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Odd Spoof Multiperfect Numbers of Higher Order

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Abstract: We extend our previous work on odd spoof multiperfect numbers to the case where spoof factor multiplicities exceed 2. This leads to the idenfitication of 11 new integers that would be odd multiperfect numbers if one of their prime factors had higher multiplicity. An example is $181\,545$, which would be an odd multiperfect number if only one of its prime factors, 3, had multiplicity 5.

Key words: odd perfect numbers, Descartes numbers, multiperfect numbers

I. Introduction

Recall that $\sigma(n)$ denotes the sum-of-divisors function of the positive integer n, and n is said to be *perfect* if $\sigma(n) = 2n$, and multiperfect (or k-perfect) if $\sigma(n) = kn$ for some positive integer $k \geq 2$. No odd perfect numbers have been found so far, but Descartes observed that

$$\mathcal{D} = 198\,585\,576\,189\,,$$

would be an odd perfect number if only one of its composite factors, $22\,021$, were prime. Regrettably, $22\,021 = 19^2 \cdot 61$, so this is not the case. Since Descartes, much effort has been devoted to finding such "spoof perfect" numbers, without success. In our previous paper [4], we presented a few numbers akin to $\mathcal{D}$, for instance

$$S = 8\,999\,757 = 3^2 \cdot 13^2 \cdot 61 \cdot 97\,,$$

which would be an odd multiperfect number if we assumed (wrongly) that one of its prime factors, 61, were a square. Indeed, if that were the case, we would have

$$\begin{aligned}\sigma(S) &= (3^2+3+1)(13^2+13+1)(97+1)(61^2+61+1) = \\ &= (13)\cdot(3\cdot 61)\cdot(2\cdot 7^2)\cdot(3\cdot 13\cdot 97) = \\ &= 98\cdot 3^2\cdot 13^2\cdot 61\cdot 97 = 98S\,.\end{aligned}$$

This led us to devise an algorithm to search for such numbers, through which we found several more. In this paper, our aim is to further develop our methods: first, by generalizing the concept of spoof k-perfect numbers, and second, by extending our search for numbers similar to $\mathcal{D}$ and S. As a result, we find 11 new odd positive integers that would be multiperfect if only one of their prime factors had higher multiplicity. One such example is

$$T = 181\,545 = 3\cdot 5\cdot 7^2\cdot 13\cdot 19\,,$$

which would be an odd multiperfect number if only one of its prime factors, 3 had multiplicity 5:

$$\begin{aligned}\sigma(T) &= (3^5+3^4+3^3+3^2+3+1)(5+1)(7^2+7+ \\ &\quad +1)(13+1)(19+1) = \\ &= (2^2\cdot 7\cdot 13)(2\cdot 3)(3\cdot 19)(2\cdot 7)(2^2\cdot 5) = \\ &= 192\cdot 3\cdot 5\cdot 7^2\cdot 13\cdot 19 = 192T\,.\end{aligned}$$

In the following sections, we generalize the concept of spoof multiperfect numbers and discuss some of their properties. We then adapt Robin's classical inequality to spoof multiperfect numbers and provide details of the algorithm used to obtain our results, including the pseudocode.

II. Generalized Spoof Multiperfect Numbers

In our previous paper [4], we defined two kinds of spoof multiperfect numbers. In particular, we designated the positive integer $s = nx$ as a spoof k-perfect number:

1) *of the first kind* if $\sigma(n)(x+1) = knx$,
2) *of the second kind* if $\sigma(n)(x^2+x+1) = knx$,

for a positive integer $k \geq 2$. We now extend this definition by allowing the spoof factor x to have any multiplicity greater than 2.

Definition 1 (Spoof k-perfect number of order α)**.** Let $s = nx$ be a positive integer such that $n, x \in \mathbb{N}$ and $n, x \geq 2$. Furthermore, let $\alpha \geq 1$ be an integer and define

$$\mathcal{S}_\alpha = \sum_{a=0}^{\alpha} x^a \, .$$

Then, if

$$\sigma(n)\mathcal{S}_\alpha = knx \, ,$$

for some positive integer k, then s is a spoof k-perfect number *of order* α.

Note that the case $\alpha = 1$ corresponds to the classical Descartes numbers, $\alpha = 2$ to the numbers in our previous work (such as 8 999 757), while the cases with $\alpha > 2$ form the basis of the present study. A trivial example of an odd spoof k-perfect number of order 3 is $s = 15$. Indeed, if we assume (incorrectly) that its prime factor 3 has multiplicity 3, then

$$\begin{aligned} \sigma(s) &= (5+1)\cdot(3^3+3^2+3+1) = \\ &= 2^4 \cdot 3 \cdot 5 = 16s \, . \end{aligned}$$

In our search for such numbers, we implemented an algorithm that finds all spoof multiperfect numbers of order α within a specified range, which we outline in Sec. IV. We were thus able to check all integers $s = nx$ with $n < 1.6 \times 10^7$, of order $\alpha \leq 10$. As a result, we identified 14 spoof multiperfect numbers, 11 of which are new, for which x is a prime that is also coprime to n. These integers are listed in Tab. 1.

Note that the integers $s = 77\,805, 92\,781$, and 8 999 757 were already identified in our previous paper. We also highlight that the numbers $s = 62\,998\,299$ and $440\,988\,093$ are remarkable because they also have $x = 61$, which now accounts for the majority of odd spoof multiperfect numbers of order 2. It also appears in Descartes' classical example, $s = 198\,585\,576\,189$, which is the only known odd spoof perfect number of order 1.

Many other odd spoof multiperfect numbers exist, for which x is either composite or prime but not coprime to n. We have omitted these numbers from the results presented in this paper.

As noted in our previous paper, one may notice at this point that multiperfect numbers of this magnitude should not appear so early, due to an inequality discovered by Guy Robin [2] in 1984, namely

$$\sigma(n) < e^\gamma n \log\log n \, ,$$

where γ is the Euler-Mascheroni constant and $n > 5040$, if and only if the Riemann Hypothesis holds true. It thus follows that we would expect a k-perfect number n to appear only after

$$n > e^{e^{ke^{-\gamma}}} \, ,$$

which is not the case in the spoof examples discussed above. This observation leads us to examine the "spoof equivalent" of this inequality, which we undertake in the next section.

Tab. 1. Odd spoof k-perfect numbers $s = nx$ of order α

s	n	x	k	α
15	5	3	16	3
33	11	3	44	4
1 911	637	3	152	5
1 989	153	13	280	3
34 485	11 495	3	56	4
36 309	12 103	3	160	5
77 805	11 115	7	16	2
92 781	1 521	61	97	2
105 435	21 087	5	256	4
181 545	60 515	3	192	5
241 395	80 465	3	64	4
8 999 757	147 537	61	98	2
62 998 299	1 032 759	61	112	2
440 988 093	7 229 313	61	114	2

III. Robin's Inequality for Spoof Multiperfect Numbers

We begin by adapting Robin's inequality to spoof k-perfect numbers in the following manner.

Lemma 1. Let $s = nx$ denote a spoof k-perfect number of order α. Furthermore, let $n > 5040$. Then, assuming the Riemann Hypothesis, we have:

$$\frac{kx}{\mathcal{S}_\alpha} < e^\gamma \log\log n \, ,$$

where

$$\mathcal{S}_\alpha = \sum_{a=0}^{\alpha} x^a \, .$$

Proof: Let $s = nx$ denote a spoof k-perfect number of order α. By Definition 1, it follows that

$$\sigma(n) = \frac{knx}{\mathcal{S}_\alpha} .$$

On the other hand, Robin's inequality states that for $n > > 5040$,

$$\sigma(n) < e^\gamma n \log\log n .$$

Combining these two above yields

$$\frac{knx}{\mathcal{S}_\alpha} < e^\gamma n \log\log n ,$$

and, after simplifying n on both sides, our claim is proved. □

A straightforward corollary of the above gives a bound on the components of the classical Descartes numbers.

Corollary 1. Let $s = nx$ denote a Descartes number with pseudo-prime factor x. Assuming the Riemann Hypothesis, we have:

$$\frac{2x}{x+1} < e^\gamma \log\log n .$$

Proof: This follows directly by applying Lemma 1 with $k = 2$ and $\alpha = 1$. Furthermore, we no longer need the restriction $n > 5040$, since no Descartes numbers exist with n smaller than 5040. □

IV. Algorithm

In this brief final section, we provide some details about the algorithm we used to obtain the results presented in this paper, which is very similar to the one in our previous work [4]. We run through positive integers n and compute the quantity

$$q = \frac{\sigma(n)}{kn} .$$

Ensuring that the fraction q is in the lowest terms possible (i.e., the numerator q_{num} and denominator q_{den} have greatest common divisor 1), we then compute their difference δ:

$$\delta = q_{\text{den}} - q_{\text{num}} .$$

Then if

$$\delta = \sum_{a=0}^{\alpha} q_{\text{num}}^a - q_{\text{num}} ,$$

we have found a spoof k-perfect number $s = nx$ of order α, where the spoof factor is $x = q_{\text{num}}$.

In practical terms, we can check whether the positive integer n is a suitable candidate, as illustrated by the following pseudo-code.

Algorithm 1. Check whether a positive integer n is an odd spoof k-perfect number of order $\alpha < \alpha_{\max}$

```
procedure CheckCandidate(n, σ_n, α_max, k)
    q ← σ_n/(k × n)
    Reduce[q]
    num ← Numerator[q]
    den ← Denominator[q]
    delta ← den − num
    for α = 1 → α_max do
        S_α ← ComputeAlphaSum[n, α]
        if delta == (S_α − num) and num > 1 then
            s ← n × num
            if Mod[s, 2] == 1 then
                Print["Found at " + s]
            end if
        end if
    end for
end procedure
```

Note that the call to `Reduce`$[q]$ ensures that the fraction q is reduced to the lowest terms, as mentioned above. Furthermore, the `ComputeAlphaSum` function is a straightforward computer implementation of the sum we defined previously,

$$\mathcal{S}_\alpha = \sum_{a=0}^{\alpha} x^a .$$

Putting everything together, we iterate through our search space in the following manner.

Algorithm 2. Finding spoof multiperfect numbers of different orders, given the defined limits $n_{\max}, k_{\max}$, and $\alpha_{\max}$

```
procedure Main(n_max, k_max, α_max)
    for n = 1 → n_max do
        σ_n ←DivisorSigma[n]
        for k = 2 → k_max do
            CheckCandidate[n, σ_n, α_max, k]
        end for
    end for
return sum
end procedure
```

Note that by computing σ_n only once for each n, considerable computing time is saved, given that this operation is the most expensive step in the algorithm in terms of computing resources.

V. Conclusion and Further Work

In this paper, we extended our previous work on odd spoof multiperfect numbers and found several new examples of odd positive integers that would be multiperfect if only one of their prime factors had higher multiplicity. Our algorithm is straightforward and can easily be used to discover other examples, given sufficient computing resources. We hope that the present work will encourage further exploration in this area by other researchers.

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