Odd Spoof Multiperfect Numbers Of Higher Order

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Abstract: We extend our previous work on odd spoof multiperfect numbers to numbers with spoof factor multiplicities greater than 2. As a result, we find 11 new integers that would be odd multiperfect, if only one of their prime factors had higher multiplicity. An example is 181545, which would be an odd multiperfect number, if only one of its prime factors, 3, had multiplicity 5.
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I. Introduction

Recall that $\sigma(n)$ denotes the sum-of-divisors function of the positive integer n, and n is said to be *perfect* if $\sigma(n) = 2n$, and multiperfect (or k-perfect) if $\sigma(n) = kn$ for some positive integer $k \geq 2$. No odd perfect numbers have been found so far, but Descartes noted that

$$\mathcal{D} = 198585576189$$

would be an odd perfect number if only one of its composite factors, 22021, were prime. Regrettably, $22021 = 19^2 \cdot 61$, so this is not the case. Since Descartes, much effort has been expended to find such "spoof perfect" numbers, without success. In our previous paper [4], we discovered a few numbers akin to $\mathcal{D}$, for instance

$$S = 8999757 = 3^2 \cdot 13^2 \cdot 61 \cdot 97,$$

which would be an odd multiperfect number if we assume (wrongly) that one of its prime factors, 61, is a square. Indeed, if that were the case, we would have

$$\begin{aligned}\sigma(S) &= (3^2+3+1)(13^2+13+1)(97+1)(61^2+61+1)\\ &= (13)\cdot(3\cdot 61)\cdot(2\cdot 7^2)\cdot(3\cdot 13\cdot 97)\\ &= 98\cdot 3^2\cdot 13^2\cdot 61\cdot 97\\ &= 98S.\end{aligned}$$

This led us to devise an algorithm to search for such numbers and found several more. In this paper, our aim is to develop our methods even further; first, by generalizing the concept of spoof k-perfect numbers, and second, by extending our search for numbers similar to $\mathcal{D}$ and S. As a result, we find 11 new odd positive integers which would be multiperfect, if only one of their prime factors had higher multiplicity. One such example is

$$T = 181545 = 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19,$$

which would be an odd multiperfect number, if only one of its prime factors, 3 had multiplicity 5:

$$\begin{aligned}\sigma(T) &= (3^5+3^4+3^3+3^2+3+1)(5+1)(7^2+7+1)\\ &\quad (13+1)(19+1)\\ &= (2^2\cdot 7\cdot 13)(2\cdot 3)(3\cdot 19)(2\cdot 7)(2^2\cdot 5)\\ &= 192\cdot 3\cdot 5\cdot 7^2\cdot 13\cdot 19\\ &= 192T.\end{aligned}$$

In the next sections, we will provide a generalization of spoof multiperfect numbers, and discuss some of their properties. We will then adapt Robin's classical inequality to spoof multiperfect numbers and provide details about the algorithm we used to find our results, including pseudo-code.

II. Generalized spoof multiperfect numbers

In our previous paper [4], we defined two kinds of spoof multiperfect numbers. In particular, we designated the positive integer $s = nx$ as a spoof k-perfect number

1. *of the first kind* if $\sigma(n)(x+1) = knx$,
2. *of the second kind* if $\sigma(n)(x^2+x+1) = knx$,

for a positive integer $k \geq 2$. We shall now introduce an extension of this definition by allowing the spoof factor x to have any multiplicity greater than 2.

Definition 1 (Spoof k-perfect number of order α)**.** Let $s = nx$ be a positive integer such that $n, x \in \mathbb{N}$ and $n, x \geq 2$. Furthermore, let $\alpha \geq 1$ be an integer and define

$$\mathcal{S}_\alpha = \sum_{a=0}^{\alpha} x^a.$$

Then, if

$$\sigma(n)\mathcal{S}_\alpha = knx,$$

for some positive integer k, then s is a spoof k-perfect number *of order* α.

Note that the case $\alpha = 1$ corresponds to the classical Descartes numbers, the case $\alpha = 2$ to the numbers in our previous work (such as 8999757), while the cases $\alpha > 2$ form the basis of our study in this paper. A trivial example of an odd spoof k-perfect number of order 3 is $s = 15$. Indeed, if we assume (incorrectly) that its prime factor 3 has multiplicity 3, then we have

$$\begin{aligned}\sigma(s) &= (5+1)\cdot(3^3+3^2+3+1)\\ &= 2^4\cdot 3\cdot 5\\ &= 16s.\end{aligned}$$

In hopes of finding such numbers, we implemented an algorithm that finds all spoof multiperfect numbers of order α within a given range, which we outline in Section IV. We were thus able to check all integers $s = nx$ with $n < 1.6\times 10^7$, of order $\alpha \leq 10$. We found 14 spoof multiperfect numbers, of which 11 are new, for which x is a prime that is also coprime to n. Table 1 shows these integers.

Table 1: Odd spoof k-perfect numbers $s = nx$ of order α

s	n	x	k	α
15	5	3	16	3
33	11	3	44	4
1911	637	3	152	5
1989	153	13	280	3
34485	11495	3	56	4
36309	12103	3	160	5
77805	11115	7	16	2
92781	1521	61	97	2
105435	21087	5	256	4
181545	60515	3	192	5
241395	80465	3	64	4
8999757	147537	61	98	2
62998299	1032759	61	112	2
440988093	7229313	61	114	2

Note that the integers $s = 77805, 92781$, and 8999757 have already been discovered in our previous paper. We also note that the numbers $s = 62998299$ and 440988093 are remarkable because they also have $x = 61$, which now accounts for the majority of odd spoof multiperfect numbers of order 2. It also appears in Descartes' classical example $s = 198585576189$, which is the only known odd spoof perfect number of order 1.
Many other odd spoof multiperfect numbers exist, for which x is either composite, or prime but not coprime to n. We have ommitted these numbers from the results that we share in this paper.
As we noted in our previous paper, one may notice at this point that multiperfect numbers of this magnitude should not exist so early, due to an inequality discovered by Guy Robin [2] in 1984, i.e., that

$$\sigma(n) < e^{\gamma} n \log\log n,$$

where γ is the Euler-Mascheroni constant and $n > 5040$, if and only if the Riemann Hypothesis is true. It thus follows that we would expect a k-perfect number n to appear only after

$$n > e^{e^{ke^{-\gamma}}},$$

which is not the case in the spoof examples above. This observation leads us to examine the "spoof equivalent" of this inequality, which we will do in the next section.

III. Robin's inequality for spoof multiperfect numbers

We begin by adapting Robin's inequality to spoof k-perfect numbers in the following manner.

Lemma 1. Let $s = nx$ denote a spoof k-perfect number of order α. Furthermore, let $n > 5040$. Then, assuming the Riemann Hypothesis, we have:

$$\frac{kx}{\mathcal{S}_\alpha} < e^{\gamma}\log\log n,$$

where

$$\mathcal{S}_\alpha = \sum_{a=0}^{\alpha} x^a.$$

Proof. Let $s = nx$ denote a spoof k-perfect number of order α. Thus, by Definition 1, we have

$$\sigma(n) = \frac{knx}{\mathcal{S}_\alpha}.$$

On the other hand, Robin's inequality states that for $n > 5040$,

$$\sigma(n) < e^{\gamma} n\log\log n.$$

Combining these two above gives

$$\frac{knx}{\mathcal{S}_\alpha} < e^{\gamma} n\log\log n,$$

and after simplifying n on both sides our claim is proved. □

A quick corollary of the above gives a bound on the components of the classical Descartes numbers.

Corollary 1. Let $s = nx$ denote a Descartes number with pseudo-prime factor x. Then, assuming the Riemann Hypothesis, we have:

$$\frac{2x}{x+1} < e^{\gamma}\log\log n.$$

Proof. We simply apply Lemma 1 with $k = 2$ and $\alpha = 1$. Furthermore, we no longer need the restriction $n > 5040$ because no Descartes numbers exist with n smaller than 5040. □

IV. Algorithm

In this small final section, we give a few details about the algorithm we used to find the results in this paper, which is very similar to the one in our previous work [4]. We run through positive integers n and compute the quantity

$$q = \frac{\sigma(n)}{kn}.$$

Taking care that the fraction q is in the lowest terms possible (i.e., the numerator q_{num} and denominator q_{den} have greatest common divisor 1), we can compute their difference δ:

$$\delta = q_{den} - q_{num}.$$

Then if

$$\delta = \sum_{a=0}^{\alpha} q_{num}^{a} - q_{num},$$

we have found a spoof k-perfect number $s = nx$ of order α, where the spoof factor is $x = q_{num}$.
In practical terms, we can check if the positive integer n is a suitable candidate as illustrated by the following pseudocode.

Algorithm 1 Check whether a positive integer n is an odd spoof k-perfect number of order $\alpha < \alpha_{max}$

```
procedure CheckCandidate(n, σ_n, α_max, k)
    q ← σ_n/(k × n)
    Reduce[q]
    num ← Numerator[q]
    den ← Denominator[q]
    delta ← den − num
    for α = 1 → α_max do
        S_α ← ComputeAlphaSum[n, α]
        if delta == (S_α − num) and num > 1 then
            s ← n × num
            if Mod[s, 2] == 1 then
                Print["Found at " + s]
            end if
        end if
    end for
end procedure
```

Note that the call to Reduce$[q]$ ensures that the fraction q is reduced to the lowest terms, as mentioned above. Furthermore, the ComputeAlphaSum function is a simple computer implementation of the sum we defined previously,

$$\mathcal{S}_{\alpha} = \sum_{a=0}^{\alpha} x^{a}.$$

Putting everything together, we iterate through our search space in the following manner.

Algorithm 2 Finding spoof multiperfect numbers of different orders, given the defined limits n_{max}, k_{max}, and α_{max}

```
procedure Main(n_max, k_max, α_max)
    for n = 1 → n_max do
        σ_n ← DivisorSigma[n]
        for k = 2 → k_max do
            CheckCandidate[n, σ_n, α_max, k]
        end for
    end for
return sum
end procedure
```

Note that by computing σ_n only once for each n, considerable computing time is gained, given that this operation is the most expensive one in the algorithm in terms of computing resources.

V. Conclusion and further work

In this paper we extended our previous work on odd spoof multiperfect numbers and found several new examples of odd positive integers that would be multiperfect, if only one of their prime factors had higher multiplicity. Since our algorithm is simple, it can easily be used to discover other examples with sufficient computing resources, and indeed we hope that the present work will encourage others to do so.

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