

6 + ∞ New Expressions for the Euler-Mascheroni Constant

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Abstract: In the first part, we review some formulae for the Euler-Mascheroni constant γ . For four of these formulae, we present a comparison of computer determinations of these expressions with the actual value of γ . Next, we provide new formulae expressing the γ constant in terms of the Ramanujan-Soldner constant μ . By employing the cosine integral, we obtain another infinity of formulae for γ . Finally, we express γ in terms of π .

Key words: Euler-Mascheroni constant, Ramanujan-Soldner constant, cosine integral

I. Introduction

The Euler-Mascheroni constant is defined by the following limit:

$$\gamma = \lim_{k \rightarrow \infty} \left(\sum_{n=1}^k \frac{1}{n} - \log(k) \right) = 0.57721566490153286 \dots, \quad (1)$$

see, e.g., [1, 2]. It is not known whether γ is irrational (see [2, 3]). It is known that if γ is rational and equal to a simple fraction p/q then $q > 10^{242\,080}$ (see [1, p. 97]).

The Euler-Mascheroni constant γ is the first element of the sequence of the Stieltjes constants γ_n defined by

$$\gamma_n = \lim_{m \rightarrow \infty} \left[\left(\sum_{k=1}^m \frac{(\log k)^n}{k} \right) - \frac{(\log m)^{n+1}}{n+1} \right]. \quad (2)$$

When $n = 0$ (which corresponds to the Euler-Mascheroni constant $\gamma = \gamma_0$), the numerator of the fraction in the first summand in (2) is formally 0^0 , which is taken to be 1. These constants are the coefficients of the Laurent series for the Riemann $\zeta(s)$ function:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n. \quad (3)$$

The limit in (1) is very slowly convergent (like n^{-1}), and in [4] it was shown that a slight modification of (1):

$$\gamma = \lim_{k \rightarrow \infty} \left(\sum_{n=1}^k \frac{1}{n} - \log\left(k + \frac{1}{2}\right) \right),$$

improves convergence to $1/n^2$. Currently, sequences converging to γ much faster are known; see [5], which presents a sequence converging to γ like n^{-6} . There are numerous formulae expressing γ as limits, series, integrals, or products; see [1] and, e.g., [2, 6, 7]. Here, we highlight the infinity of formulae for γ [6, p. 4]:

$$\gamma = \sum_{k=1}^n \frac{1}{k} - \log(n) - \sum_{k=2}^{\infty} \frac{\zeta(k, n+1)}{k}, \quad n = 2, 3, \dots, \quad (4)$$

where the Hurwitz zeta function is defined as:

$$\zeta(s, k) = \sum_{n=0}^{\infty} \frac{1}{(n+k)^s}, \quad \Re(s) > 1, \quad k \neq -1, -2, -3, \dots \quad (5)$$

A second infinite set of formulae for γ is found in [8, Eq. (9.3.10)]:

$$\gamma = \sum_{k=1}^n \frac{1}{k} - \log n - \int_n^{\infty} \frac{\{x\}}{x^2} dx, \quad n = 1, 2, 3, \dots, \quad (6)$$

where $\{x\}$ is the fractional part of x .

Tab. 1. Values of the product in (9) up to $n = 1000, 10\,000, \dots, 10^{13}$ (second column) and values from Mertens's formula (third column), their ratio (fourth column), and finite approximations to γ (last column). The fluctuations in the last digits of the computed values obtained are possibly caused by irregularities in the prime distribution or by the accumulation of floating-point errors

n	$\prod_{p < n} (1 + 1/p_n)$	$6e^\gamma \log(n)/\pi^2$	ratio	$\gamma(n)$
10^3	7.5094464	7.4891425	1.0027111	0.57992110
10^4	9.9849904	9.9733461	1.0011675	0.57838053
10^5	12.4756558	12.4721158	1.0002838	0.57749746
10^6	14.9651229	14.9643917	1.0000489	0.57726252
10^7	17.4570890	17.4568441	1.0000140	0.57722769
10^8	19.9494269	19.9493052	1.0000061	0.57721977
10^9	22.4418428	22.4417674	1.0000034	0.57721703
10^{10}	24.9342956	24.9342295	1.0000027	0.57721631
10^{11}	27.4267504	27.4266917	1.0000021	0.57721581
10^{12}	29.9192150	29.9191539	1.0000020	0.57721571
10^{13}	32.4116846	32.4116161	1.0000021	0.57721578

The third example includes uncountable many formulae for γ ; see, e.g., [9]: for real $r > 0$

$$\gamma = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{\infty} \left\{ \left(\frac{n^k}{k!} \right)^r \left[\sum_{j=1}^k \frac{1}{j} - \log(k) \right] \right\}}{\sum_{k=0}^{\infty} \left(\frac{n^k}{k!} \right)^r}. \quad (7)$$

That this formula gives γ for each $r > 0$ means that the derivative of the rhs with respect to r is zero. There are also doubly uncountable formulae for γ . As the fourth example, we present the formula (3.13) from [7]:

$$\gamma = r \int_0^\infty \left(\frac{1}{1+x^q} - \exp(-x^r) \right) \frac{dx}{x}, \quad q > 0, r > 0. \quad (8)$$

The numerical value of the Euler-Mascheroni constant has been calculated repeatedly to ever increasing decimal places; see, e.g., [9]. The current world record (as of 14 June 2023) is 700 000 000 000 decimal digits of γ and belongs to Jordan Ranous and Kevin O'Brien (see [link](https://ehfd.github.io/world-record/euler-mascheroni-constant)¹).

The Euler-Mascheroni constant appears in numerous places in number theory, including the theory of the Riemann zeta function, for example in Nicolas' and Robin's criteria for the Riemann Hypothesis (see, e.g., [10, vol.1, chapters 5 and 7]). One of the most remarkable appearances of the γ constant is in F. Mertens's two products over primes [11, p. 351], one of which involves the constants π , e , γ (the so-called "holy trinity"):

$$\lim_{n \rightarrow \infty} \frac{1}{\log(n)} \prod_{p < n} \left(1 + \frac{1}{p} \right) = \frac{6e^\gamma}{\pi^2}, \quad (9)$$

from which we obtain

$$\gamma = \log \left[\frac{\pi^2}{6} \lim_{n \rightarrow \infty} \frac{1}{\log(n)} \prod_{p < n} \left(1 + \frac{1}{p} \right) \right]. \quad (10)$$

With today's computers, we can verify the accuracy of the above relation. In the Tab. 1, we present a numerical test of (9) alongside the values of $\gamma(n)$ computed from finite products over primes:

$$\gamma(n) = \log \left[\frac{\pi^2}{6} \frac{1}{\log(n)} \prod_{p < n} \left(1 + \frac{1}{p} \right) \right]. \quad (11)$$

We present these numerical calculations as an illustration of (9), not as a method to compute γ , since more efficient algorithms are known (see, e.g., [9]).

Another appearance of γ is found in the formula for the average value of the divisor function $d(n)$, which counts the number of divisors of n , including 1 and n , as given by the theorem proved by Dirichlet (see, e.g., [11, Th.320]):

$$\frac{1}{n} \sum_{k=1}^n d(k) = \log n + 2\gamma - 1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right). \quad (12)$$

Values of γ obtained from the formula above for $n = 2^{15}, \dots, 2^{23}$ are presented in Tab. 2.

Relation (12) suggests that there is a connection between γ and the distribution of primes. Another example of this relation is found in [12, Corollary 1]:

$$\gamma = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1 - \Lambda(n)}{n} \right), \quad (13)$$

where the von Mangoldt function $\Lambda(n)$ is defined as

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

¹ <https://ehfd.github.io/world-record/euler-mascheroni-constant>

Tab. 2. Values of γ obtained from (12) for $n = 2^{20}, 2^{22}, \dots, 2^{32}$

n	$\sum_{k=1}^n d(k)$	γ from (12)
$2^{20} = 1\,048\,576$	14 698 342	0.57724382397818362
$2^{22} = 4\,194\,304$	64 607 782	0.57722880551235453
$2^{24} = 16\,777\,216$	281 689 074	0.57722242972001786
$2^{26} = 67\,108\,864$	1 219 788 256	0.57721736522986625
$2^{28} = 268\,435\,456$	5 251 282 902	0.57721609681052878
$2^{30} = 1\,073\,741\,824$	22 493 653 324	0.57721585470866853
$2^{32} = 4\,294\,967\,296$	95 928 700 948	0.57721570434208188

The rhs of (13) is a sum of two divergent series: the harmonic series and a decimated harmonic series where, instead of ones in the nominator values of $\Lambda(n) > 1$ appear; see Tab. 3.

Tab. 3. Values of γ obtained from (13) for $n = 10, \dots, 19^7$ compared with the actual value of the Euler-Mascheroni constant

n	Eq. (13)	ratio
10 000	0.576533359060	1.001183462901
100 000	0.576946987410	1.000465688352
1 000 000	0.577417595935	0.999650285972
10 000 000	0.577287376546	0.999875778257

A surprising appearance of γ is in the “harmonic” sum of the reciprocals of the non-trivial zeros ρ of the Riemann zeta function [13, p. 67 and p. 159], [14, pp. 80–82]:

$$\sum_{\rho} \frac{1}{\rho} = 1 + \frac{1}{2}\gamma - \frac{1}{2}\log(4\pi) = 0.023095708966\dots \quad (15)$$

Above, the three constants π, e, γ appear: e is hidden in the natural logarithm. The sum (15) is real and convergent when zeros ρ and the complex conjugate $\bar{\rho}$ are paired and summed according to the increasing absolute values of the imaginary parts of ρ . Several years ago, using the L-function calculator written by Michael Rubinstein (see [15]), we calculated 100 000 000 imaginary parts of zeros of $\zeta(s)$; the last obtained zero has the value $\rho_{100\,000\,000} = \frac{1}{2} + i42653549.7609515$. In Tab. 4, we present approximations to γ obtained from (15) after summing over 1000, 10 000, $\dots$, 100 000 000 zeros of the zeta function.

Another connection with the Riemann zeta function $\zeta(s)$ is given by the astonishing fact: the Riemann Hypothesis is true iff the following relation holds (see [16]):

$$\frac{1}{\pi} \int_0^\infty \frac{2t \arg[\zeta(1/2 + it)]}{(1/4 + t^2)^2} dt = \gamma - 3. \quad (16)$$

The largest known prime numbers are of the form $2^p - 1$ where p is also a prime; these are called Mersenne primes

(see, e.g., <https://www.mersenne.org/>). In [17, p. 101], [18, p. 388] (see also [19, §3.5]), the Lenstra-Pomerance-Wagstaff conjecture was formulated: if $\mathcal{M}_n$ denotes the n^{th} Mersenne prime, then $\mathcal{M}_n$ grows doubly exponentially with n :

$$\log_2 \log_2 \mathcal{M}_n \sim ne^{-\gamma}. \quad (17)$$

The presence of γ here comes from Mertens’s result (9). In Fig. 1, we compare the Lenstra-Pomerance-Wagstaff conjecture with all 51 currently known Mersenne primes ([link](#)²). The least squares fit yields a line with a slope of 0.54, which leads to a rather poor value 0.61 for γ .

Fast-converging formulae for γ were presented in [9]; these are the most commonly utilized formulae for high-precision numerical calculations of γ .

In this paper, we present some new formulae for γ obtained by using special values of the arguments of the logarithmic and cosine integrals. A similar idea appeared in [20], where the series for the exponential integral was used to calculate γ up to 3566 decimal places. A few of these new expressions represent the Euler-Mascheroni constant as the difference of two numbers, one of which is transcendental. This offers hope for proving the irrationality and perhaps even the transcendence of γ .

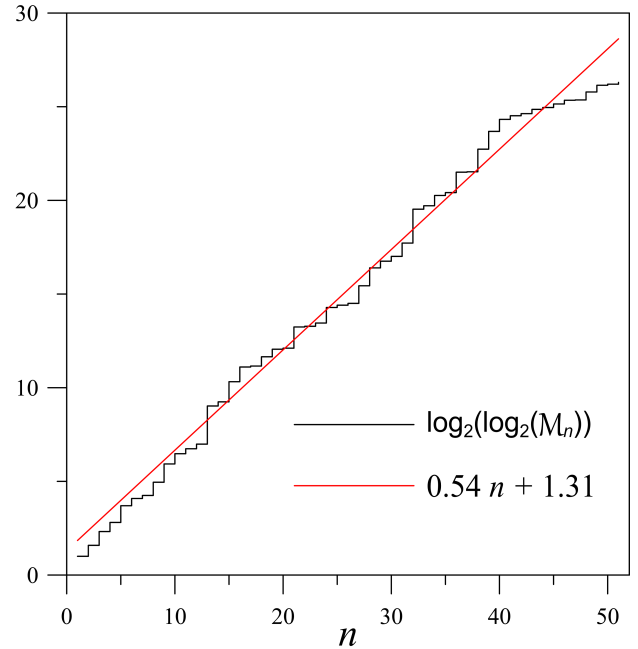


Fig. 1. Plot illustrating the Lenstra-Pomerance-Wagstaff conjecture. The least-squares fit was performed on all known $\mathcal{M}_n$ and is given by $0.54n + 1.31$, which yields a rather poor value for γ of 0.61

An astonishing formula was found by A. Kawalec [21], who expressed γ in terms of the imaginary parts t_l of the nontrivial zeros of the Riemann zeta function on the critical line $\zeta(\frac{1}{2} + it_n) = 0$:

² <https://www.mersenne.org/>

$$\gamma = \lim_{k \rightarrow \infty} \left[2 \sum_{n=1}^k \sum_{m=n+1}^k \frac{(-1)^m (-1)^{n+1}}{\sqrt{mn}} \cos \left(t_l \log(m/n) \right) + \log(k) \right], \quad l = 1, 2, 3, \dots \quad (18)$$

There is an infinitude of formulae, as there are infinitely many nontrivial zeta zeros. The above formula does not depend on the Riemann Hypothesis: if there are any zeros off the critical line, they do not enter (18).

II. Logarithmic Integral

The logarithmic integral is defined for all positive real numbers $x \neq 1$ by the definite integral

$$\text{li}(x) \equiv \begin{cases} p.v. \int_0^x \frac{du}{\log(u)}, & \text{for } x > 1, \\ \int_0^x \frac{du}{\log(u)}, & \text{for } 0 < x < 1, \end{cases} \quad (19)$$

where *p.v.* stands for the Cauchy principal value around $u = 1$:

$$p.v. \int_0^x \frac{du}{\log(u)} = \lim_{\epsilon \rightarrow 0} \left(\int_0^{1-\epsilon} \frac{du}{\log(u)} + \int_{1+\epsilon}^x \frac{du}{\log(u)} \right). \quad (20)$$

There is a series giving the logarithmic integral $\text{li}(x)$ for all $x > 1$ (see [22, formulae 5.1.3 and 5.1.10]):

$$\text{li}(x) = \gamma + \log \log x + \sum_{n=1}^{\infty} \frac{\log^n x}{n \cdot n!} \quad \text{for } x > 1. \quad (21)$$

This series is quickly convergent because it has $nn!$ in the denominator, which eventually overwhelms the $\log^n(x)$ term in the numerator. The above expansion was known to C.F. Gauss and F.W. Bessel; see the remarks by R. Dedekind after the famous paper “Über die Anzahl der Primzahlen unter einer gegebenen Grösse” by B. Riemann in [23, p. 168].

Tab. 4. Values of γ obtained from (15) after summing over $n = 1000, 10\,000, \dots, 100\,000\,000$ zeros of $\zeta(s)$

n	γ
1 000	0.5757765
10 000	0.5769463
100 000	0.5771715
1 000 000	0.5772091
10 000 000	0.5772147
100 000 000	0.5772155

After a change of variables, a variant of the above series is given by:

$$\int_x^{\infty} \frac{e^{-t}}{t} dt = -\gamma - \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!}. \quad (22)$$

This was used in [20] for large $x > 0$, when the lhs of the above equation is practically zero (in fact, it is $\mathcal{O}(e^{-x}/x)$), to compute 3566 digits of γ , see also [9].

The logarithmic integral takes the value 0 at only one real number which is denoted by μ and called the Ramanujan-Soldner constant

$$\int_0^{\mu} \frac{du}{\log u} = 0, \quad (23)$$

see, e.g., [24, entry 14, p. 126, Eq. (11.3)], and its numerical value is:

$$\mu = 1.45136923488338105028396848589202745 \dots$$

Thus, for $x > \mu$, we have:

$$\text{li}(x) = \int_{\mu}^x \frac{du}{\log(u)}. \quad (24)$$

Inserting $x = \mu > 1$ into (21), we obtain the first formula expressing the Euler-Mascheroni constant via the Ramanujan-Soldner constant:

$$\gamma = -\log \log \mu - \sum_{n=1}^{\infty} \frac{\log^n \mu}{n \cdot n!}. \quad (25)$$

The constant $\log \mu = 0.37250741078136663446 \dots$ appearing here is the zero of the exponential integral $\text{Ei}(x)$; see (A1). The series in (25) is very quickly convergent. Using PARI [25], we verified that summing (25) up to only $n = 20$ reproduces 31 digits of γ . In the Appendix, we provide the script to check (25) to any desired number of digits. In contrast, calculating (1) at $k = 1\,000\,000$ yields only 5 digits of γ .

An even faster-converging series was discovered by Ramanujan [24, p. 130]:

$$\begin{aligned} \int_{\mu}^x \frac{du}{\log u} &= \gamma + \log \log x + \\ &+ \sqrt{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (\log x)^n}{n! 2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}, \quad \text{for } x > 1. \end{aligned} \quad (26)$$

Putting here $x = \mu$, we obtain a second formula for the Euler-Mascheroni constant:

$$\gamma = -\log \log \mu + \sqrt{\mu} \sum_{n=1}^{\infty} \frac{(-1)^n (\log \mu)^n}{n! 2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}. \quad (27)$$

We verified using PARI that summing the series above to $n = 20$ reproduces correctly 37 digits of γ .

Using $x = e$ in (21) greatly simplifies the series, leading to the third expression for the Euler-Mascheroni constant:

$$\gamma = \int_{\mu}^e \frac{du}{\log u} - \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} := \alpha - \beta, \quad (28)$$

where the numbers

$$\alpha := \int_{\mu}^e \frac{du}{\log u} = 1.89511781635593675546652 \dots, \quad (29)$$

$$\beta := \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} = 1.31790215145440389486 \dots \quad (30)$$

The number β is irrational by the same reasoning which proves the irrationality of $e = \sum_{n=0}^{\infty} 1/n!$ (see, e.g., [26, p. 65]), which can be repeated here *mutatis mutandis*. In fact, from the Siegel-Shidlovsky theorem [27, see Eq. 5.2 for $k = 1$], it follows that β (30) is transcendental.

Putting $x = e$ in (26) yields the fourth expression for the Euler-Mascheroni constant:

$$\gamma = \int_{\mu}^e \frac{du}{\log u} + \sqrt{e} \sum_{n=1}^{\infty} \frac{(-1)^n}{n! 2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}. \quad (31)$$

Finally, let us note that in [1], at several places (e.g., pp. 52, 104), we find that Euler expressed hope that γ is the logarithm of some important number. Above, we have presented two series for γ in terms of the logarithm of the Ramanujan-Soldner constant μ .

III. Cosine Integral

Many special functions involve the Euler-Mascheroni constant in their expansions. The cosine integral function $\text{Ci}(x)$ for $x > 0$ has a series expansion also containing γ (see, e.g., [22, §5.2, formula 5.2.16]):

$$\begin{aligned} \text{Ci}(x) &= - \int_x^{\infty} \frac{\cos u}{u} du = \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{2n(2n)!} = \\ &= \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{2^{n+1} n n! (2n-1)!!}, \end{aligned} \quad (32)$$

because $(2n)! = 2^n n! (2n-1)!!$, where the odd factorial $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)$. Using $x = 1$, we obtain the fifth expression for the Euler-Mascheroni constant:

$$\gamma = - \int_1^{\infty} \frac{\cos u}{u} du + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n)!}, \quad (33)$$

where:

$$\int_1^{\infty} \frac{\cos u}{u} du = -0.3374039229009681346626 \dots, \quad (34)$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n)!} = 0.2398117420005647259439 \dots \quad (35)$$

Again, γ can be expressed simply in terms of two other numbers. The cosine integral $\text{Ci}(x)$ has infinitely many zeros that do not have individual names and are non-periodic. They are usually denoted by c_k (see [28, Eq. (9)]). The first zeros are $c_0 = 0.61650548562 \dots$, $c_1 = 3.38418042255 \dots$, $c_2 = 6.42704774405 \dots$, ... In [28], A.J. MacLeod provides the asymptotic expansion for these zeros:

$$\begin{aligned} c_k \approx k\pi + \frac{1}{k\pi} - \frac{16}{3(k\pi)^3} + \frac{1673}{15(k\pi)^5} + \\ - \frac{507746}{105(k\pi)^7} + \frac{111566353}{315(k\pi)^9} \dots \end{aligned} \quad (36)$$

By substituting any specific zero c_k into (32), we obtain an infinity of expressions for γ

$$\gamma = - \sum_{n=1}^{\infty} \frac{(-c_k^2)^n}{2n(2n)!} - \log c_k, \quad k = 0, 1, 2, \dots \quad (37)$$

In Tab. A1, we present values of γ obtained from this formula using c_k calculated from (36). In the last column, the differences between the values in third column and γ are given.

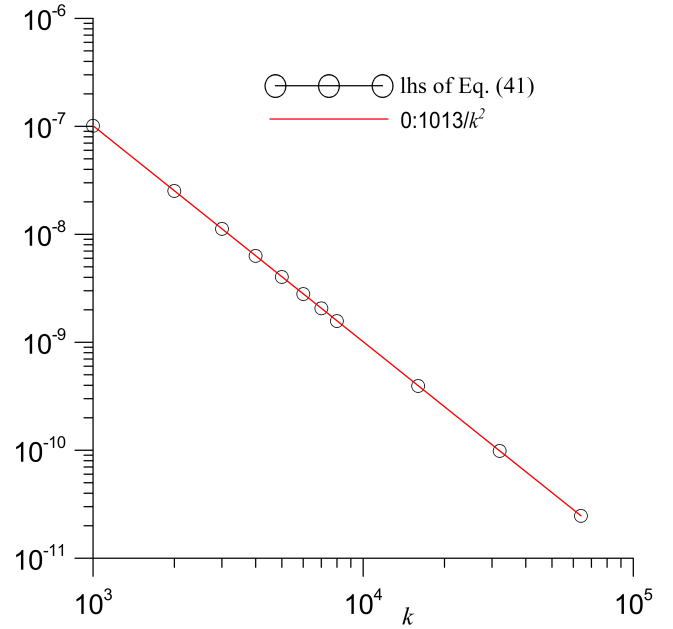


Fig. 2. The double-logarithmic plot of lhs of Eq. (41) for $k = 1000, 2000, \dots, 8000, 16000, 32000, 64000$

From the MacLeod formula (36), we see that the large zeros of $\text{Ci}(x)$ approach the zeros of $\sin(x) = \int \cos(x) dx$: $c_k \sim k\pi$ for large k . Thus, we have our sixth formula for the Euler-Mascheroni constant:

$$\gamma = \lim_{k \rightarrow \infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (k\pi)^{2n}}{2n(2n)!} - \log(k\pi) \right). \quad (38)$$

Denoting $x = k\pi$, the above formula can also be written as

$$\gamma = \lim_{x \rightarrow \infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2n(2n)!} - \log(x) \right). \quad (39)$$

In [29, p. 98], we found a similar formula obtained by S. Ramanujan:

$$\gamma = \lim_{x \rightarrow \infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n!n} - \log(x) \right). \quad (40)$$

It appears to be a subtle problem to reconcile Eqs. (39) and (40). Formula (39) in some sense resembles the original definition (1). Using PARI we computed the expression in the large parentheses in (38) for $k = 1000, 2000, 4000, 8000, \dots, 64\,000$. For example, for $k = 64\,000$, the expression in the large parentheses on the rhs of (38) gives $0.577215664926\dots$, i.e., it reproduces correctly the first 10 digits of γ . The differences between γ and the values obtained from (38) for the aforementioned set of k were perfectly arranged on a straight line on the double logarithmic plot (see Fig.2). A least-squares fit gave the equation of the line as $0.101321k^{-1.9999}$, which suggests that:

$$\left| \gamma + \log(k\pi) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (k\pi)^{2n}}{2n(2n)!} \right| = \frac{0.1013211}{k^2}. \quad (41)$$

In [22], we found the asymptotic expansion of $\text{Ci}(x)$; see formulae (5.2.9), (5.2.35), and (5.2.34):

$$\begin{aligned} \text{Ci} \sim & \frac{1}{x} \left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \frac{6!}{x^6} + \dots \right) \sin(x) + \\ & + \frac{1}{x^2} \left(1 - \frac{3!}{x^2} + \frac{5!}{x^4} - \frac{7!}{x^6} + \dots \right) \cos(x). \end{aligned} \quad (42)$$

Putting the above $x = k\pi$, we obtain:

$$\left| \gamma - \left(-\log(k\pi) - \sum_{n=1}^{\infty} \frac{(-1)^n (k\pi)^{2n}}{2n(2n)!} \right) \right| \sim \frac{1}{\pi^2 k^2}, \quad (43)$$

and $1/\pi^2 = 0.1013211836\dots$, which agrees with the constant on the rhs of (41).

Concluding remarks: We believe that the existence of countable many formulae for a given constant is a necessary condition for its irrationality, whereas the existence of uncountable many formulae is a necessary condition for transcendence. For example, there is an expression for π depending on arbitrary complex number z (see [30, not labelled formulae on top of p. 15]):

$$\pi = \sum_{k=0}^{\infty} \left(\frac{z-4}{4k+3} + \frac{z+4}{4k+1} + \frac{z}{4k+4} - \frac{3z}{4k+2} \right). \quad (44)$$

The fact that, for each $z \in \mathbb{C}$, the value of the rhs is constant means that the derivative of the rhs with respect to z is zero. In fact, the Cauchy-Riemann equations should be satisfied.

However, the existence of infinitely many expressions for a given number $r \in \mathbb{R}$ is not sufficient for irrationality. As a counterexample, we can give the following infinite sequence of telescoping series:

$$1 = \sum_{n=1}^{\infty} \frac{(n+1)^k - n^k}{n^k(n+1)^k}, \quad k = 1, 2, 3, \dots$$

Here is a less trivial example:

$$\frac{1}{2} = \sin\left(\frac{\pi}{6} + 2k\pi\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/6 + 2k\pi)^{2n+1}}{(2n+1)!},$$

$$k = 0, \pm 1, \pm 2, \dots$$

As for γ , there exist uncountable many expressions of the form given in (7), which provides a strong argument that γ is not only irrational but even transcendental.

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References

- [1] J. Havil, *Gamma: Exploring Euler's Constant*, Princeton University Press, Princeton, NJ (2003).
- [2] J.C. Lagarias, *Euler's constant: Euler's work and modern developments*, Bulletin of the American Mathematical Society **50**(4), 527–628 (2013).
- [3] J. Sondow, *Criteria for Irrationality of Euler's Constant*, Proceedings of the American Mathematical Society **131**(11), 3335–3345 (2003).
- [4] D.W. DeTemple, *A quicker convergence to Euler's constant*, The American Mathematical Monthly **100**(5), 468–470 (1993).
- [5] C. Mortici, *On new sequences converging towards the Euler-Mascheroni constant*, Computers and Mathematics with Applications **59**(8), 2610–2614 (2010).
- [6] X. Gourdon, P. Sebah, *Collection of formulae for the Euler constant* (2008), <http://numbers.computation.free.fr/Constants/Gamma/gammaFormulas.pdf>.
- [7] J. Choi, H.M. Srivastava, *Integral Representations for the Euler-Mascheroni Constant γ* , Integral Transforms and Special Functions **21**(9), 675–690 (2010).
- [8] G. Boros, V. Moll, *Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals*, Cambridge University Press (2004).
- [9] R.P. Brent, E.M. McMillan, *Some New Algorithms for High-Precision Computation of Euler's Constant*, Mathematics of Computation **34**(149), 305–312 (1980).
- [10] K. Broughan, *Equivalents of the Riemann Hypothesis, volume 1 and 2*, Cambridge University Press (2017).
- [11] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Science Publications (1980).
- [12] M.W. Coffey, *The stieltjes constants, their relation to the η_j coefficients, and representation of the Hurwitz zeta function*, Analysis (Oldenbourg Wissenschaftsverlag) **30** (2010).

- [13] H.M. Edwards, *Riemann's zeta function*, Pure and Applied Mathematics **58**, Academic Press (1974).
- [14] H. Davenport, *Multiplicative Number Theory*, Graduate Texts in Mathematics **74**, Springer-Verlag, New York (1980).
- [15] Rubinstein's, *L-function calculator*, <http://doc.sagemath.org/html/en/reference/lfunctions/sage/lfunctions/lcalc.html>.
- [16] S.K. Sekatskii, S. Beltraminelli, D. Merlini, *On equalities involving integrals of the logarithm of the Riemann ζ -function and equivalent to the Riemann hypothesis*, Ukrainian Mathematical Journal **64**(2), 247–261 (2012).
- [17] C. Pomerance, *Recent developments in primality testing*, The Mathematical Intelligencer **3**(3), 97–105 (1981).
- [18] S.S. Wagstaff, *Divisors of Mersenne Numbers*, Mathematics of Computation **40**(161), 385–397 (1983).
- [19] M.R. Schroeder, *Number Theory In Science And Communication, With Applications In Cryptography, Physics, Digital Information, Computing, And Self-Similarity*, Springer-Verlag New York, Inc (2006).
- [20] D.W. Sweeney, *On the Computation of Euler's Constant*, Mathematics of Computation **17**(82), 170 (1963).
- [21] A. Kawalec, *Asymptotic formulas for harmonic series in terms of a non-trivial zero on the critical line*, Computational Methods in Science and Technology, 161–166 (2019).
- [22] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York (1964).
- [23] B. Riemann, *Gesammelte Mathematische Werke, Wissenschaftlicher Nachlass und Nachträge – Collected Papers*, Springer-Verlag, Berlin, Heidelberg (1991).
- [24] B.C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer Verlag (1994).
- [25] *PARI/GP, version 2.3.0, 64 bits* (2018), <http://pari.math.u-bordeaux.fr/>.
- [26] W. Rudin, *Principles of mathematical analysis*, McGraw-Hill Book Co., New York (1976).
- [27] N.I. Fel'dman, A.B. Shidlovskii, *The development and present state of the theory of transcendental numbers*, Russian Mathematical Surveys **22**(3), 1–79 (1967).
- [28] A.J. MacLeod, *Asymptotic expansions for the zeros of certain special functions*, Journal of Computational and Applied Mathematics **145**(2), 261–267 (2002).
- [29] B.C. Berndt, *Ramanujan's Notebooks, Part I*, Springer Verlag (1985).
- [30] V.S. Adamchik, S. Wagon, π : A 2000-year search changes direction, Mathematica in Education and Research **5**(1), 11–19 (1996).
- [31] H. Cohen, R. Villegas, F. Fernando, D. Zagier, *Convergence acceleration of alternating series*, Experiment. Math. **9**(1), 3–12 (2000).

Appendix

Below is a simple PARI/GP script checking Eq. (25) to any arbitrary accuracy declared by \p precision. In the example shown, it is set to 2222. The output demonstrates agreement between the lhs and rhs of (25) up to the number of digits dictated by precision. The computation takes only a fraction of a second.

```
allocatemem()
\p 2222
Soldner=solve(x=1.4, 1.5, real(eint1(-log(x)))));
tmp=log(Soldner);
ss=suminf(n=1, tmp^n/(n*n!));
write("EMRS.txt", Euler+log(tmp)+ss);
```

Tab. A1. Values of expression (37) when for c_k the series (36) are substituted for $k = 10, 20, \dots, 100$

k	c_k from Eq. (36)	Eq. (37) for this c_k	Eq. (37) for this $c_k - \gamma $
10	31.447589011629313	0.5772156649004098	1.123×10^{-12}
20	62.847747177749027	0.5772156649015328	1.953×10^{-17}
30	94.258383581485718	0.5772156649015328	2.888×10^{-20}
40	125.67166120666795	0.5772156649015328	2.657×10^{-22}
50	157.08599750231211	0.5772156649015328	6.519×10^{-24}
60	188.50086358429127	0.5772156649015328	2.871×10^{-25}
70	219.91603253410894	0.5772156649015328	1.771×10^{-26}
80	251.33139082491842	0.5772156649015328	1.180×10^{-27}
90	282.74687536370536	0.5772156649015328	2.181×10^{-29}
100	314.16244828586940	0.5772156649015328	2.861×10^{-29}

In the above script, we used the fact that the logarithmic integral is related to the exponential integral $\text{Ei}(x)$; see, e.g., [22, formula (5.1.3)]:

$$\text{li}(x) = \text{Ei}(\log x), \quad x > 1, \quad (\text{A1})$$

where

$$\text{Ei}(x) \equiv \begin{cases} -p.v. \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, & \text{for } x > 0, \\ -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt, & \text{for } x < 0, \end{cases} \quad (\text{A2})$$

and the principal value is needed to avoid the singularity of the integrand at $t = 0$. The logarithmic integral is not implemented in PARI, whereas the exponential integral is implemented as `eint1(x)`. Running the above script produces the number 4.27×10^{-2235} . To check (27), modify the last lines as follows:

```
ss=suminf(n=1, (-1)^n*tmp^n/(2^(n-1.0)*n!)*
          sum(k=0, floor((n-1)*0.5), 1.0/(2.0*k+1.0)));
write("EMRS.txt", Euler+log(tmp)-sqrt(Soldner)*ss);
```

This time, the output obtained was 2.7328×10^{-2233} .

The Eq. (33) can be checked in PARI using the following commands:

```
allocatemem()
\p 2222
tmp=sumalt(n=1, (-1)^(n-1)/(2*n*(2*n)!));
c_i=intnum(u=1, [oo, I], cos(u)/u);
print(Euler+c_i-tmp);
```

We provide further explanations: PARI contains the numerical routine `sumalt` for summing infinite alternating series in which the extremely efficient algorithm of Cohen, Villegas, and Zagier [31] is implemented; in PARI, `oo` denotes infinity $+\infty$; `intnum(.)` is the function for numerical integration, and the flag `k*I` ($I = i$, i.e. $i^2 = -1$) tells the procedure that the integrand is an oscillating function of the type $\cos(kx)$, with $k = 1$ in this case. After a few minutes, the computation yielded $1.42335 \times 10^{-2235}$. This result shows the power of PARI's procedures: the value of the cosine integral at 1 is indeed calculated numerically without using the expansion (32) and the value of γ , so the circular reasoning (tautology) is avoided.



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