$3n + 3^k$ Problem

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Abstract: The Collatz problem is generalized to the $3n + 3^k$ case. It is shown that as long as the Collatz function iterates converge to the cycle passing through the number 1, the $3n + 3^k$ sequence converges to the cycle passing through the number 3^k for arbitrary positive integers n and k. The proof demonstrates that the sequence of $3n + 3^k$ function iterates for a number $3^k n$ corresponds exactly to the sequence of Collatz iterates for n multiplied by 3^k.

Key words: Collatz conjecture, number theory, dynamical systems

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The Collatz problem [3, 4] is a number theory problem that provides an algorithm for generating a sequence. The algorithm is as follows. Start with an arbitrary positive integer n. If n is even, divide it by two; if n is odd, multiply it by three and add one. The conjecture states that the sequence will always reach the number 1.

More formally, the Collatz conjecture asserts that a sequence defined by repeatedly applying the function

$$T_0(n) = \begin{cases} (3n+1)/2, & \text{for odd } n, \\ n/2, & \text{for even } n, \end{cases} \tag{1}$$

always converges to the cycle passing through the number 1 for arbitrary positive integer n.

Theorem 1. As long as the T_0 iterates converge to 1, the sequence defined by repeatedly applying the function

$$T_k(n) = \begin{cases} (3n+3^k)/2, & \text{for odd } n, \\ n/2, & \text{for even } n, \end{cases} \tag{2}$$

converges to the cycle passing through the number 3^k for any arbitrary positive integers n and k.

Note that the $3n + 3^k$ problem was recently discussed in [1].

Proof: The function $T_k(n)$ can be adjusted for multiples of three using

$$3 \cdot T_k(n) = \begin{cases} (3 \cdot 3n + 3^k \cdot 3)/2, & \text{for odd } n, \\ 3n/2, & \text{for even } n. \end{cases} \tag{3}$$

Now, substitute $3n$ for m (so that m is a multiple of 3). Then $3 \cdot T_k(n) = T_{k+1}(3n)$, and therefore

$$T_{k+1}(m) = \begin{cases} (3m+3^{k+1})/2, & \text{for odd } m, \\ m/2, & \text{for even } m. \end{cases} \tag{4}$$

Note that $3n$ is odd when n is odd, and $3n$ is even when n is even. It means that the sequence of T_{k+1} iterates for $m = 3n$ corresponds exactly to the sequence of T_k iterates for n multiplied by 3.

Now we know the behavior of the trajectory of T_{k+1} when m is a multiple of 3. It remains to determine what happens to this trajectory when m is not a multiple of 3. If such a number is even, then we can repeatedly pull out all the factors of 2 (the even branch of T_{k+1}) until we reach an odd number. Therefore, we focus on odd m. A single iterate of the T_{k+1} function yields $(3m + 3)/2$ (the odd branch of T_{k+1}), where $(3m + 3)/2$ is a multiple of 3 ($3m + 3$ is obviously a multiple of 3, the division by 2 has no effect on divisibility by 3). Note that the iterates of T_{k+1} converge to the cycle passing through the number 3^{k+1}, which corresponds to $T_0(n) = 1$. □

Corollary 1.1. T_k iterates always lead to the cycle $1 \cdot 3^k \to 2 \cdot 3^k \to 1 \cdot 3^k$, assuming the Collatz conjecture holds.

The iterates of T_0 lead to the cycle $1 \to 2 \to 1$, where as the iterates of T_1 go to $3 \to 6 \to 3$, etc.

Corollary 1.2. The sequence of T_k iterates for a number $3^k n$ corresponds exactly to the sequence of T_0 iterates for n multiplied by 3^k.

In short, $T_k(3^k n) = 3^k T_0(n)$. See an example for T_0, T_1, and T_2 in Tab. 1.

Now, consider T_1 starting with an odd number n. The first T_1 takes us to $(3n + 3)/2$, which is a number of the form $3m/2$. Meanwhile, consider the even number $n + 1$ and subject it to iteration T_0. The result is $(n + 1)/2$ (since the $n + 1$ is even), which is the number of the form $m/2$. We can see that subsequent T_1 iterates correspond exactly to the T_0 iterates multiplied by 3. Thus, there is a relationship between the initial number n for T_1 and the number $n + 1$ for T_0. An example for the numbers 63 and 64 is given in Tab. 2.

In [2], Lagarias extends $T_0(n)$ to the rational numbers with odd denominators, and shows that any rational element q of a cycle of $T_0(n)$ must be of the form

$$q = \frac{\sum_{i=0}^{l-1} 3^i 2^{a_i}}{2^m - 3^l} , \tag{5}$$

for some $l \geq 0$, $m > l$ and $m > a_0 > a_1 > \ldots > a_{l-1} \geq 0$.

Lemma 1.1. The map $T_0(n)$ has no non-zero rational cycle elements with a denominator which is a multiple of 3.

Proof: Suppose that $l = 0$. Then $2^m - 3^0 = 2^m - 1$ can be a multiple of 3 (for instance, for $m = 4$). However, in this case the sum in the numerator is 0, because we sum from $i = 0$ to -1, which is an empty sum; therefore, $q = 0$.

Now suppose that $l > 0$. Then the numerator is $2^m - 3^l \equiv \equiv 2^m \pmod 3$, and 2^m is never a multiple of 3, by the unique prime factorization of integers. □

Another way of looking at $T_k(n)$ is by introducing the function $L_{3^k}(n) = n/3^k$. Notice that $L_{3^k}(n)$ is a bijection on the rational numbers with the inverse function ${L_{3^k}}^{-1}(n) = 3^k n$.

We have

$$T_k(n) = {L_{3^k}}^{-1} \circ T_0 \circ L_{3^k}(n). \tag{6}$$

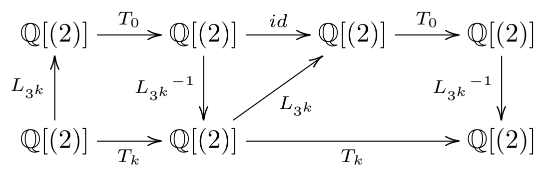
We now show this:

$$\begin{aligned} T_k(n) &= \begin{cases} (3n + 3^k)/2, & \text{for odd } n, \\ n/2, & \text{for even } n \end{cases} = \\ &= \begin{cases} 3^k(3(n/3^k) + 1)/2, & \text{for odd } n, \\ 3^k(n/3^k)/2, & \text{for even } n \end{cases} = \\ &= {L_{3^k}}^{-1} \circ T_0 \circ L_{3^k}(n). \end{aligned} \tag{7}$$

Considering our equation in (7), we obtain the following commutative diagram

$$\begin{CD} n/3^k @>{T_0}>> T_0(n/3^k) \\ @A{L_{3^k}}AA @VV{{L_{3^k}}^{-1}}V \\ n @>>{T_k}> T_k(n) \end{CD}$$

Using the notation $\mathbb{Q}[(2)]$ for all rational numbers with an odd denominator, iterating the above diagram yields the following diagram



giving us

$$T_k(T_k(n)) = {L_{3^k}}^{-1}(T_0(T_0(L_{3^k}(n)))), \tag{8}$$

and we may continue this process, since L_{3^k} and ${L_{3^k}}^{-1}$ cancel each other out. We will use this fact extensively in the proof of the next corollary.

Corollary 1.3. $T_k(n)$ has exactly the same integer cycles as $T_0(n)$, except that they are scaled by 3^k.

Proof: Let q be an element of an integral cycle of $T_k(n)$ of length l. Then

$$\begin{aligned} q &= {T_k}^{(l)}(q) = \\ &= {L_{3^k}}^{-1} \circ T_0 \circ L_{3^k} \circ L_{3^k}{-1} \circ T_0 \circ L_{3^k} \circ \\ &\quad \underbrace{\circ \ldots \circ {L_{3^k}}^{-1} \circ T_0 \circ L_{3^k}(q)}_{l} = \\ &= {L_{3^k}}^{-1} \circ \underbrace{T_0 \circ \ldots \circ T_0}_{l} \circ L_{3^k}(q) = \\ &= {L_{3^k}}^{-1} \circ {T_0}^{(l)} \circ L_{3^k}(q), \end{aligned} \tag{9}$$

where ${L_{3^k}}^{-1} \circ T_0 \circ L_{3^k}$ is composed exactly l times on the second and third lines. Applying L_{3^k} from the left to the first part of the equation above and the last part of the equation yields:

$$q/3^k = {T_0}^{(l)}(q/3^k), \tag{10}$$

so $q/3^k$ is a rational cycle element of T_0. However, by the preceding result, there are no rational cycle elements whose denominator is a multiple of 3. Therefore, $3^k | q$ and hence $q = 3^k q'$. It follows that q' is a fixed point of ${T_0}^{(l)}$ since

$$\begin{aligned} q/3^k &= {T_0}^{(l)}(q/3^k), \\ 3^k q'/3^k &= {T_0}^{(l)}(3^k q'/3^k), \\ q' &= {T_0}^{(l)}(q'). \end{aligned} \tag{11}$$

Tab. 1. T_0, T_1 and T_2 function iterates

Function	Iteration						
	0	1	2	3	4	5	6
T_2	189	288	144	72	36	18	9
T_1	63	96	48	24	12	6	3
T_0	21	32	16	8	4	2	1

Tab. 2. T_0 and T_1 function iterates for starting values $n+1$ and n

Function	Iteration						
	0	1	2	3	4	5	6
T_1	63	96	48	24	12	6	3
T_0	64	32	16	8	4	2	1

Hence, any integral cycle element of T_k is a integral cycle element of T_0 multiplied by 3^k.

Now, suppose that q is an integral cycle element of T_0. We claim that ${T_k}^{(l)}(3^k q) = 3^k q$. As shown above,

$$ {T_k}^{(l)}(n) = {L_{3^k}}^{-1} \circ {T_0}^{(l)} \circ L_{3^k}(n), \tag{12} $$

hence

$$ \begin{aligned} {T_k}^{(l)}(3^k q) &= {L_{3^k}}^{-1} \circ {T_0}^{(l)} \circ L_{3^k}(3^k q) = \\ &= {L_{3^k}}^{-1} \circ {T_0}^{(l)}(3^k q/3^k) = \\ &= {L_{3^k}}^{-1} \circ {T_0}^{(l)}(q) = \\ &= {L_{3^k}}^{-1}(q) = \\ &= 3^k q \,. \end{aligned} \tag{13} $$

□

In [2], a special type of generalization of $T_0(n)$ is considered, namely

$$ T_O(n) = \begin{cases} (3n+O)/2, & \text{for odd } n, \\ n/2, & \text{for even } n, \end{cases} \tag{14} $$

where $O \equiv \pm 1 \pmod 6$. Thus, O is either of the form $O = 6o+1$ or $O = 6o-1$ for some integer o. Hence, in particular, O will never be a multiple of 3. We now extend the result of the previous corollary to this generalization $T_O(n)$.

Theorem 2. The integral cycles of the map $T_{3^k O}(n)$ are exactly the integral cycles of $T_O(n)$, multiplied by 3^k.

Proof: First, note that for any non-zero integers A and B, it holds that

$$ \begin{aligned} L_A(L_B(n)) &= n/B/A = n/(B\cdot A) = \\ &= n/(A\cdot B) = n/A/B = L_B(L_A(n)), \end{aligned} \tag{15} $$

likewise, we have ${L_B}^{-1}({L_A}^{-1}(n)) = {L_A}^{-1}({L_B}^{-1}(n))$.

Just as in the case of $T_k(n)$ we have $T_O(n) = {L_O}^{-1}(T_0(L_O(n)))$ and $T_{3^k O}(n) = {L_{3^k O}}^{-1}(T_0(L_{3^k O}(n)))$. And since $L_{3^k O}(n) = L_{3^k}(L_O(n))$ we have

$$ \begin{aligned} T^{(l)}_{3^k O}(n) &= {L_{3^k O}}^{-1} \circ T_0 \circ L_{3^k O} \circ {L_{3^k O}}^{-1} \circ T_0 \circ L_{3^k O} \circ \\ &\quad \circ \underbrace{\ldots \circ {L_{3^k O}}^{-1} \circ T_0 \circ L_{3^k O}(n)}_{l} = \\ &= {L_{3^k}}^{-1} \circ {L_O}^{-1} \circ T_0 \circ L_O \circ L_{3^k} \circ \\ &\quad \circ \underbrace{\ldots \circ {L_{3^k}}^{-1} \circ {L_O}^{-1} \circ T_0 \circ L_O \circ L_{3^k}}_{l}(n) = \\ &= {L_{3^k}}^{-1} \circ T_O \circ L_{3^k} \circ {L_{3^k}}^{-1} \circ T_O \circ L_{3^k} \circ \\ &\quad \circ \underbrace{\ldots \circ {L_{3^k}}^{-1} \circ T_O \circ L_{3^k}}_{l}(n) = \\ &= {L_{3^k}}^{-1} \circ \underbrace{T_O \circ \ldots T_O}_{l} \circ L_{3^k}(n) = \\ &= {L_{3^k}}^{-1} \circ {T_O}^{(l)} \circ L_{3^k}(n). \end{aligned} \tag{16} $$

Where the compositions on the first and second lines are exactly l times, the compositions on the third and fourth lines are exactly l times, and the compositions on lines five and six are l times. Accordingly, we have

$$ {T_O}^{(l)}(n) = {L_O}^{-1}({T_k}^{(l)}(L_O(n))), \tag{17} $$

and

$$ {T_O}^{(l)}(n) = {L_{3^k O}}^{-1}({T_0}^{(l)}(L_{3^k O}(n))). \tag{18} $$

Now, suppose that q is a integral cycle element of $T_{3^k O}(n)$, and let l be the length of this cycle. Then

$$ q = L_{3^k}({T_O}^{(l)}(L_{3^k}(q))) \quad \Leftrightarrow \quad q/3^k = {T_O}^{(l)}(q/3^k), \tag{19} $$

hence $q/3^k$ is a (fractional) cycle element of $T_O(n)$. But we also have that

$$\begin{aligned} q/3^k = {T_O}^{(l)}(q/3^k) = {L_O}^{-1}(T_0^{(l)}(L_O(q/3^k))) \quad \Leftrightarrow \\ \Leftrightarrow \quad q/(3^k O) = {T_0}^{(l)}(q/(3^k O)). \end{aligned} \tag{20}$$

Hence, $q/(3^k O)$ is a (rational) cycle element of $T_0(n)$. However, as shown above, there are no rational cycle elements of $T_0(n)$ with a denominator divisible by 3. Therefore, $q = = 3^k \cdot q'$ and q'/O is a (rational) cycle element of $T_0(n)$. It follows that $q/3^k = 3^k \cdot q'/3^k = q'$ is an integral cycle element of $T_O(n)$.

To complete the proof, we have

$$\begin{aligned} {T_{3^k O}}^{(l)}(q) &= L_{3^k}({T_O}^{(l)}(q/3^k)) = \\ &= 3^k \cdot {T_O}^{(l)}(q') = \\ &= 3^k \cdot q' = q \ , \end{aligned} \tag{21}$$

so any cycle element of $T_{3^k O}(n)$ is 3^k times a cycle element of $T_O(n)$. Now, suppose that q' is a cycle element of $T_O(n)$ then

$$\begin{aligned} {T_{3^k O}}^{(l)}(3^k \cdot q') &= {L_{3^k}}^{-1}({L_O}^{(l)}(L_{3^k}(3^k \cdot q'))) = \\ &= 3^k \cdot {L_O}^{(l)}(3^k \cdot q'/3^k) = \\ &= 3^k \cdot {L_O}^{(l)}(q') = \\ &= 3^k \cdot q' \ . \end{aligned} \tag{22}$$

Thus, $3^k \cdot q'$ is an integral cycle element of $T_{3^k O}(n)$. $\square$

Conclusion

We have shown that $T_k(n)$ behaves on multiples of 3^k exactly as $T_0(n)$, and that any n becomes a multiple of 3^k after finitely many steps. Therefore, studying $T_k(n)$ yields no new information about $T_0(n)$. However, it might be the case that if one were able to prove the conjecture for $T_k(n)$, this would automatically mean that they prove the conjecture for $T_0(n)$. Likewise, we have shown that $T_{3^k O}(n)$ behaves on multiples of 3^k exactly like $T_O(n)$, and that they have the same integral cycles (up to a factor 3^k). Consequently, all heuristics for $T_O(n)$ also hold for $T_{3^k O}(n)$.

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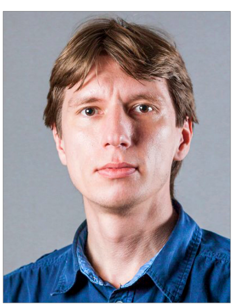
References

[1] N. Boulkaboul, $3n + 3^k$*: New perspective on Collatz conjecture* (2022). arXiv:2212.00073.

[2] J.C. Lagarias, *The set of rational cycles for the 3x+1 problem*, Acta Arithmetica **56**(1), 33–53 (1990).

[3] J.C. Lagarias, *The* $3x + 1$ *problem: An annotated bibliography (1963–1999) (sorted by author)* (2003). arXiv:math/0309224.

[4] J.C. Lagarias, *The* $3x + 1$ *problem: An annotated bibliography, II (2000–2009)* (2012). arXiv:math/0608208.



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