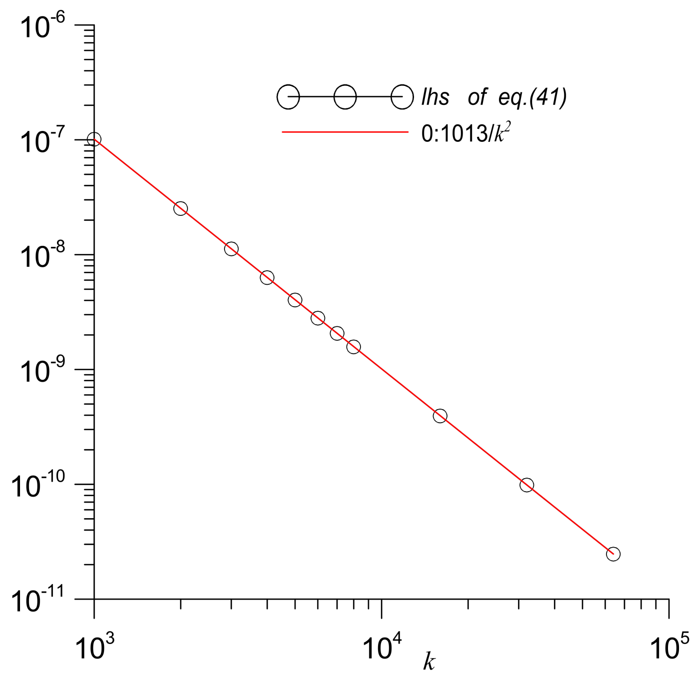


Fig.2 The double–logarithmic plot of lhs of equation (41) for
$k = 1000, 2000, \ldots, 8000, 16000, 32000, 64000$.

From the MacLeod formula (36) we see that large zeros of $\text{Ci}(x)$ approach just zeros of $\sin(x) = \int \cos(x)dx$: $c_k \sim k\pi$ for large $k$. Thus we have our sixth formula for the Euler—Mascheroni constant:

$$\gamma = \lim_{k\to\infty} \Big( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(k\pi)^{2n}}{2n(2n)!} - \log(k\pi) \Big). \tag{38}$$

Denoting $x = k\pi$ the above formula can be written also as

$$\gamma = \lim_{x\to\infty} \Big( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{2n(2n)!} - \log(x) \Big). \tag{39}$$

In [5, p.98] we found similar formula obtained by S. Ramanujan:

$$\gamma = \lim_{x\to\infty} \Big( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n}}{n!n} - \log(x) \Big). \tag{40}$$

It seems to be a subtle problem to reconcile equations (39) and (40). The formula (39) in some sense resembles the original definition (1). Using Pari we calculated the expression in the big parentheses in (38) for $k = 1000, 2000, 4000, 8000, \ldots, 64000$. E.g. for $k = 64000$ the expression in big parentheses on rhs of (38) gives $0.577215664926\ldots$, i.e. it reproduces correctly first 10 digits of $\gamma$. Differences between $\gamma$ and values obtained from (38) for mentioned above set of $k$ were perfectly arranged on the straight line on the double logarithmic plot, see Fig.2. Fitting by least–square method gave equation of the line $0.101321k^{-1.9999}$, thus it suggests:

$$\left| \gamma + \log(k\pi) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(k\pi)^{2n}}{2n(2n)!} \right| = \frac{0.1013211}{k^2}. \tag{41}$$

In [3] we found asymptotic expansion of Ci$(x)$, see formulae (5.2.9), (5.2.35) and (5.2.34):

$$\text{Ci} \sim \frac{1}{x}\Big(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\frac{6!}{x^6}+\dots\Big)\sin(x)+\frac{1}{x^2}\Big(1-\frac{3!}{x^2}+\frac{5!}{x^4}-\frac{7!}{x^6}+\dots\Big)\cos(x) \tag{42}$$

Putting above $x = k\pi$ we obtain:

$$\left|\gamma-\Big(-\log(k\pi)-\sum_{n=1}^{\infty}\frac{(-1)^n(k\pi)^{2n}}{2n(2n)!}\Big)\right| \sim \frac{1}{\pi^2k^2} \tag{43}$$

and $1/\pi^2 = 0.1013211836\dots$ what agrees with constant on rhs of (41).

**Concluding remarks:** We think that the existence of countable many formulae for a given constant is necessary condition for its irrationality while the existence of uncountable many formulae is necessary condition for transcendentality. For example, there is an expression for the $\pi$ depending on arbitrary complex number $z$ (see [4, not labelled formuls on top of p.15]) Erdős:

$$\pi = \sum_{k=0}^{\infty}\left(\frac{z-4}{4k+3}+\frac{z+4}{4k+1}+\frac{z}{4k+4}-\frac{3z}{4k+2}\right) \tag{44}$$

The fact, that for each $z \in \mathbb{C}$ the value of rhs is constant means that the derivative of rhs with respect to $z$ is zero. In fact there should be Cauchy–Riemann's equations satisfied.

The existence of infinitely many expressions for a given number $r \in R$ is not sufficient for irrationality. As a counter–example we have following infinite sequence of telescoping series

$$1 = \sum_{n=1}^{\infty}\frac{(n+1)^k-n^k}{n^k(n+1)^k}, \qquad k = 1,2,3,\dots.$$

Here is less trivial example:

$$\frac{1}{2} = \sin(\frac{\pi}{6}+2k\pi) = \sum_{n=0}^{\infty}\frac{(-1)^n(\pi/6+2k\pi)^{2n+1}}{(2n+1)!}, \; k = 0,\pm1,\pm2,\dots.$$

As for $\gamma$ there exist uncountable many expressions like in (7) it is a strong argument tht $\gamma$ is not only irrational but even transendental.

**Appendix:** Below is a simple PARI/GP script checking eq. (25) to arbitrary accuracy declared by `\p` precision. In the example below it is set to 2222. The output gives agreement between the lhs and rhs of (25) up to the number of digits given by precision. It takes a fraction of a second to get results.

```
allocatemem()

\p 2222

Soldner=solve(x=1.4, 1.5, real(eint1(-log(x))));
tmp=log(Soldner);
ss=suminf(n=1,  tmp^n/(n*n!));
write("EMRS.txt", Euler+log(tmp)+ss);
```

TABLE 5 The values of expression (37) when for $c_k$ the series (36) are substituted for $k = 10, 20, \ldots, 100$.

| $k$ | $c_k$ from eq.(36) | eq.(37) for this $c_k$ | $\lvert$eq.(37) for this $c_k - \gamma\rvert$ |
|---|---|---|---|
| 10 | 31.447589011629313 | 0.5772156649004098 | $1.123 \times 10^{-12}$ |
| 20 | 62.847747177749027 | 0.5772156649015328 | $1.953 \times 10^{-17}$ |
| 30 | 94.258383581485718 | 0.5772156649015328 | $2.888 \times 10^{-20}$ |
| 40 | 125.67166120666795 | 0.5772156649015328 | $2.657 \times 10^{-22}$ |
| 50 | 157.08599750231211 | 0.5772156649015328 | $6.519 \times 10^{-24}$ |
| 60 | 188.50086358429127 | 0.5772156649015328 | $2.871 \times 10^{-25}$ |
| 70 | 219.91603253410894 | 0.5772156649015328 | $1.771 \times 10^{-26}$ |
| 80 | 251.33139082491842 | 0.5772156649015328 | $1.180 \times 10^{-27}$ |
| 90 | 282.74687536370536 | 0.5772156649015328 | $2.181 \times 10^{-29}$ |
| 100 | 314.16244828586940 | 0.5772156649015328 | $2.861 \times 10^{-29}$ |

In the above script we used the fact that logarithmic integral is related to the exponential integral $\mathrm{Ei}(x)$, see e.g. [3, formula (5.1.3)]:

$$\mathrm{li}(x) = \mathrm{Ei}(\log x), \qquad x > 1, \tag{45}$$

where

$$\mathrm{Ei}(x) =\equiv \begin{cases} -p.v. \displaystyle\int_{-x}^{\infty} \frac{e^{-t}}{t} dt, & \text{for} \quad x > 0. \\ -\displaystyle\int_{-x}^{\infty} \frac{e^{-t}}{t} dt, & \text{for} \quad x < 0 \end{cases} \tag{46}$$

and the principal value is needed to avoid a singularity of the integrand at $t = 0$. The logarithmic integral is not implemented in Pari while exponential integral is implemented as `eint1(x)`. We obtained as a result of running the above script the number $4.27 \times 10^{-2235}$. To check (27) change last lines to

```
ss=suminf(n=1, (-1)^n*tmp^n/(2^(n-1.0)*n!)*
         sum(k=0, floor((n-1)*0.5),  1.0/(2.0*k+1.0)));
write("EMRS.txt", Euler+log(tmp)-sqrt(Soldner)*ss);
```

The output we obtained this time was $2.7328 \times 10^{-2233}$.

The equation (33) can be checked in Pari using the following commands:

```
allocatemem()
\p 2222

tmp=sumalt(n=1, (-1)^(n-1)/(2*n*(2*n)!));
c_i=intnum(u=1, [oo, I], cos(u)/u);
print(Euler+c_i-tmp);
```

We give further explanations: PARI contains the numerical routine `sumalt` for summing infinite alternating series in which extremely efficient algorithm of Cohen, Villegas and Zagier [12] is implemented; `oo` denotes in Pari infinity $+\infty$; `intnum`$(\cdot)$ is the function for numerical integration and flag `k*I` (`I`$= \imath$, i.e. $\imath^2 = -1$) tells the procedure that the integrand is an oscillating function of the type $\cos(kx)$, here $k = 1$. After a few minutes we obtained $1.42335 \times 10^{-2235}$. This result shows the power of Pari's procedures: the value of the cosine integral at 1 is indeed calculated numerically without using the expansion (32) and the value of $\gamma$ so the vicious circle (tautology) is avoided.

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