

Fundamental Solutions in the Theory of Micromorphic Thermoelastic Diffusion for Triple Porosity Materials with Microtemperature and Microconcentration Effects

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ABSTRACT

The main goal of this article is to derive the basic governing equations for the linearized theory of a micromorphic thermoelastic diffusion medium. This encompasses the analysis of microtemperatures, microconcentrations, and triple porosity. Moreover, the objective is to establish the fundamental solution for these equations in situations involving steady oscillations and equilibrium, presented in elementary functions.

Keywords: Thermoelastic diffusion; Triple porosity; Steady oscillations.

1. INTRODUCTION

Eringen [1-5] introduced continuum theories with microstructure, which Grot [6] extended to thermodynamics of elastic bodies with varying microelement temperatures by modifying the Clausius-Duhem inequality and incorporating energy balance laws for microtemperatures. Iesan [7, 8] advanced the theory by formulating field equations for microstretch thermoelastic bodies and deriving a linear theory incorporating additional mechanical degrees of freedom.

Aouadi et al. [9] developed nonlinear and linear theories for thermoelastic diffusion materials with microtemperatures and microconcentrations, proving well-posedness and studying solution behavior. Chiril and Marin [10] and Kansal [11] further refined constitutive relations and field equations for materials with microstructure, emphasizing microtemperatures and microconcentrations.

Triple porosity, characterized by macro, meso, and micro levels, is pivotal in elasticity and thermoelasticity, with significant contributions by Svanadze [12], Straughan [13], and Kansal [14] in deriving governing equations. Fundamental solutions, crucial for solving boundary value problems, have been extensively explored in elasticity and thermoelasticity theories, leveraging elementary functions for practical and numerical applications [11, 12, 14-17].

This paper develops constitutive relations and field equations for anisotropic micromorphic thermoelastic diffusion materials with microtemperatures, microconcentrations, and triple porosity. It reduces the anisotropic equations to isotropic forms, derives fundamental solutions for steady oscillations using elementary functions, and constructs solutions for equilibrium conditions.

2. BASIC EQUATIONS

We consider a scenario where, at time t_0 , the body occupies the bounded regular region B within three-dimensional space. Our focus is specifically on the linear theory of elastic bodies. Following Iesan [7], Aouadi et al. [9] and Kansal [14], the dynamical equations are:

Balance of linear momentum:

$$t_{ji,j} + \rho f_i = \rho \ddot{u}_i, \quad (1)$$

Balance of first stress moments:

$$m_{gij,g} + t_{ji} - s_{ji} + \rho l_{ij} = \rho \dot{\sigma}_{ij}, \quad (2)$$

Balance for energy:

$$\rho \dot{U} = t_{ij} \dot{u}_{j,i} + (s_{ij} - t_{ij}) \dot{\phi}_{ji} + m_{gij} \dot{\phi}_{ij,g} + q_{j,j} + \Omega_{ij} \dot{v}_{i,j} - p_i \dot{v}_i, \quad (3)$$

Balance of first moment of energy:

$$\rho \dot{\varepsilon}_i = Q_{ji,j} + q_i - \varpi_i, \quad (4)$$

Balance of first moment of mass diffusion:

$$\rho \dot{\chi}_i = \xi_{ji,j} + \eta_i - \upsilon_i, \quad (5)$$

where p_i , $i = 1, 2, 3$ satisfy the relation

$$\begin{aligned} \Omega_{1j,j} + p_1 + \rho \Lambda_1 &= \rho \kappa_1 \ddot{v}_1, \\ \Omega_{2j,j} + p_2 + \rho \Lambda_2 &= \rho \kappa_2 \ddot{v}_2, \\ \Omega_{3j,j} + p_3 + \rho \Lambda_3 &= \rho \kappa_3 \ddot{v}_3. \end{aligned} \quad (6)$$

Here t_{ij} are the components of stress tensor, u_i are the components of displacement vector $\mathbf{u}$, ρ is the density, f_i is the body force, m_{gij} are components of first stress moment tensor, s_{ij} are the components of microstress tensor, σ_{ij} are the components of inertial spin tensor per unit mass, l_{ij} is the first body moment density, U is the internal energy density, ε_i, χ_i are the first moments of energy vector and mass diffusion respectively, ϕ_{ij} are the components of microdeformation tensor, κ_i are the coefficients of equilibrated inertia, v_i are the volume fraction fields corresponding to macro-, meso-, micro-pores respectively, Ω_{ij} are the components of equilibrated stress vectors corresponding to v_i , Λ_i are extrinsic equilibrated body forces per unit mass associated to macro-, meso-, micro-pores respectively, q_i, η_i are the components of heat flux and mass flux vectors, Q_{ij}, ξ_{ij} are the first heat flux and mass diffusion moment tensors, ϖ_i, υ_i are the microheat flux and micromass flux averages.

Following Aouadi [9], the local form of the principle of entropy is given by the following expression:

$$\rho \dot{S} - \left(\frac{q_j}{T} + \frac{Q_{ji} T_i}{T} \right)_{,j} + \left(\frac{P \eta_j}{T} + \frac{P \xi_{ji} T_i}{T} \right)_{,j} \geq 0, \quad (7)$$

where S , P are the entropy and chemical potential per unit mass respectively, T is the absolute temperature, and T_i is the microtemperature vector.

The local form of the mass concentration law is

$$\eta_{i,i} = \dot{C}, \quad (8)$$

where C is the concentration of the diffusion material in the elastic body.

For each micro element, the mass conservation law becomes

$$\dot{C} = (\eta_j + C_i \xi_{ji})_{,j}. \quad (9)$$

The spin inertia is given by

$$\sigma_{ij} = \tilde{n}_{gi} \dot{\phi}_{iy} \dot{\phi}_{yg}, \quad (10)$$

where $\tilde{n}_{gi}$ is the microinertia tensor.

Eringen [18] introduced a special kind of micromorphic solids called microstretch solids. In this case, for all motions, we have

$$\phi_{ij} = \phi \delta_{ij}, m_{gij} = \frac{1}{3} \mathfrak{R}_g \delta_{ij}, l_{ij} = \frac{1}{3} \tau_1 \delta_{ij}, \tilde{n}_{ij} = \frac{1}{3} \tau_2 \delta_{ij}, \quad (11)$$

where ϕ is the dilatation function, $\mathfrak{R}_g$ is the microstress vector, τ_1 is the generalized external body load, and τ_2 is a given constant.

Equations (2) and (3) with the help of equations (4), (5), (9)-(11) and inequality (7) become

$$\mathfrak{R}_{g,g} - s + \rho \tau_1 = \rho \tau_2 \ddot{\phi}, \quad (12)$$

$$\begin{aligned} & \rho [\dot{S}T - \dot{U} - T_i \dot{\varepsilon}_i - C_i \dot{\chi}_i] + t_{ij} \dot{e}_{ij} + \mathfrak{R}_g \dot{\phi}_{,g} + s \dot{\phi} + \Omega_{ij} \dot{v}_{i,j} - p_i \dot{v}_i + \frac{1}{T} q_j T_{,j} + \frac{1}{T} T_j Q_{ji} T_i - Q_{ji} T_{i,j} + P \dot{C} \\ & + (q_i - \varpi_i) T_i - P C_i \xi_{ji,j} - P C_{i,j} \xi_{ji} + P_{,j} \eta_j - \frac{1}{T} P \eta_j T_{,j} + T \left(\frac{P}{T} T_i \xi_{ji} \right)_{,j} + \xi_{ji,j} C_i + (\eta_i - v_i) C_i \geq 0. \end{aligned} \quad (13)$$

where $s = s_{ii} - t_{ii}$ is the intrinsic body load, and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are the components of strain tensor.

Introducing the function Γ by $\Gamma = U + T_i \varepsilon_i + C_i \chi_i - TS$, the inequality (13) in the context of linear theory can be expressed as:

$$\begin{aligned} & -\rho [\dot{\Gamma} + S \dot{T} - \dot{T}_i \varepsilon_i - \dot{C}_i \chi_i] + t_{ij} \dot{e}_{ij} + \mathfrak{R}_g \dot{\phi}_{,g} + s \dot{\phi} + \Omega_{ij} \dot{v}_{i,j} - p_i \dot{v}_i + \frac{1}{T} q_j T_{,j} - Q_{ji} T_{i,j} + P \dot{C} \\ & + (q_i - \varpi_i) T_i + P_{,j} \eta_j + \xi_{ji,j} C_i + (\eta_i - v_i) C_i \geq 0. \end{aligned} \quad (14)$$

The function Γ can be written in terms of independent variables $e_{ij}, \phi, \phi_{,i}, v_i, v_{i,j}, T, T_{,i}, T_{i,j}, C, C_{,i}, C_i$ and $C_{i,j}$. Therefore, we obtain:

$$\begin{aligned} \dot{\Gamma} = & \frac{\partial \Gamma}{\partial e_{ij}} \dot{e}_{ij} + \frac{\partial \Gamma}{\partial v_i} \dot{v}_i + \frac{\partial \Gamma}{\partial v_{i,j}} \dot{v}_{i,j} + \frac{\partial \Gamma}{\partial \phi} \dot{\phi} + \frac{\partial \Gamma}{\partial \phi_{,i}} \dot{\phi}_{,i} + \frac{\partial \Gamma}{\partial T} \dot{T} + \frac{\partial \Gamma}{\partial T_{,i}} \dot{T}_{,i} + \frac{\partial \Gamma}{\partial T_{i,j}} \dot{T}_{i,j} + \\ & \frac{\partial \Gamma}{\partial T_{i,j}} \dot{T}_{i,j} + \frac{\partial \Gamma}{\partial C} \dot{C} + \frac{\partial \Gamma}{\partial C_{,i}} \dot{C}_{,i} + \frac{\partial \Gamma}{\partial C_i} \dot{C}_i + \frac{\partial \Gamma}{\partial C_{i,j}} \dot{C}_{i,j}. \end{aligned} \quad (15)$$

Inequality (14) with the help of (15) becomes

$$\begin{aligned} & \left[t_{ij} - \rho \frac{\partial \Gamma}{\partial e_{ij}} \right] \dot{e}_{ij} + \left[\Omega_{ij} - \rho \frac{\partial \Gamma}{\partial v_{i,j}} \right] \dot{v}_{i,j} - \left[p_i + \rho \frac{\partial \Gamma}{\partial v_i} \right] \dot{v}_i + \left[s - \rho \frac{\partial \Gamma}{\partial \phi} \right] \dot{\phi} + \left[\mathfrak{R}_g - \rho \frac{\partial \Gamma}{\partial \phi_{,g}} \right] \dot{\phi}_{,g} \\ & + \rho \left[\varepsilon_i - \frac{\partial \Gamma}{\partial T_i} \right] \dot{T}_i + \rho \left[\chi_i - \frac{\partial \Gamma}{\partial C_i} \right] \dot{C}_i - \rho \left[S + \frac{\partial \Gamma}{\partial T} \right] \dot{T} + \left[P - \rho \frac{\partial \Gamma}{\partial C} \right] \dot{C} - \rho \frac{\partial \Gamma}{\partial T_{,i}} \dot{T}_{,i} - \rho \frac{\partial \Gamma}{\partial T_{i,j}} \dot{T}_{i,j} \\ & - \rho \frac{\partial \Gamma}{\partial C_{,i}} \dot{C}_{,i} - \rho \frac{\partial \Gamma}{\partial C_{i,j}} \dot{C}_{i,j} + \frac{1}{T} q_j T_{,j} - Q_{ji} T_{i,j} + (q_i - \varpi_i) T_i + P_{,j} \eta_j + \xi_{ji,j} C_i + (\eta_i - v_i) C_i \geq 0. \end{aligned}$$

The inequality must hold for all rates $\dot{e}_{ij}, \dot{\phi}, \dot{\phi}_{,i}, \dot{v}_i, \dot{v}_{i,j}, \dot{T}, \dot{T}_{,i}, \dot{T}_{i,j}, \dot{C}, \dot{C}_{,i}, \dot{C}_i$ and $\dot{C}_{i,j}$.

Therefore, the coefficients of the aforementioned variables must equate to zero, indicating:

$$\begin{aligned} t_{ij} &= \rho \frac{\partial \Gamma}{\partial e_{ij}}, \Omega_{ij} = \rho \frac{\partial \Gamma}{\partial v_{i,j}}, p_i = -\rho \frac{\partial \Gamma}{\partial v_i}, s = \rho \frac{\partial \Gamma}{\partial \phi}, \mathfrak{R}_g = \rho \frac{\partial \Gamma}{\partial \phi_{,g}}, \varepsilon_i = \frac{\partial \Gamma}{\partial T_i}, \chi_i = \frac{\partial \Gamma}{\partial C_i}, \\ S &= -\frac{\partial \Gamma}{\partial T}, P = \rho \frac{\partial \Gamma}{\partial C}, \frac{\partial \Gamma}{\partial T_{,i}} = 0, \frac{\partial \Gamma}{\partial T_{i,j}} = 0, \frac{\partial \Gamma}{\partial C_{,i}} = 0, \frac{\partial \Gamma}{\partial C_{i,j}} = 0, \\ q_j T_{,j} - T Q_{ji} T_{i,j} + T(q_i - \varpi_i) T_i + T P_{,j} \eta_j + T \xi_{ji,j} C_i + T(\eta_i - v_i) C_i &\geq 0. \end{aligned} \quad (16)$$

Let us introduce the notations

$$\phi = \mathbf{v} - \mathbf{v}_0, \quad \theta = T - T_0,$$

where $\phi = (\phi_1, \phi_2, \phi_3)$, T_0 is the reference temperature of the body chosen such that $\left| \frac{\theta}{T_0} \right| \ll 1$,

$\mathbf{v}_0$ are the volume fraction fields in reference configuration.

In the linear theory of materials possessing a centre of symmetry, we can take Γ in the form

$$\begin{aligned} 2\rho\Gamma = & c_{ijgy} e_{ij} e_{gy} + 2a_{ij} e_{ij} \theta + 2b_{ij} e_{ij} C + 2c_{ij} e_{ij} \phi_1 + 2d_{ij} e_{ij} \phi_2 + 2f_{ij} e_{ij} \phi_3 + 2v_{ij} e_{ij} \phi - 2l_i \phi_i \theta - 2n_i \phi_i C \\ & - \frac{\rho C_e \theta^2}{T_0} - 2a\theta C + bC^2 + \alpha_i \phi_i^2 + 2\alpha_4 \phi_1 \phi_2 + 2\alpha_5 \phi_2 \phi_3 + 2\alpha_6 \phi_3 \phi_1 + A_{ij} \phi_{1,i} \phi_{1,j} + B_{ij} \phi_{2,i} \phi_{2,j} + C_{ij} \phi_{3,i} \phi_{3,j} \\ & + 2D_{ij} \phi_{1,i} \phi_{2,j} + 2E_{ij} \phi_{2,i} \phi_{3,j} + 2S_{ij} \phi_{3,i} \phi_{1,j} - \alpha_{ij} T_i T_j - \beta_{ij} C_i C_j - 2\gamma_{ij} T_i C_j - 2P_{ij} \phi_{1,j} T_i - 2U_{ij} \phi_{2,j} T_i - 2\lambda_{ij} \phi_{3,j} T_i \\ & - 2M_{ij} \phi_{1,j} C_i - 2T_{ij} \phi_{2,j} C_i - 2\chi_{ij} \phi_{3,j} C_i + \alpha \phi^2 - 2c\theta\phi - 2dC\phi - 2s_i \phi_i \phi + V_{ij} \phi_{,i} \phi_{,j} - 2W_{ij} \phi_{,j} T_i - 2X_{ij} \phi_{,j} C_i. \end{aligned} \quad (17)$$

Using the above equation (17) in the system of equations (16), the following consecutive equations are obtained:

$$\begin{aligned}
t_{ij} &= c_{ijgy} e_{gy} + c_{ij}\phi_1 + d_{ij}\phi_2 + f_{ij}\phi_3 + v_{ij}\phi + a_{ij}\theta + b_{ij}C, \\
\Omega_{1j} &= A_{ij}\phi_{1,i} + D_{ij}\phi_{2,i} + S_{ij}\phi_{3,i} - P_{ij}T_i - M_{ij}C_i, \\
\Omega_{2j} &= D_{ij}\phi_{1,i} + B_{ij}\phi_{2,i} + E_{ij}\phi_{3,i} - U_{ij}T_i - T_{ij}C_i, \\
\Omega_{3j} &= S_{ij}\phi_{1,i} + E_{ij}\phi_{2,i} + C_{ij}\phi_{3,i} - \lambda_{ij}T_i - \chi_{ij}C_i, \\
p_1 &= -c_{ij}e_{ij} - \alpha_1\phi_1 - \alpha_4\phi_2 - \alpha_6\phi_3 + s_1\phi + l_1\theta + n_1C, \\
p_2 &= -d_{ij}e_{ij} - \alpha_4\phi_1 - \alpha_2\phi_2 - \alpha_5\phi_3 + s_2\phi + l_2\theta + n_2C, \\
p_3 &= -f_{ij}e_{ij} - \alpha_6\phi_1 - \alpha_5\phi_2 - \alpha_3\phi_3 + s_3\phi + l_3\theta + n_3C, \\
\rho\varepsilon_i &= -P_{ij}\phi_{1,j} - U_{ij}\phi_{2,j} - \lambda_{ij}\phi_{3,j} - W_{ij}\phi_{,j} - \alpha_{ij}T_j - \gamma_{ij}C_j, \\
\rho\chi_i &= -M_{ij}\phi_{1,j} - T_{ij}\phi_{2,j} - \chi_{ij}\phi_{3,j} - X_{ij}\phi_{,j} - \gamma_{ji}T_j - \beta_{ij}C_j, \\
\rho S &= -a_{ij}e_{ij} + l_1\phi_1 + c\phi + \frac{\rho C_e \theta}{T_0} + aC, P = b_{ij}e_{ij} - n_1\phi_1 - d\phi - a\theta + bC, \\
s &= v_{ij}e_{ij} - s_1\phi_1 + \alpha\phi - c\theta - dC, \mathfrak{R}_g = V_{ig}\phi_{,i} - W_{ig}T_i - X_{ig}C_i.
\end{aligned} \tag{18}$$

The linear expressions for $q_i, Q_{ij}, \varpi_i, \eta_i, \xi_{ij}, v_i$ are

$$\begin{aligned}
q_i &= k_{ij}\theta_{,j} + \kappa_{ij}T_j, Q_{ij} = -m_{ijyg}T_{g,y}, \varpi_i = (k_{ij} - K_{ij})\theta_{,j} + (\kappa_{ij} - L_{ij})T_j, \\
\eta_i &= h_{ij}P_{,j} + m_{ij}C_j, \xi_{ij} = -n_{ijyg}C_{g,y}, v_i = (h_{ij} - H_{ij})P_{,j} + (m_{ij} - O_{ij})C_j.
\end{aligned} \tag{19}$$

The linearized form of inequality (7) is $\rho T_0 \dot{S} = q_{i,i}$. (20)

Considering equations (18) and (19), equations (1), (4)-(6), (8), (12), and (20) can be reformulated as:

$$\begin{aligned}
c_{ijgy} e_{gy,j} + c_{ij}\phi_{1,j} + d_{ij}\phi_{2,j} + f_{ij}\phi_{3,j} + v_{ij}\phi_{,j} + a_{ij}\theta_{,j} + b_{ij}C_{,j} + \rho f_i &= \rho \ddot{u}_i, \\
m_{jyig} T_{g,yj} - \alpha_{ij}\dot{T}_j - \gamma_{ij}\dot{C}_j - P_{ij}\dot{\phi}_{1,j} - U_{ij}\dot{\phi}_{2,j} - \lambda_{ij}\dot{\phi}_{3,j} - W_{ij}\dot{\phi}_{,j} &= K_{ij}\theta_{,j} + L_{ij}T_j, \\
n_{jyig} C_{g,yj} - \gamma_{ji}\dot{T}_j - \beta_{ij}\dot{C}_j - M_{ij}\dot{\phi}_{1,j} - T_{ij}\dot{\phi}_{2,j} - \chi_{ij}\dot{\phi}_{3,j} - X_{ij}\dot{\phi}_{,j} &= H_{ij}P_{,j} + O_{ij}C_j, \\
-c_{ij}e_{ij} - P_{ij}T_{i,j} - M_{ij}C_{i,j} + A_{ij}\phi_{1,ij} + D_{ij}\phi_{2,ij} + S_{ij}\phi_{3,ij} - \alpha_1\phi_1 - \alpha_4\phi_2 \\
- \alpha_6\phi_3 + s_1\phi + l_1\theta + n_1C + \rho\Lambda_1 &= \rho\kappa_1\ddot{\phi}_1, \\
-d_{ij}e_{ij} - U_{ij}T_{i,j} - T_{ij}C_{i,j} + D_{ij}\phi_{1,ij} + B_{ij}\phi_{2,ij} + E_{ij}\phi_{3,ij} - \alpha_4\phi_1 - \alpha_2\phi_2 \\
- \alpha_5\phi_3 + s_2\phi + l_2\theta + n_2C + \rho\Lambda_2 &= \rho\kappa_2\ddot{\phi}_2, \\
-f_{ij}e_{ij} - \lambda_{ij}T_{i,j} - \chi_{ij}C_{i,j} + S_{ij}\phi_{1,ij} + E_{ij}\phi_{2,ij} + C_{ij}\phi_{3,ij} - \alpha_6\phi_1 - \alpha_5\phi_2 \\
- \alpha_3\phi_3 + s_3\phi + l_3\theta + n_3C + \rho\Lambda_3 &= \rho\kappa_3\ddot{\phi}_3, \\
-v_{ij}e_{ij} - W_{ig}T_{i,g} - X_{ig}C_{i,g} + s_1\phi_i + V_{ig}\phi_{,ig} - \alpha\phi + c\theta + dC + \rho\tau_1 &= \rho\tau_2\ddot{\phi}, \\
T_0[-a_{ij}\dot{e}_{ij} + l_1\dot{\phi}_1 + c\dot{\phi} + a\dot{C}] + \rho C_e \dot{\theta} &= k_{ij}\theta_{,ij} + \kappa_{ij}T_{j,i}, \\
h_{ij}[b_{zy}e_{zy} - n_g\phi_g - d\phi - a\theta + bC]_{,ij} + m_{ij}C_{j,i} &= \dot{C}.
\end{aligned} \tag{21}$$

For an isotropic and homogeneous material, the subsequent equations (18) and (19) are simplified to:

$$\begin{aligned}
t_{ij} &= \lambda e_{yy} \delta_{ij} + 2\mu e_{ij} + [\tilde{\lambda}_g \phi_g + f\phi - \beta_1 \theta - \beta_2 C] \delta_{ij}, \\
\Omega_{1j} &= A_1 \phi_{1,j} + A_4 \phi_{2,j} + A_6 \phi_{3,j} - B_1 T_j - B_4 C_j, \\
\Omega_{2j} &= A_4 \phi_{1,j} + A_2 \phi_{2,j} + A_5 \phi_{3,j} - B_2 T_j - B_5 C_j, \\
\Omega_{3j} &= A_6 \phi_{1,j} + A_5 \phi_{2,j} + A_3 \phi_{3,j} - B_3 T_j - B_6 C_j, \\
p_1 &= -\tilde{\lambda}_1 e_{ii} - \alpha_1 \phi_1 - \alpha_4 \phi_2 - \alpha_6 \phi_3 + s_1 \phi + l_1 \theta + n_1 C, \\
p_2 &= -\tilde{\lambda}_2 e_{ii} - \alpha_4 \phi_1 - \alpha_2 \phi_2 - \alpha_5 \phi_3 + s_2 \phi + l_2 \theta + n_2 C, \\
p_3 &= -\tilde{\lambda}_3 e_{ii} - \alpha_6 \phi_1 - \alpha_5 \phi_2 - \alpha_3 \phi_3 + s_3 \phi + l_3 \theta + n_3 C, \\
\rho \varepsilon_i &= -B_1 \phi_{1,i} - B_2 \phi_{2,i} - B_3 \phi_{3,i} - E_1 \phi_{,i} - D_1 T_i - D_3 C_i, \\
\rho \chi_i &= -B_4 \phi_{1,i} - B_5 \phi_{2,i} - B_6 \phi_{3,i} - E_2 \phi_{,i} - D_3 T_i - D_2 C_i, \\
\rho S &= \beta_1 e_{gg} + l_1 \phi_i + c\phi + \frac{\rho C_e \theta}{T_0} + aC, P = -\beta_2 e_{gg} - n_1 \phi_i - d\phi - a\theta + bC, \\
s &= f e_{gg} - s_1 \phi_i + \alpha \phi - c\theta - dC, \mathfrak{R}_g = -E_1 T_g - E_2 C_g + \gamma \phi_{,g}, \\
q_i &= k\theta_{,i} + k_1 T_i, Q_{ij} = -k_4 T_{y,y} \delta_{ij} - k_5 T_{1,j} - k_6 T_{j,i}, \varpi_i = (k - k_3)\theta_{,i} + (k_1 - k_2)T_i, \\
\eta_i &= hP_{,i} + h_1 C_i, \xi_{ij} = -h_4 C_{y,y} \delta_{ij} - h_5 C_{i,j} - h_6 C_{j,i}, v_i = (h - h_3)P_{,i} + (h_1 - h_2)C_i.
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
c_{ijgy} &= \lambda \delta_{ij} \delta_{gy} + \mu \delta_{ig} \delta_{jy} + \mu \delta_{iy} \delta_{jg}, a_{ij} = -\beta_1 \delta_{ij}, b_{ij} = -\beta_2 \delta_{ij}, c_{ij} = \tilde{\lambda}_1 \delta_{ij}, d_{ij} = \tilde{\lambda}_2 \delta_{ij}, f_{ij} = \tilde{\lambda}_3 \delta_{ij}, \\
v_{ij} &= f \delta_{ij}, A_{ij} = A_1 \delta_{ij}, B_{ij} = A_2 \delta_{ij}, C_{ij} = A_3 \delta_{ij}, D_{ij} = A_4 \delta_{ij}, E_{ij} = A_5 \delta_{ij}, S_{ij} = A_6 \delta_{ij}, P_{ij} = B_1 \delta_{ij}, \\
U_{ij} &= B_2 \delta_{ij}, \tilde{\lambda}_{ij} = B_3 \delta_{ij}, M_{ij} = B_4 \delta_{ij}, T_{ij} = B_5 \delta_{ij}, \chi_{ij} = B_6 \delta_{ij}, \alpha_{ij} = D_1 \delta_{ij}, \beta_{ij} = D_2 \delta_{ij}, \gamma_{ij} = D_3 \delta_{ij}, \\
W_{ij} &= E_1 \delta_{ij}, X_{ij} = E_2 \delta_{ij}, V_{ij} = \gamma \delta_{ij}, k_{ij} = k \delta_{ij}, \kappa_{ij} = k_1 \delta_{ij}, L_{ij} = k_2 \delta_{ij}, K_{ij} = k_3 \delta_{ij}, h_{ij} = h \delta_{ij}, m_{ij} = h_1 \delta_{ij}, \\
H_{ij} &= h_3 \delta_{ij}, m_{ijgy} = k_4 \delta_{ij} \delta_{gy} + k_6 \delta_{ig} \delta_{jy} + k_5 \delta_{iy} \delta_{jg}, O_{ij} = h_2 \delta_{ij}, n_{ijgy} = h_4 \delta_{ij} \delta_{gy} + h_6 \delta_{ig} \delta_{jy} + h_5 \delta_{iy} \delta_{jg}.
\end{aligned} \tag{23}$$

Here δ_{ij} is Kronecker's delta and $\lambda, \mu, f, \beta_1, \beta_2, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, A_1, \dots, A_6, B_1, \dots, B_6, D_1, D_2,$

$D_3, E_1, E_2, \gamma, k, h, k_1, \dots, k_6$ and $h_1, \dots, h_6$ are material constants.

Using equation (23) in equation (21), we obtain the governing equations for homogeneous isotropic micromorphic thermoelastic diffusion materials with microtemperature, microconcentration and triple porosity effects in the absence of heat and mass diffusive sources:

$$\begin{aligned}
& \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \tilde{\lambda}_i \nabla \phi_i + f \nabla \phi - \beta_1 \nabla \theta - \beta_2 \nabla C = \rho \ddot{\mathbf{u}}, \\
& (k_6 \Delta - k_2) \mathbf{v} + (k_4 + k_5) \nabla (\nabla \cdot \mathbf{v}) - k_3 \nabla \theta = D_1 \dot{\mathbf{v}} + D_3 \dot{\mathbf{w}} + B_i \nabla \dot{\phi}_i + E_1 \nabla \dot{\phi}, \\
& (h_6 \Delta - h_2) \mathbf{w} + (h_4 + h_5) \nabla (\nabla \cdot \mathbf{w}) - h_3 \nabla P = D_3 \dot{\mathbf{v}} + D_2 \dot{\mathbf{w}} + B_{i+3} \nabla \dot{\phi}_i + E_2 \nabla \dot{\phi}, \\
& -\tilde{\lambda}_1 (\nabla \cdot \mathbf{u}) - B_1 (\nabla \cdot \mathbf{v}) - B_4 (\nabla \cdot \mathbf{w}) + (A_1 \Delta - \alpha_1) \phi_1 + (A_4 \Delta - \alpha_4) \phi_2 + (A_6 \Delta - \alpha_6) \phi_3 \\
& + s_1 \phi + l_1 \theta + n_1 C = \rho \kappa_1 \ddot{\phi}_1, \\
& -\tilde{\lambda}_2 (\nabla \cdot \mathbf{u}) - B_2 (\nabla \cdot \mathbf{v}) - B_5 (\nabla \cdot \mathbf{w}) + (A_4 \Delta - \alpha_4) \phi_1 + (A_2 \Delta - \alpha_2) \phi_2 + (A_5 \Delta - \alpha_5) \phi_3 \\
& + s_2 \phi + l_2 \theta + n_2 C = \rho \kappa_2 \ddot{\phi}_2, \\
& -\tilde{\lambda}_3 (\nabla \cdot \mathbf{u}) - B_3 (\nabla \cdot \mathbf{v}) - B_6 (\nabla \cdot \mathbf{w}) + (A_6 \Delta - \alpha_6) \phi_1 + (A_5 \Delta - \alpha_5) \phi_2 + (A_3 \Delta - \alpha_3) \phi_3 \\
& + s_3 \phi + l_3 \theta + n_3 C = \rho \kappa_3 \ddot{\phi}_3, \\
& -f (\nabla \cdot \mathbf{u}) - E_1 (\nabla \cdot \mathbf{v}) - E_2 (\nabla \cdot \mathbf{w}) + s_i \phi_i + (\gamma \Delta - \alpha) \phi + c \theta + d C = \rho \tau_2 \ddot{\phi}, \\
& T_0 [\beta_1 (\nabla \cdot \dot{\mathbf{u}}) + l_1 \dot{\phi}_1 + c \dot{\phi} + a \dot{C}] + \rho C_e \dot{\theta} = k \Delta \theta + k_1 (\nabla \cdot \mathbf{v}), \\
& h \Delta [-\beta_2 (\nabla \cdot \mathbf{u}) - n_1 \phi_i - d \phi - a \theta + b C] + h_1 (\nabla \cdot \mathbf{w}) = \dot{C},
\end{aligned} \tag{24}$$

where Δ, ∇ are, Laplacian and Del operators respectively.

In the following sections, the chemical potential has been adopted as a state variable rather than concentration. For this purpose, the eleventh equation of (22) can be rewritten as:

$$C = \frac{1}{b} [\beta_2 e_{gg} + n_i \phi_i + d \phi + a \theta + P]. \tag{25}$$

Hence, the system of equations (24), with the assistance equation (25) is transformed into:

$$\begin{aligned}
& \mu \Delta \mathbf{u} + (\lambda' + \mu) \nabla (\nabla \cdot \mathbf{u}) + \tilde{\lambda}_i \nabla \phi_i + \wp \nabla \phi - \vartheta_1 \nabla \theta - \vartheta_2 \nabla P = \rho \ddot{\mathbf{u}}, \\
& (k_6 \Delta - k_2) \mathbf{v} + (k_4 + k_5) \nabla (\nabla \cdot \mathbf{v}) - k_3 \nabla \theta = D_1 \dot{\mathbf{v}} + D_3 \dot{\mathbf{w}} + B_i \nabla \dot{\phi}_i + E_1 \nabla \dot{\phi}, \\
& (h_6 \Delta - h_2) \mathbf{w} + (h_4 + h_5) \nabla (\nabla \cdot \mathbf{w}) - h_3 \nabla P = D_3 \dot{\mathbf{v}} + D_2 \dot{\mathbf{w}} + B_{i+3} \nabla \dot{\phi}_i + E_2 \nabla \dot{\phi}, \\
& -\tilde{\lambda}_1 (\nabla \cdot \mathbf{u}) - B_1 (\nabla \cdot \mathbf{v}) - B_4 (\nabla \cdot \mathbf{w}) + (A_1 \Delta - \zeta_1) \phi_1 + (A_4 \Delta - \zeta_4) \phi_2 + (A_6 \Delta - \zeta_6) \phi_3 \\
& + \delta_1 \phi + \xi_1 \theta + v_1 P = \rho \kappa_1 \ddot{\phi}_1, \\
& -\tilde{\lambda}_2 (\nabla \cdot \mathbf{u}) - B_2 (\nabla \cdot \mathbf{v}) - B_5 (\nabla \cdot \mathbf{w}) + (A_4 \Delta - \zeta_4) \phi_1 + (A_2 \Delta - \zeta_2) \phi_2 + (A_5 \Delta - \zeta_5) \phi_3 \\
& + \delta_2 \phi + \xi_2 \theta + v_2 P = \rho \kappa_2 \ddot{\phi}_2, \\
& -\tilde{\lambda}_3 (\nabla \cdot \mathbf{u}) - B_3 (\nabla \cdot \mathbf{v}) - B_6 (\nabla \cdot \mathbf{w}) + (A_6 \Delta - \zeta_6) \phi_1 + (A_5 \Delta - \zeta_5) \phi_2 + (A_3 \Delta - \zeta_3) \phi_3 \\
& + \delta_3 \phi + \xi_3 \theta + v_3 P = \rho \kappa_3 \ddot{\phi}_3, \\
& -\wp (\nabla \cdot \mathbf{u}) - E_1 (\nabla \cdot \mathbf{v}) - E_2 (\nabla \cdot \mathbf{w}) + \delta_i \phi_i + (\gamma \Delta - \beta) \phi + n \theta + \xi P = \rho \tau_2 \ddot{\phi}, \\
& -T_0 [\vartheta_1 (\nabla \cdot \dot{\mathbf{u}}) + \xi_i \dot{\phi}_i + n \dot{\phi} + \varsigma \dot{P} + \eta \dot{\theta}] + k_1 (\nabla \cdot \mathbf{v}) + k \Delta \theta = 0, \\
& -[\vartheta_2 (\nabla \cdot \dot{\mathbf{u}}) + v_i \dot{\phi}_i + \xi \dot{\phi} + \varsigma \dot{\theta} + \wp \dot{P}] + h_1 (\nabla \cdot \mathbf{w}) + h \Delta P = 0,
\end{aligned} \tag{26}$$

where

$$\varpi = b^{-1}, \vartheta_2 = \varpi \beta_2, \vartheta_1 = \beta_1 + a \vartheta_2, \tilde{\lambda}_i = \tilde{\lambda}_i - n_i \vartheta_2, \lambda' = \lambda - \vartheta_2 \beta_2, \varsigma = a \varpi, v_i = n_i \varpi,$$

$$\zeta_i = \alpha_i - n_i v_i, \zeta_4 = \alpha_4 - n_1 v_2, \zeta_5 = \alpha_5 - n_2 v_3, \zeta_6 = \alpha_6 - n_3 v_1, \xi_i = l_i + n_i \zeta,$$

$$\wp = f - \vartheta_2 d, \delta_i = s_i + v_i d, \xi = d\varpi, \beta = \alpha - d\xi, n = c + d\zeta, \eta = \frac{\rho C_e}{T_0} + a\zeta.$$

3. STEADY OSCILLATIONS

The displacement vector, microtemperature, microconcentration, volume fraction fields, microstretch, temperature change and chemical potential functions are presumed as

$$[\mathbf{u}(\mathbf{x}, t), \mathbf{v}(\mathbf{x}, t), \mathbf{w}(\mathbf{x}, t), \phi(\mathbf{x}, t), \phi(\mathbf{x}, t), \theta(\mathbf{x}, t), P(\mathbf{x}, t)] = \text{Re}[(\mathbf{u}^*, \mathbf{v}^*, \mathbf{w}^*, \phi^*, \phi^*, \theta^*, P^*)e^{-i\omega t}], \quad (27)$$

where, ω is oscillation frequency.

By employing equation (27) in the system of equations (26) and omitting asterisks (*) for simplicity, we derive the system of equations for steady oscillations as:

$$\begin{aligned} & [\mu\Delta + \rho\omega^2]\mathbf{u} + (\lambda' + \mu)\nabla(\nabla \cdot \mathbf{u}) + \tilde{\lambda}_1 \nabla \phi_1 + \wp \nabla \phi - \vartheta_1 \nabla \theta - \vartheta_2 \nabla P = \mathbf{0}, \\ & [k_6 \Delta - k_2 + i\omega D_1]\mathbf{v} + (k_4 + k_5)\nabla(\nabla \cdot \mathbf{v}) + i\omega D_3 \mathbf{w} + i\omega B_1 \nabla \phi_1 + i\omega E_1 \nabla \phi - k_3 \nabla \theta = \mathbf{0}, \\ & i\omega D_3 \mathbf{v} + [h_6 \Delta - h_2 + i\omega D_2]\mathbf{w} + (h_4 + h_5)\nabla(\nabla \cdot \mathbf{w}) + i\omega B_{1+3} \nabla \phi_1 + i\omega E_2 \nabla \phi - h_3 \nabla P = \mathbf{0}, \\ & -\tilde{\lambda}_1 (\nabla \cdot \mathbf{u}) - B_1 (\nabla \cdot \mathbf{v}) - B_4 (\nabla \cdot \mathbf{w}) + (A_1 \Delta - \gamma_1)\phi_1 + (A_4 \Delta - \zeta_4)\phi_2 + (A_6 \Delta - \zeta_6)\phi_3 \\ & + \delta_1 \phi + \xi_1 \theta + v_1 P = 0, \\ & -\tilde{\lambda}_2 (\nabla \cdot \mathbf{u}) - B_2 (\nabla \cdot \mathbf{v}) - B_5 (\nabla \cdot \mathbf{w}) + (A_4 \Delta - \zeta_4)\phi_1 + (A_2 \Delta - \gamma_2)\phi_2 + (A_5 \Delta - \zeta_5)\phi_3 \\ & + \delta_2 \phi + \xi_2 \theta + v_2 P = 0, \\ & -\tilde{\lambda}_3 (\nabla \cdot \mathbf{u}) - B_3 (\nabla \cdot \mathbf{v}) - B_6 (\nabla \cdot \mathbf{w}) + (A_6 \Delta - \zeta_6)\phi_1 + (A_5 \Delta - \zeta_5)\phi_2 + (A_3 \Delta - \gamma_3)\phi_3 \\ & + \delta_3 \phi + \xi_3 \theta + v_3 P = 0, \\ & -\wp (\nabla \cdot \mathbf{u}) - E_1 (\nabla \cdot \mathbf{v}) - E_2 (\nabla \cdot \mathbf{w}) + \delta_i \phi_i + (\gamma \Delta - \tilde{\beta})\phi + n\theta + \xi P = 0, \\ & i\omega T_0 [\vartheta_1 (\nabla \cdot \mathbf{u}) + \xi_i \phi_i + n\phi + \varsigma P] + k_1 (\nabla \cdot \mathbf{v}) + [k\Delta + i\omega \eta T_0] \theta = 0, \\ & i\omega [\vartheta_2 (\nabla \cdot \mathbf{u}) + v_i \phi_i + \xi \phi + \varsigma \theta] + h_1 (\nabla \cdot \mathbf{w}) + [h\Delta + i\omega \varpi] P = 0, \end{aligned} \quad (28)$$

where, $\gamma_i = \zeta_i - \rho \kappa_i \omega^2$, $\tilde{\beta} = \beta - \rho \omega^2 \tau_2$, $i = 1, 2, 3$.

We introduce the second order matrix differential operators with constant coefficients

$$\mathbf{F}(\mathbf{D}_x) = (\mathbf{F}_{gz}(\mathbf{D}_x))_{15 \times 15},$$

where

$$\begin{aligned} F_{ij}(\mathbf{D}_x) &= [\mu\Delta + \rho\omega^2]\delta_{ij} + (\lambda' + \mu)\frac{\partial^2}{\partial x_i \partial x_j}, F_{i;j+3}(\mathbf{D}_x) = F_{i;j+6}(\mathbf{D}_x) = 0, F_{i;j+9}(\mathbf{D}_x) = \tilde{\lambda}_j \frac{\partial}{\partial x_i}, \\ F_{i;13}(\mathbf{D}_x) &= \wp \frac{\partial}{\partial x_i}, F_{i;14}(\mathbf{D}_x) = -\vartheta_1 \frac{\partial}{\partial x_i}, F_{i;15}(\mathbf{D}_x) = -\vartheta_2 \frac{\partial}{\partial x_i}, F_{i+3;j}(\mathbf{D}_x) = 0, \end{aligned}$$

$$\begin{aligned}
F_{i+3;j+3}(\mathbf{D}_x) &= [k_6\Delta + k_8]\delta_{ij} + (k_4 + k_5)\frac{\partial^2}{\partial x_i \partial x_j}, F_{i+3;j+6}(\mathbf{D}_x) = \imath\omega D_3\delta_{ij}, F_{i+3;j+9}(\mathbf{D}_x) = \imath\omega B_j \frac{\partial}{\partial x_i}, \\
F_{i+3;13}(\mathbf{D}_x) &= \imath\omega E_1 \frac{\partial}{\partial x_i}, F_{i+3;14}(\mathbf{D}_x) = -k_3 \frac{\partial}{\partial x_i}, F_{i+3;15}(\mathbf{D}_x) = 0, F_{i+6;j}(\mathbf{D}_x) = 0, F_{i+6;j+3}(\mathbf{D}_x) = \imath\omega D_3\delta_{ij}, \\
F_{i+6;j+6}(\mathbf{D}_x) &= [h_6\Delta + h_8]\delta_{ij} + (h_4 + h_5)\frac{\partial^2}{\partial x_i \partial x_j}, F_{i+6;j+9}(\mathbf{D}_x) = \imath\omega B_{j+3} \frac{\partial}{\partial x_i}, F_{i+6;13}(\mathbf{D}_x) = \imath\omega E_2 \frac{\partial}{\partial x_i}, \\
F_{i+6;14}(\mathbf{D}_x) &= 0, F_{i+6;15}(\mathbf{D}_x) = -h_3 \frac{\partial}{\partial x_i}, F_{i+9;j}(\mathbf{D}_x) = -\tilde{\lambda}_i \frac{\partial}{\partial x_j}, F_{i+9;j+3}(\mathbf{D}_x) = -B_i \frac{\partial}{\partial x_j}, \\
F_{i+9;j+6}(\mathbf{D}_x) &= -B_{i+3} \frac{\partial}{\partial x_j}, F_{i+9;i+9}(\mathbf{D}_x) = A_i\Delta - \gamma_i, F_{10;11}(\mathbf{D}_x) = F_{11;10}(\mathbf{D}_x) = A_4\Delta - \zeta_4, \\
F_{10;12}(\mathbf{D}_x) &= F_{12;10}(\mathbf{D}_x) = A_6\Delta - \zeta_6, F_{11;12}(\mathbf{D}_x) = F_{12;11}(\mathbf{D}_x) = A_5\Delta - \zeta_5, F_{i+9;13}(\mathbf{D}_x) = \delta_i, F_{i+9;14}(\mathbf{D}_x) = \xi_i, \\
F_{i+9;15}(\mathbf{D}_x) &= v_i, F_{13;j}(\mathbf{D}_x) = -\wp \frac{\partial}{\partial x_j}, F_{13;j+3}(\mathbf{D}_x) = -E_1 \frac{\partial}{\partial x_j}, F_{13;j+6}(\mathbf{D}_x) = -E_2 \frac{\partial}{\partial x_j}, F_{13;j+9}(\mathbf{D}_x) = \delta_j, \\
F_{13;13}(\mathbf{D}_x) &= \gamma\Delta - \tilde{\beta}, F_{13;14}(\mathbf{D}_x) = n, F_{13;15}(\mathbf{D}_x) = \xi, F_{14;j}(\mathbf{D}_x) = \imath\omega \vartheta_1 T_0 \frac{\partial}{\partial x_j}, F_{14;j+3}(\mathbf{D}_x) = k_1 \frac{\partial}{\partial x_j}, \\
F_{14;j+6}(\mathbf{D}_x) &= 0, F_{14;j+9}(\mathbf{D}_x) = \imath\omega \xi_j T_0, F_{14;13}(\mathbf{D}_x) = \imath\omega n T_0, F_{14;14}(\mathbf{D}_x) = k\Delta + \imath\omega \eta T_0, \\
F_{14;15}(\mathbf{D}_x) &= \imath\omega \varsigma T_0, F_{15;j}(\mathbf{D}_x) = \imath\omega \vartheta_2 \frac{\partial}{\partial x_j}, F_{15;j+3}(\mathbf{D}_x) = 0, F_{15;j+6}(\mathbf{D}_x) = h_1 \frac{\partial}{\partial x_j}, F_{15;j+9}(\mathbf{D}_x) = \imath\omega v_j, \\
F_{15;13}(\mathbf{D}_x) &= \imath\omega \xi, F_{15;14}(\mathbf{D}_x) = \imath\omega \varsigma, F_{15;15}(\mathbf{D}_x) = h\Delta + \imath\omega \varpi_i, j = 1, 2, 3
\end{aligned}$$

where $k_8 = \imath\omega D_1 - k_2, h_8 = \imath\omega D_2 - h_2$.

and

$$\tilde{\mathbf{F}}(\mathbf{D}_x) = (\tilde{\mathbf{F}}_{gz}(\mathbf{D}_x))_{15 \times 15},$$

where

$$\begin{aligned}
\tilde{F}_{ij}(\mathbf{D}_x) &= \mu\Delta \delta_{ij} + (\lambda' + \mu)\frac{\partial^2}{\partial x_i \partial x_j}, \tilde{F}_{i+3;j+3}(\mathbf{D}_x) = k_6\Delta \delta_{ij} + (k_4 + k_5)\frac{\partial^2}{\partial x_i \partial x_j}, \\
\tilde{F}_{i+6;j+6}(\mathbf{D}_x) &= h_6\Delta \delta_{ij} + (h_4 + h_5)\frac{\partial^2}{\partial x_i \partial x_j}, \tilde{F}_{i+6;i+6}(\mathbf{D}_x) = A_i\Delta, \tilde{F}_{10;11}(\mathbf{D}_x) = \\
\tilde{F}_{11;10}(\mathbf{D}_x) &= A_4\Delta, \tilde{F}_{10;12}(\mathbf{D}_x) = \tilde{F}_{12;10}(\mathbf{D}_x) = A_6\Delta, \tilde{F}_{11;12}(\mathbf{D}_x) = \tilde{F}_{12;11}(\mathbf{D}_x) = A_5\Delta, \\
\tilde{F}_{13;13}(\mathbf{D}_x) &= \gamma\Delta, \tilde{F}_{14;14}(\mathbf{D}_x) = k\Delta, \tilde{F}_{15;15}(\mathbf{D}_x) = h\Delta, i, j = 1, 2, 3
\end{aligned}$$

and remaining elements of matrix $\tilde{\mathbf{F}}(\mathbf{D}_x)$ are zero.

The system of equations (28) can be represented as

$$\mathbf{F}(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \phi, \theta, P)$ is a fifteen-component vector function for E^3 .

The matrix $\tilde{\mathbf{F}}(\mathbf{D}_x)$ is called the principal part of the operator $\mathbf{F}(\mathbf{D}_x)$.

DEFINITION 1: The operator $\mathbf{F}(\mathbf{D}_x)$ is said to be elliptic if $|\tilde{\mathbf{F}}(\mathbf{m})| \neq 0$, where $\mathbf{m} = (m_1, m_2, m_3)$.

Since $|\tilde{\mathbf{F}}(\mathbf{m})| = \mu^2 \tilde{\lambda} k k_6 k_7 h h_6 h_7 \gamma \vartheta |\mathbf{m}|^{30}$, $\tilde{\lambda} = \lambda' + 2\mu$, $k_7 = k_4 + k_5 + k_6$, $h_7 = h_4 + h_5 + h_6$,

$$\vartheta = \begin{vmatrix} A_1 & A_4 & A_6 \\ A_4 & A_2 & A_5 \\ A_6 & A_5 & A_3 \end{vmatrix}, \text{ therefore operator } \mathbf{F}(\mathbf{D}_x) \text{ is an elliptic differential operator iff}$$

$$\mu \tilde{\lambda} k k_6 k_7 h h_6 h_7 \gamma \vartheta \neq 0. \quad (29)$$

DEFINITION 2: The fundamental solution of the system of equations (28) (the fundamental matrix of operator $\mathbf{F}$) is the matrix $\mathbf{G}(\mathbf{x}) = (G_{gz}(\mathbf{x}))_{15 \times 15}$ satisfying condition

$$\mathbf{F}(\mathbf{D}_x) \mathbf{G}(\mathbf{x}) = \delta(\mathbf{x}) \mathbf{I}(\mathbf{x}), \quad (30)$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{I}(\mathbf{x}) = (\delta_{gz})_{15 \times 15}$ is the unit matrix and $\mathbf{x} \in E^3$.

Now, we construct $\mathbf{G}(\mathbf{x})$ in terms of elementary functions.

4. CONSTRUCTION OF $\mathbf{G}(\mathbf{x})$ IN TERMS OF ELEMENTARY FUNCTIONS

Let us consider the system of non-homogeneous equations

$$(\mu \Delta + \rho \omega^2) \mathbf{u} + (\lambda' + \mu) \nabla(\nabla \cdot \mathbf{u}) - \tilde{\lambda}_1 \nabla \phi_i - \wp \nabla \phi + \imath \omega T_0 \vartheta_1 \nabla \theta + \imath \omega \vartheta_2 \nabla P = \mathbf{H}, \quad (31)$$

$$(k_6 \Delta + k_8) \mathbf{v} + (k_4 + k_5) \nabla(\nabla \cdot \mathbf{v}) + \imath \omega D_3 \mathbf{w} - B_i \nabla \phi_i - E_1 \nabla \phi + k_1 \nabla \theta = \mathbf{V}, \quad (32)$$

$$\imath \omega D_3 \mathbf{v} + (h_6 \Delta + h_8) \mathbf{w} + (h_4 + h_5) \nabla(\nabla \cdot \mathbf{w}) - B_{i+3} \nabla \phi_i - E_2 \nabla \phi + h_1 \nabla P = \mathbf{W}, \quad (33)$$

$$\begin{aligned} & \tilde{\lambda}_1 (\nabla \cdot \mathbf{u}) + \imath \omega B_1 (\nabla \cdot \mathbf{v}) + \imath \omega B_4 (\nabla \cdot \mathbf{w}) + (A_1 \Delta - \gamma_1) \phi_1 + (A_4 \Delta - \zeta_4) \phi_2 + (A_6 \Delta - \zeta_6) \phi_3 \\ & + \delta_1 \phi + \imath \omega \xi_1 T_0 \theta + \imath \omega v_1 P = X_1, \end{aligned} \quad (34)$$

$$\begin{aligned} & \tilde{\lambda}_2 (\nabla \cdot \mathbf{u}) + \imath \omega B_2 (\nabla \cdot \mathbf{v}) + \imath \omega B_5 (\nabla \cdot \mathbf{w}) + (A_4 \Delta - \zeta_4) \phi_1 + (A_2 \Delta - \gamma_2) \phi_2 + (A_5 \Delta - \zeta_5) \phi_3 \\ & + \delta_2 \phi + \imath \omega \xi_2 T_0 \theta + \imath \omega v_2 P = X_2, \end{aligned} \quad (35)$$

$$\begin{aligned} & \tilde{\lambda}_3 (\nabla \cdot \mathbf{u}) + \imath \omega B_3 (\nabla \cdot \mathbf{v}) + \imath \omega B_6 (\nabla \cdot \mathbf{w}) + (A_6 \Delta - \zeta_6) \phi_1 + (A_5 \Delta - \zeta_5) \phi_2 + (A_3 \Delta - \gamma_3) \phi_3 \\ & + \delta_3 \phi + \imath \omega \xi_3 T_0 \theta + \imath \omega v_3 P = X_3, \end{aligned} \quad (36)$$

$$\wp (\nabla \cdot \mathbf{u}) + \imath \omega E_1 (\nabla \cdot \mathbf{v}) + \imath \omega E_2 (\nabla \cdot \mathbf{w}) + \delta_i \phi_i + (\gamma \Delta - \beta) \phi + \imath \omega n T_0 \theta + \imath \omega \xi P = L, \quad (37)$$

$$- \vartheta_1 (\nabla \cdot \mathbf{u}) - k_3 (\nabla \cdot \mathbf{v}) + \xi_i \phi_i + n \phi + [k \Delta + \imath \omega T_0 \eta] \theta + \imath \omega \zeta P = Y, \quad (38)$$

$$- \vartheta_2 (\nabla \cdot \mathbf{u}) - h_3 (\nabla \cdot \mathbf{w}) + v_i \phi_i + \xi \phi + \imath \omega T_0 \zeta \theta + [h \Delta + \imath \omega \varpi] P = Z, \quad (39)$$

where $\mathbf{H}, \mathbf{V}, \mathbf{W}$ are three-component vector functions on E^3 ; X_i, L, Y and Z are scalar functions on E^3 .

The system of equations (31)–(39) may be written in the form

$$\mathbf{F}^{\text{tr}}(\mathbf{D}_{\mathbf{x}})\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \quad (40)$$

where $\mathbf{F}^{\text{tr}}$ is the transpose of matrix $\mathbf{F}$, $\mathbf{Q} = (\mathbf{H}, \mathbf{V}, \mathbf{W}, X_i, L, Y, Z)$, $\mathbf{x} \in E^3$.

Applying operator $(\nabla \cdot)$ to the equations (31)–(33), we obtain

$$[\tilde{\lambda}\Delta + \rho\omega^2](\nabla \cdot \mathbf{u}) - \tilde{\lambda}_1\Delta\phi_i - \wp\Delta\phi + \imath\omega T_0\vartheta_1\Delta\theta + \imath\omega\vartheta_2\Delta P = \nabla \cdot \mathbf{H}, \quad (41)$$

$$[k_7\Delta + k_8](\nabla \cdot \mathbf{v}) + \imath\omega D_3(\nabla \cdot \mathbf{w}) - B_1\Delta\phi_i - E_1\Delta\phi + k_1\Delta\theta = \nabla \cdot \mathbf{V}, \quad (42)$$

$$\imath\omega D_3(\nabla \cdot \mathbf{v}) + [h_7\Delta + h_8](\nabla \cdot \mathbf{w}) - B_{i+3}\Delta\phi_i - E_2\Delta\phi + h_1\Delta P = \nabla \cdot \mathbf{W}. \quad (43)$$

The equations (34)–(39) and (41)–(43) may be expressed in the form

$$\mathbf{N}(\Delta)\mathbf{S} = \tilde{\mathbf{Q}}, \quad (44)$$

where $\mathbf{S} = (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{w}, \phi, \phi, \theta, P)$, $\tilde{\mathbf{Q}} = (w_1, \dots, w_9) = (\nabla \cdot \mathbf{H}, \nabla \cdot \mathbf{V}, \nabla \cdot \mathbf{W}, X_i, L, Y, Z)$ and

$$\mathbf{N}(\Delta) = \left(N_{gz}(\Delta) \right)_{9 \times 9} = \begin{pmatrix} \tilde{\lambda}\Delta + \rho\omega^2 & 0 & 0 & -\tilde{\lambda}_1\Delta & -\tilde{\lambda}_2\Delta & -\tilde{\lambda}_3\Delta & -\wp\Delta & \imath\omega T_0\vartheta_1\Delta & \imath\omega\vartheta_2\Delta \\ 0 & k_7\Delta + k_8 & \imath\omega D_3 & -B_1\Delta & -B_2\Delta & -B_3\Delta & -E_1\Delta & k_1\Delta & 0 \\ 0 & \imath\omega D_3 & h_7\Delta + h_8 & -B_4\Delta & -B_5\Delta & -B_6\Delta & -E_2\Delta & 0 & h_1\Delta \\ \tilde{\lambda}_1 & \imath\omega B_1 & \imath\omega B_4 & A_1\Delta - \gamma_1 & A_4\Delta - \zeta_4 & A_6\Delta - \zeta_6 & \delta_1 & \imath\omega T_0\xi_1 & \imath\omega v_1 \\ \tilde{\lambda}_2 & \imath\omega B_2 & \imath\omega B_5 & A_4\Delta - \zeta_4 & A_2\Delta - \gamma_2 & A_5\Delta - \zeta_5 & \delta_2 & \imath\omega T_0\xi_2 & \imath\omega v_2 \\ \tilde{\lambda}_3 & \imath\omega B_3 & \imath\omega B_6 & A_6\Delta - \zeta_6 & A_5\Delta - \zeta_5 & A_3\Delta - \gamma_3 & \delta_3 & \imath\omega T_0\xi_3 & \imath\omega v_3 \\ \wp & \imath\omega E_1 & \imath\omega E_2 & \delta_1 & \delta_2 & \delta_3 & \gamma\Delta - \tilde{\beta} & \imath\omega T_0n & \imath\omega \xi \\ -\vartheta_1 & -k_3 & 0 & \xi_1 & \xi_2 & \xi_3 & n & k\Delta + \imath\omega T_0\eta & \imath\omega \varsigma \\ -\vartheta_2 & 0 & -h_3 & v_1 & v_2 & v_3 & \xi & \imath\omega T_0\varsigma & h\Delta + \imath\omega \varpi \end{pmatrix}_{9 \times 9}$$

Equation (44) may also be written in determinant form as

$$\Gamma_1(\Delta)\mathbf{S} = \Psi, \quad (45)$$

where $\Psi = (\Psi_1, \dots, \Psi_9)$, $\Psi_j = \frac{1}{\tilde{A}} \sum_{i=1}^9 N_{ij}^* w_i$, $\Gamma_1(\Delta) = \frac{|\mathbf{N}(\Delta)|}{\tilde{A}}$, $\tilde{A} = \tilde{\lambda} k k_7 h h_7 \gamma \vartheta$ $j = 1, \dots, 9$

and N_{ij}^* is the cofactor of the element N_{ij} of the matrix $\mathbf{N}$.

On expanding $\Gamma_1(\Delta)$, we see that

$$\Gamma_1(\Delta) = \prod_{i=1}^9 (\Delta + \lambda_i^2),$$

where λ_i^2 , $i = 1, \dots, 9$ are the roots of the equation $\Gamma_1(-m) = 0$, w.r.t. m .

Applying operator $\Gamma_1(\Delta)$ to equation (31) with the assistance of equation (45), we obtain:

$$\Gamma_1(\Delta)(\Delta + \lambda_{10}^2)\mathbf{u} = \mathbf{\Psi}', \quad (46)$$

where $\lambda_{10}^2 = \frac{\rho\omega^2}{\mu}$, $\mathbf{\Psi}' = \frac{1}{\mu} \left\{ \Gamma_1(\Delta)\mathbf{H} - \nabla[(\lambda' + \mu)\psi_1 - \tilde{\lambda}_i\psi_{i+3} - \wp\psi_7 + \imath\omega\mathfrak{S}_1T_0\psi_8 + \imath\omega\mathfrak{S}_2\psi_9] \right\}$.

Multiplying equations (32) and (33) by $h_6\Delta + h_8$ and $\imath\omega D_3$ respectively, we obtain

$$(h_6\Delta + h_8)[(k_6\Delta + k_8)\mathbf{v} + (k_4 + k_5)\nabla(\nabla \cdot \mathbf{v})] + (h_6\Delta + h_8)\imath\omega D_3\mathbf{w} = \quad (47)$$

$$(h_6\Delta + h_8)[\mathbf{V} + B_i\nabla\phi_i + E_1\nabla\phi - k_1\nabla\theta],$$

$$(\imath\omega D_3)^2\mathbf{v} + \imath\omega D_3[(h_6\Delta + h_8)\mathbf{w} + (h_4 + h_5)\nabla(\nabla \cdot \mathbf{w})] = \imath\omega D_3[\mathbf{W} + B_{i+3}\nabla\phi_i + E_2\nabla\phi - h_1\nabla P]. \quad (48)$$

Using equation (48) in (47) and then applying $\Gamma_1(\Delta)$ to the resulting equation with the assistance of equation (45), we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{v} = \mathbf{\Psi}'', \quad (49)$$

where $\Gamma_2(\Delta) = \frac{1}{\tilde{B}} \begin{vmatrix} k_6\Delta + k_8 & \imath\omega D_3 \\ \imath\omega D_3 & h_6\Delta + h_8 \end{vmatrix}$, $\tilde{B} = k_6h_6$ and

$$\mathbf{\Psi}'' = \frac{1}{\tilde{B}} \left\{ (h_6\Delta + h_8)[\Gamma_1(\Delta)\mathbf{V} - (k_4 + k_5)\nabla\psi_2 + B_i\nabla\psi_{i+3} + E_1\nabla\psi_7 - k_1\nabla\psi_8] \right. \\ \left. - \imath\omega D_3[\Gamma_1(\Delta)\mathbf{W} - (h_4 + h_5)\nabla\psi_3 + B_{i+3}\nabla\psi_{i+3} + E_2\nabla\psi_7 - h_1\nabla\psi_9] \right\}.$$

It can be seen that $\Gamma_2(\Delta) = (\Delta + \lambda_{11}^2)(\Delta + \lambda_{12}^2)$.

Multiplying equations (32) and (33) by $\imath\omega D_3$ and $k_6\Delta + k_8$ respectively, we obtain

$$\imath\omega D_3[(k_6\Delta + k_8)\mathbf{v} + (k_4 + k_5)\nabla(\nabla \cdot \mathbf{v})] + (\imath\omega D_3)^2\mathbf{w} = \imath\omega D_3[\mathbf{V} + B_i\nabla\phi_i + E_1\nabla\phi - k_1\nabla\theta], \quad (50)$$

$$\imath\omega D_3(k_6\Delta + k_8)\mathbf{v} + (k_6\Delta + k_8)[(h_6\Delta + h_8)\mathbf{w} + (h_4 + h_5)\nabla(\nabla \cdot \mathbf{w})] = \quad (51)$$

$$(k_6\Delta + k_8)[\mathbf{W} + B_{i+3}\nabla\phi_i + E_2\nabla\phi - h_1\nabla P].$$

Using equation (50) in (51) and applying $\Gamma_1(\Delta)$ to the resulting equation with the help of equation (45), we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{w} = \mathbf{\Psi}''', \quad (52)$$

where

$$\Psi''' = \frac{1}{\tilde{B}} \left\{ (k_6\Delta + k_8)[\Gamma_1(\Delta)\mathbf{W} - (h_4 + h_5)\nabla\psi_3 + B_{i+3}\nabla\psi_{i+3} + E_2\nabla\psi_7 - h_1\nabla\psi_9] \right. \\ \left. - \imath\omega D_3[\Gamma_1(\Delta)\mathbf{V} - (k_4 + k_5)\nabla\psi_2 + B_i\nabla\psi_{i+3} + E_1\nabla\psi_7 - k_1\nabla\psi_8] \right\}.$$

From equations (45), (46), (49) and (52), we have

$$\Theta(\Delta)\mathbf{U}(\mathbf{x}) = \hat{\Psi}(\mathbf{x}), \quad (53)$$

where

$$\hat{\Psi}(\mathbf{x}) = (\Psi', \Psi'', \Psi''', \Psi_4, \dots, \Psi_9) \text{ and } \Theta(\Delta) = (\Theta_{gz}(\Delta))_{15 \times 15}, \\ \Theta_{ii}(\Delta) = \Gamma_1(\Delta)(\Delta + \lambda_{10}^2) = \prod_{y=1}^{10}(\Delta + \lambda_y^2), \quad \Theta_{i+3i+3}(\Delta) = \Theta_{i+6i+6}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{y=1}^{9,11,12}(\Delta + \lambda_y^2), \\ \Theta_{jj}(\Delta) = \Gamma_1(\Delta) = \prod_{y=1}^9(\Delta + \lambda_y^2), \quad \Theta_{gz}(\Delta) = 0 \quad i = 1, 2, 3 \quad j = 10, \dots, 15 \quad g, z = 1, \dots, 15 \quad g \neq z.$$

The expressions for $\Psi', \Psi'', \Psi''', \Psi_j, j = 4, \dots, 9$ can be rewritten in the form

$$\Psi' = \frac{1}{\mu} \Gamma_1(\Delta) \mathbf{JH} + w_{11}(\Delta) \nabla(\nabla \cdot \mathbf{H}) + w_{21}(\Delta) \nabla(\nabla \cdot \mathbf{V}) + w_{31}(\Delta) \nabla(\nabla \cdot \mathbf{W}) + \sum_{i=4}^9 w_{i1}(\Delta) \nabla w_i, \\ \Psi'' = w_{12}(\Delta) \nabla(\nabla \cdot \mathbf{H}) + \frac{1}{\tilde{B}} (h_6\Delta + h_8) \Gamma_1(\Delta) \mathbf{JV} + w_{22}(\Delta) \nabla(\nabla \cdot \mathbf{V}) \\ - \frac{1}{\tilde{B}} \imath\omega D_3 \Gamma_1(\Delta) \mathbf{JW} + w_{32}(\Delta) \nabla(\nabla \cdot \mathbf{W}) + \sum_{i=4}^9 w_{i2}(\Delta) \nabla w_i, \\ \Psi''' = w_{13}(\Delta) \nabla(\nabla \cdot \mathbf{H}) - \frac{1}{\tilde{B}} \imath\omega D_3 \Gamma_1(\Delta) \mathbf{JV} + w_{23}(\Delta) \nabla(\nabla \cdot \mathbf{V}) \\ + \frac{1}{\tilde{B}} (k_6\Delta + k_8) \Gamma_1(\Delta) \mathbf{JW} + w_{33}(\Delta) \nabla(\nabla \cdot \mathbf{W}) + \sum_{i=4}^9 w_{i3}(\Delta) \nabla w_i, \quad (54) \\ \Psi_j = w_{1j}(\Delta) \nabla \cdot \mathbf{H} + w_{2j}(\Delta) \nabla \cdot \mathbf{V} + w_{3j}(\Delta) \nabla \cdot \mathbf{W} + \sum_{i=4}^9 w_{ij}(\Delta) w_i,$$

where $\mathbf{J} = (\delta_{ij})_{3 \times 3}$ is the unit matrix and

$$w_{i1}(\Delta) = -\frac{1}{\tilde{A}\mu} \left[(\lambda' + \mu) N_{i1}^*(\Delta) - [\tilde{\lambda}_j N_{i;j+3}^*(\Delta) + \wp N_{i7}^*(\Delta) - \imath\omega \vartheta_1 T_0 N_{i8}^*(\Delta) - \imath\omega \vartheta_2 N_{i9}^*(\Delta)] \right], \\ w_{i2}(\Delta) = -\frac{1}{\tilde{A}\tilde{B}} \left\{ (h_6\Delta + h_8) [(k_4 + k_5) N_{i2}^*(\Delta) - B_j N_{i;j+3}^*(\Delta) - E_1 N_{i7}^*(\Delta) + k_1 N_{i8}^*(\Delta)] \right. \\ \left. - \imath\omega D_3 [(h_4 + h_5) N_{i3}^*(\Delta) - B_{j+3} N_{i;j+3}^*(\Delta) - E_2 N_{i7}^*(\Delta) + h_1 N_{i9}^*(\Delta)] \right\}, \\ w_{i3}(\Delta) = -\frac{1}{\tilde{A}\tilde{B}} \left\{ -\imath\omega D_3 [(k_4 + k_5) N_{i2}^*(\Delta) - B_j N_{i;j+3}^*(\Delta) - E_1 N_{i7}^*(\Delta) + k_1 N_{i8}^*(\Delta)] + \right. \\ \left. (k_6\Delta + k_8) [(h_4 + h_5) N_{i3}^*(\Delta) - B_{j+3} N_{i;j+3}^*(\Delta) - E_2 N_{i7}^*(\Delta) + h_1 N_{i9}^*(\Delta)] \right\}, \\ w_{ig}(\Delta) = \frac{N_{ig}^*(\Delta)}{\tilde{A}}, i = 1, \dots, 9 \quad g = 4, \dots, 9.$$

From equation (54), we have

$$\hat{\Psi}(\mathbf{x}) = \mathbf{R}^{\text{tr}}(\mathbf{D}_x) \mathbf{Q}(\mathbf{x}), \quad (55)$$

where

$$\begin{aligned}
\mathbf{R}(\mathbf{D}_x) &= (\mathbf{R}_{gz}(\mathbf{D}_x))_{15 \times 15}, \\
R_{ij}(\mathbf{D}_x) &= \frac{1}{\mu} \Gamma_1(\Delta) \delta_{ij} + w_{11}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, R_{i;j+3}(\mathbf{D}_x) = w_{12}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, \\
R_{i;j+6}(\mathbf{D}_x) &= w_{13}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, R_{i;y+6}(\mathbf{D}_x) = w_{1y}(\Delta) \frac{\partial}{\partial x_i}, \\
R_{i+3;j}(\mathbf{D}_x) &= w_{21}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, R_{i+3;j+3}(\mathbf{D}_x) = \frac{1}{B} (h_6 \Delta + h_8) \Gamma_1(\Delta) \delta_{ij} + w_{22}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, \\
R_{i+3;j+6}(\mathbf{D}_x) &= -\frac{1}{B} \iota \omega D_3 \Gamma_1(\Delta) \delta_{ij} + w_{23}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, R_{i+3;y+6}(\mathbf{D}_x) = w_{2y}(\Delta) \frac{\partial}{\partial x_i}, \\
R_{i+6;j}(\mathbf{D}_x) &= w_{31}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, R_{i+6;j+3}(\mathbf{D}_x) = -\frac{1}{B} \iota \omega D_3 \Gamma_1(\Delta) \delta_{ij} + w_{32}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, \\
R_{i+6;j+6}(\mathbf{D}_x) &= \frac{1}{B} (k_6 \Delta + k_8) \Gamma_1(\Delta) \delta_{ij} + w_{33}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, R_{i+6;y+6}(\mathbf{D}_x) = w_{3y}(\Delta) \frac{\partial}{\partial x_i}, \\
R_{y+6;i}(\mathbf{D}_x) &= w_{y1}(\Delta) \frac{\partial}{\partial x_i}, R_{y+6;i+3}(\mathbf{D}_x) = w_{y2}(\Delta) \frac{\partial}{\partial x_i}, R_{y+6;i+6}(\mathbf{D}_x) = w_{y3}(\Delta) \frac{\partial}{\partial x_i}, \\
R_{y+6;\hbar+6}(\mathbf{D}_x) &= w_{y\hbar}(\Delta) \quad i, j = 1, 2, 3 \quad y, \hbar = 4, \dots, 9.
\end{aligned}$$

From equations (40), (53) and (55), we obtain

$$\mathbf{F}(\mathbf{D}_x) \mathbf{R}(\mathbf{D}_x) = \mathbf{\Theta}(\Delta). \quad (56)$$

We assume that $\lambda_i^2 \neq \lambda_j^2 \neq 0$ $i, j = 1, \dots, 12$.

Let

$$\begin{aligned}
\mathbf{Y}(\mathbf{x}) &= (\mathbf{Y}_{ij}(\mathbf{x}))_{15 \times 15}, Y_{zz}(\mathbf{x}) = \sum_{g=1}^{10} r_{1g} \varsigma_g(\mathbf{x}), \\
Y_{z+3;z+3}(\mathbf{x}) &= Y_{z+6;z+6}(\mathbf{x}) = \sum_{g=1}^{9,11,12} r_{2g} \varsigma_g(\mathbf{x}), Y_{\ell\ell}(\mathbf{x}) = \sum_{g=1}^9 r_{3g} \varsigma_g(\mathbf{x}), \\
Y_{ij}(\mathbf{x}) &= 0 \quad z = 1, 2, 3 \quad \ell = 10, \dots, 15 \quad i, j = 1, \dots, 15 \quad i \neq j
\end{aligned}$$

where

$$\begin{aligned}
\varsigma_g(\mathbf{x}) &= -\frac{e^{\iota \lambda_g |\mathbf{x}|}}{4\pi |\mathbf{x}|}, r_{1y} = \prod_{i=1, i \neq y}^{10} (\lambda_i^2 - \lambda_y^2)^{-1}, r_{2\ell} = \prod_{i=1, i \neq \ell}^{9,11,12} (\lambda_i^2 - \lambda_\ell^2)^{-1}, \\
r_{3z} &= \prod_{i=1, i \neq z}^9 (\lambda_i^2 - \lambda_z^2)^{-1} \quad g = 1, \dots, 12 \quad y = 1, \dots, 10 \\
\ell &= 1, \dots, 9, 11, 12 \quad z = 1, \dots, 9.
\end{aligned} \quad (57)$$

LEMMA 1: The matrix $\mathbf{Y}$ is the fundamental matrix of the operator $\mathbf{\Theta}(\Delta)$ i.e.

$$\Theta(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}). \quad (58)$$

PROOF: To establish the lemma, it is adequate to prove that

$$\Gamma_1(\Delta)(\Delta + \lambda_{10}^2)\mathbf{Y}_{11}(\mathbf{x}) = \delta(\mathbf{x}), \quad (59)$$

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{Y}_{44}(\mathbf{x}) = \delta(\mathbf{x}), \quad (60)$$

$$\Gamma_1(\Delta)\mathbf{Y}_{10;10}(\mathbf{x}) = \delta(\mathbf{x}). \quad (61)$$

Consider

$$\sum_{i=1}^9 r_{3i} = \frac{1}{z_{10}} \sum_{j=1}^9 (-1)^{j+1} z_j,$$

where

$$\begin{aligned} z_1 &= \prod_{i=3}^9 (\lambda_2^2 - \lambda_i^2) \prod_{j=4}^9 (\lambda_3^2 - \lambda_j^2) \prod_{z=5}^9 (\lambda_4^2 - \lambda_z^2) \prod_{y=6}^9 (\lambda_5^2 - \lambda_y^2) \prod_{g=7}^9 (\lambda_6^2 - \lambda_g^2) \prod_{\ell=8}^9 (\lambda_7^2 - \lambda_\ell^2) (\lambda_8^2 - \lambda_9^2), \\ z_2 &= \prod_{i=3}^9 (\lambda_1^2 - \lambda_i^2) \prod_{j=4}^9 (\lambda_3^2 - \lambda_j^2) \prod_{z=5}^9 (\lambda_4^2 - \lambda_z^2) \prod_{y=6}^9 (\lambda_5^2 - \lambda_y^2) \prod_{g=7}^9 (\lambda_6^2 - \lambda_g^2) \prod_{\ell=8}^9 (\lambda_7^2 - \lambda_\ell^2) (\lambda_8^2 - \lambda_9^2), \\ z_3 &= \prod_{i=2}^9 (\lambda_1^2 - \lambda_i^2) \prod_{j=4}^9 (\lambda_2^2 - \lambda_j^2) \prod_{z=5}^9 (\lambda_4^2 - \lambda_z^2) \prod_{y=6}^9 (\lambda_5^2 - \lambda_y^2) \prod_{g=7}^9 (\lambda_6^2 - \lambda_g^2) \prod_{\ell=8}^9 (\lambda_7^2 - \lambda_\ell^2) (\lambda_8^2 - \lambda_9^2), \\ z_4 &= \prod_{i=2}^9 (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^9 (\lambda_2^2 - \lambda_j^2) \prod_{z=5}^9 (\lambda_3^2 - \lambda_z^2) \prod_{y=6}^9 (\lambda_4^2 - \lambda_y^2) \prod_{g=7}^9 (\lambda_6^2 - \lambda_g^2) \prod_{\ell=8}^9 (\lambda_7^2 - \lambda_\ell^2) (\lambda_8^2 - \lambda_9^2), \\ z_5 &= \prod_{i=2}^9 (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^9 (\lambda_2^2 - \lambda_j^2) \prod_{z=4}^9 (\lambda_3^2 - \lambda_z^2) \prod_{y=6}^9 (\lambda_4^2 - \lambda_y^2) \prod_{g=7}^9 (\lambda_6^2 - \lambda_g^2) \prod_{\ell=8}^9 (\lambda_7^2 - \lambda_\ell^2) (\lambda_8^2 - \lambda_9^2), \\ z_6 &= \prod_{i=2}^9 (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^9 (\lambda_2^2 - \lambda_j^2) \prod_{z=4}^9 (\lambda_3^2 - \lambda_z^2) \prod_{y=5}^9 (\lambda_4^2 - \lambda_y^2) \prod_{g=7}^9 (\lambda_6^2 - \lambda_g^2) \prod_{\ell=8}^9 (\lambda_7^2 - \lambda_\ell^2) (\lambda_8^2 - \lambda_9^2), \\ z_7 &= \prod_{i=2}^9 (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^9 (\lambda_2^2 - \lambda_j^2) \prod_{z=4}^9 (\lambda_3^2 - \lambda_z^2) \prod_{y=5}^9 (\lambda_4^2 - \lambda_y^2) \prod_{g=6}^9 (\lambda_5^2 - \lambda_g^2) \prod_{\ell=8}^9 (\lambda_6^2 - \lambda_\ell^2) (\lambda_8^2 - \lambda_9^2), \\ z_8 &= \prod_{i=2}^9 (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^9 (\lambda_2^2 - \lambda_j^2) \prod_{z=4}^9 (\lambda_3^2 - \lambda_z^2) \prod_{y=5}^9 (\lambda_4^2 - \lambda_y^2) \prod_{g=6}^9 (\lambda_5^2 - \lambda_g^2) \prod_{\ell=7}^9 (\lambda_6^2 - \lambda_\ell^2) (\lambda_7^2 - \lambda_9^2), \\ z_9 &= \prod_{i=2}^8 (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^8 (\lambda_2^2 - \lambda_j^2) \prod_{z=4}^8 (\lambda_3^2 - \lambda_z^2) \prod_{y=5}^8 (\lambda_4^2 - \lambda_y^2) \prod_{g=6}^8 (\lambda_5^2 - \lambda_g^2) \prod_{\ell=7}^8 (\lambda_6^2 - \lambda_\ell^2) (\lambda_7^2 - \lambda_8^2), \\ z_{10} &= \prod_{i=2}^9 (\lambda_1^2 - \lambda_i^2) \prod_{j=3}^9 (\lambda_2^2 - \lambda_j^2) \prod_{z=4}^9 (\lambda_3^2 - \lambda_z^2) \prod_{y=5}^9 (\lambda_4^2 - \lambda_y^2) \prod_{g=6}^9 (\lambda_5^2 - \lambda_g^2) \prod_{\ell=7}^9 (\lambda_6^2 - \lambda_\ell^2) \prod_{h=8}^9 (\lambda_7^2 - \lambda_h^2) (\lambda_8^2 - \lambda_9^2). \end{aligned}$$

Upon simplifying the R.H.S. of the above relation, we obtain $\sum_{i=1}^9 r_{3i} = 0.$ (62)

Similarly, we find that

$$\begin{aligned}
\sum_{i=2}^9 r_{3i} (\lambda_1^2 - \lambda_i^2) &= 0, \sum_{i=3}^9 r_{3i} \left[\prod_{j=1}^2 (\lambda_j^2 - \lambda_i^2) \right] = 0, \sum_{i=4}^9 r_{3i} \left[\prod_{j=1}^3 (\lambda_j^2 - \lambda_i^2) \right] = 0, \\
\sum_{i=5}^9 r_{3i} \left[\prod_{j=1}^4 (\lambda_j^2 - \lambda_i^2) \right] &= 0, \sum_{i=6}^9 r_{3i} \left[\prod_{j=1}^5 (\lambda_j^2 - \lambda_i^2) \right] = 0, \sum_{i=7}^9 r_{3i} \left[\prod_{j=1}^6 (\lambda_j^2 - \lambda_i^2) \right] = 0, \\
\sum_{i=8}^9 r_{3i} \left[\prod_{j=1}^7 (\lambda_j^2 - \lambda_i^2) \right] &= 0, r_{39} \left[\prod_{i=1}^8 (\lambda_i^2 - \lambda_9^2) \right] = 1.
\end{aligned} \tag{63}$$

$$\text{Also, we have } (\Delta + \lambda_y^2) \varsigma_g(\mathbf{x}) = \delta(\mathbf{x}) + (\lambda_y^2 - \lambda_g^2) \varsigma_g(\mathbf{x}) \quad y, g = 1, \dots, 12. \tag{64}$$

Now, let us consider

$$\Gamma_1(\Delta) Y_{10;10}(\mathbf{x}) = \prod_{i=1}^9 (\Delta + \lambda_i^2) \sum_{g=1}^9 r_{3g} \varsigma_g(\mathbf{x}) = \prod_{i=2}^9 (\Delta + \lambda_i^2) \sum_{g=1}^9 r_{3g} [\delta(\mathbf{x}) + (\lambda_1^2 - \lambda_g^2) \varsigma_g(\mathbf{x})].$$

Using equations (62)-(64) in the above relation, we obtain

$$\begin{aligned}
&\Gamma_1(\Delta) Y_{10;10}(\mathbf{x}) \\
&= \prod_{i=2}^9 (\Delta + \lambda_i^2) \sum_{g=2}^9 r_{3g} (\lambda_1^2 - \lambda_g^2) \varsigma_g(\mathbf{x}) = \prod_{i=3}^9 (\Delta + \lambda_i^2) \sum_{g=2}^9 r_{3g} (\lambda_1^2 - \lambda_g^2) [\delta(\mathbf{x}) + (\lambda_2^2 - \lambda_g^2) \varsigma_g(\mathbf{x})] \\
&= \prod_{i=3}^9 (\Delta + \lambda_i^2) \sum_{g=3}^9 r_{3g} \left[\prod_{j=1}^2 (\lambda_j^2 - \lambda_g^2) \right] \varsigma_g(\mathbf{x}) = \prod_{i=4}^9 (\Delta + \lambda_i^2) \sum_{g=3}^9 r_{3g} \left[\prod_{j=1}^2 (\lambda_j^2 - \lambda_g^2) \right] [\delta(\mathbf{x}) + (\lambda_3^2 - \lambda_g^2) \varsigma_g(\mathbf{x})] \\
&= \prod_{i=4}^9 (\Delta + \lambda_i^2) \sum_{g=4}^9 r_{3g} \left[\prod_{j=1}^3 (\lambda_j^2 - \lambda_g^2) \right] \varsigma_g(\mathbf{x}) = \prod_{i=5}^9 (\Delta + \lambda_i^2) \sum_{g=4}^9 r_{3g} \left[\prod_{j=1}^3 (\lambda_j^2 - \lambda_g^2) \right] [\delta(\mathbf{x}) + (\lambda_4^2 - \lambda_g^2) \varsigma_g(\mathbf{x})] \\
&= \prod_{i=5}^9 (\Delta + \lambda_i^2) \sum_{g=5}^9 r_{3g} \left[\prod_{j=1}^4 (\lambda_j^2 - \lambda_g^2) \right] \varsigma_g(\mathbf{x}) = \prod_{i=6}^9 (\Delta + \lambda_i^2) \sum_{g=5}^9 r_{3g} \left[\prod_{j=1}^4 (\lambda_j^2 - \lambda_g^2) \right] [\delta(\mathbf{x}) + (\lambda_5^2 - \lambda_g^2) \varsigma_g(\mathbf{x})] \\
&= \prod_{i=6}^9 (\Delta + \lambda_i^2) \sum_{g=6}^9 r_{3g} \left[\prod_{j=1}^5 (\lambda_j^2 - \lambda_g^2) \right] \varsigma_g(\mathbf{x}) = \prod_{i=7}^9 (\Delta + \lambda_i^2) \sum_{g=6}^9 r_{3g} \left[\prod_{j=1}^5 (\lambda_j^2 - \lambda_g^2) \right] [\delta(\mathbf{x}) + (\lambda_6^2 - \lambda_g^2) \varsigma_g(\mathbf{x})] \\
&= \prod_{i=7}^9 (\Delta + \lambda_i^2) \sum_{g=7}^9 r_{3g} \left[\prod_{j=1}^6 (\lambda_j^2 - \lambda_g^2) \right] \varsigma_g(\mathbf{x}) = \prod_{i=8}^9 (\Delta + \lambda_i^2) \sum_{g=7}^9 r_{3g} \left[\prod_{j=1}^6 (\lambda_j^2 - \lambda_g^2) \right] [\delta(\mathbf{x}) + (\lambda_7^2 - \lambda_g^2) \varsigma_g(\mathbf{x})] \\
&= \prod_{i=8}^9 (\Delta + \lambda_i^2) \sum_{g=8}^9 r_{3g} \left[\prod_{j=1}^7 (\lambda_j^2 - \lambda_g^2) \right] \varsigma_g(\mathbf{x}) = (\Delta + \lambda_9^2) \sum_{g=8}^9 r_{3g} \left[\prod_{j=1}^7 (\lambda_j^2 - \lambda_g^2) \right] [\delta(\mathbf{x}) + (\lambda_8^2 - \lambda_g^2) \varsigma_g(\mathbf{x})] \\
&= (\Delta + \lambda_9^2) \varsigma_9(\mathbf{x}) = \delta(\mathbf{x}).
\end{aligned}$$

Equations (60) and (61) can be proved in the similar way.

We introduce the matrix

$$\mathbf{G}(\mathbf{x}) = \mathbf{R}(\mathbf{D}_x) \mathbf{Y}(\mathbf{x}). \tag{65}$$

From equations (56), (58) and (65), we obtain

$$\mathbf{F}(\mathbf{D}_x) \mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{D}_x) \mathbf{R}(\mathbf{D}_x) \mathbf{Y}(\mathbf{x}) = \mathbf{\Theta}(\Delta) \mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x}) \mathbf{I}(\mathbf{x}).$$

Therefore, $\mathbf{G}(\mathbf{x})$ is a solution to equation (30).

THEOREM 1: If the condition (29) is satisfied, then the matrix $\mathbf{G}(\mathbf{x})$ defined by the equation (65) is the fundamental solution of the system of equations (28) and each element of the matrix $\mathbf{G}(\mathbf{x})$ is represented in the following form:

$$\begin{aligned} G_{gz}(\mathbf{x}) &= R_{gz}(\mathbf{D}_x)Y_{11}(\mathbf{x}), G_{gy}(\mathbf{x}) = R_{gy}(\mathbf{D}_x)Y_{44}(\mathbf{x}), \\ G_{gj}(\mathbf{x}) &= R_{gj}(\mathbf{D}_x)Y_{10;10}(\mathbf{x}) \quad g=1,\dots,15 \quad z=1,2,3 \quad y=4,\dots,9 \quad j=10,\dots,15 \end{aligned}$$

5. BASIC PROPERTIES OF MATRIX $\mathbf{G}(\mathbf{x})$

THEOREM 2: Each column of the matrix $\mathbf{G}(\mathbf{x})$ is a solution of system of equations (28) at every point $\mathbf{x} \in E^3$ except at the origin.

THEOREM 3: If the condition (29) is satisfied, then the fundamental solution of the system

$$\tilde{\mathbf{F}}(\mathbf{D}_x) \mathbf{U}(\mathbf{x}) = \mathbf{0}$$

is the matrix $\tilde{\mathbf{G}}(\mathbf{x}) = \left(\tilde{G}_{gz}(\mathbf{x}) \right)_{15 \times 15}$, where

$$\begin{aligned} \tilde{G}_{ij}(\mathbf{x}) &= \frac{1}{\lambda} \nabla (\nabla \cdot \zeta_2^*(\mathbf{x})) - \frac{1}{\mu} \nabla \times (\nabla \times \zeta_2^*(\mathbf{x})), \tilde{G}_{i+3;j+3}(\mathbf{x}) = \frac{1}{k_7} \nabla (\nabla \cdot \zeta_2^*(\mathbf{x})) - \frac{1}{k_6} \nabla \times (\nabla \times \zeta_2^*(\mathbf{x})), \\ \tilde{G}_{i+6;j+6}(\mathbf{x}) &= \frac{1}{h_7} \nabla (\nabla \cdot \zeta_2^*(\mathbf{x})) - \frac{1}{h_6} \nabla \times (\nabla \times \zeta_2^*(\mathbf{x})), \tilde{G}_{10;10}(\mathbf{x}) = \frac{A_2 A_3 - A_5^2}{9} \zeta_1^*(\mathbf{x}), \\ \tilde{G}_{10;11}(\mathbf{x}) &= \tilde{G}_{11;10}(\mathbf{x}) = \frac{A_5 A_6 - A_4 A_3}{9} \zeta_1^*(\mathbf{x}), \tilde{G}_{10;12}(\mathbf{x}) = \tilde{G}_{12;10}(\mathbf{x}) = \frac{A_4 A_5 - A_2 A_6}{9} \zeta_1^*(\mathbf{x}), \\ \tilde{G}_{11;11}(\mathbf{x}) &= \frac{A_1 A_3 - A_6^2}{9} \zeta_1^*(\mathbf{x}), \tilde{G}_{11;12}(\mathbf{x}) = \tilde{G}_{12;11}(\mathbf{x}) = \frac{A_4 A_6 - A_1 A_5}{9} \zeta_1^*(\mathbf{x}), \\ \tilde{G}_{12;12}(\mathbf{x}) &= \frac{A_1 A_2 - A_4^2}{9} \zeta_1^*(\mathbf{x}), \tilde{G}_{13;13}(\mathbf{x}) = \frac{\zeta_1^*(\mathbf{x})}{\gamma}, \tilde{G}_{14;14}(\mathbf{x}) = \frac{\zeta_1^*(\mathbf{x})}{k}, \tilde{G}_{15;15}(\mathbf{x}) = \frac{\zeta_1^*(\mathbf{x})}{h}, \\ \zeta_1^*(\mathbf{x}) &= -\frac{1}{4\pi|\mathbf{x}|}, \zeta_2^*(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, i, j = 1, 2, 3 \end{aligned}$$

and the remaining elements of matrix $\tilde{\mathbf{G}}(\mathbf{x})$ are zero.

6. FUNDAMENTAL SOLUTION OF SYSTEM OF EQUATIONS IN EQUILIBRIUM THEORY

If we substitute $\omega = 0$ into the system of equations (28), we derive the system of equations for the equilibrium theory of micromorphic thermoelastic diffusion with microtemperatures, microconcentrations, and triple porosity as:

$$\begin{aligned}
& \mu \Delta \mathbf{u} + (\lambda' + \mu) \nabla(\nabla \cdot \mathbf{u}) + \tilde{\lambda}_i \nabla \phi_i + \wp \nabla \phi - \mathfrak{S}_1 \nabla \theta - \mathfrak{S}_2 \nabla P = \mathbf{0}, \\
& (k_6 \Delta - k_2) \mathbf{v} + (k_4 + k_5) \nabla(\nabla \cdot \mathbf{v}) - k_3 \nabla \theta = \mathbf{0}, \\
& (h_6 \Delta - h_2) \mathbf{w} + (h_4 + h_5) \nabla(\nabla \cdot \mathbf{w}) - h_3 \nabla P = \mathbf{0}, \\
& -\tilde{\lambda}_1(\nabla \cdot \mathbf{u}) - B_1(\nabla \cdot \mathbf{v}) - B_4(\nabla \cdot \mathbf{w}) + (A_1 \Delta - \zeta_1) \phi_1 + (A_4 \Delta - \zeta_4) \phi_2 + (A_6 \Delta - \zeta_6) \phi_3 \\
& + \delta_1 \phi + \xi_1 \theta + v_1 P = 0, \\
& -\tilde{\lambda}_2(\nabla \cdot \mathbf{u}) - B_2(\nabla \cdot \mathbf{v}) - B_5(\nabla \cdot \mathbf{w}) + (A_4 \Delta - \zeta_4) \phi_1 + (A_2 \Delta - \zeta_2) \phi_2 + (A_5 \Delta - \zeta_5) \phi_3 \\
& + \delta_2 \phi + \xi_2 \theta + v_2 P = 0, \\
& -\tilde{\lambda}_3(\nabla \cdot \mathbf{u}) - B_3(\nabla \cdot \mathbf{v}) - B_6(\nabla \cdot \mathbf{w}) + (A_6 \Delta - \zeta_6) \phi_1 + (A_5 \Delta - \zeta_5) \phi_2 + (A_3 \Delta - \zeta_3) \phi_3 \\
& + \delta_3 \phi + \xi_3 \theta + v_3 P = 0, \\
& -\wp(\nabla \cdot \mathbf{u}) - E_1(\nabla \cdot \mathbf{v}) - E_2(\nabla \cdot \mathbf{w}) + \delta_i \phi_i + (\gamma \Delta - \beta) \phi + n \theta + \xi P = 0, \\
& k_1(\nabla \cdot \mathbf{v}) + k \Delta \theta = 0, \\
& h_1(\nabla \cdot \mathbf{w}) + h \Delta P = 0.
\end{aligned} \tag{66}$$

We introduce the second order matrix differential operators with constant coefficients

$$\mathbf{E}(\mathbf{D}_x) = \left(E_{gz}(\mathbf{D}_x) \right)_{15 \times 15},$$

where matrix $\mathbf{E}(\mathbf{D}_x)$ can be obtained from $\mathbf{F}(\mathbf{D}_x)$ by taking $\omega = 0$.

The system of equations (66) can be represented as

$$\mathbf{E}(\mathbf{D}_x) \mathbf{U}(\mathbf{x}) = \mathbf{0}.$$

DEFINITION 3: The operator $\mathbf{E}(\mathbf{D}_x)$ is an elliptic differential operator iff equation (29) is satisfied.

DEFINITION 4: The fundamental solution of the system of equations (66) is the matrix $\mathbf{G}'(\mathbf{x}) = \left(G'_{gz}(\mathbf{x}) \right)_{15 \times 15}$ satisfying condition

$$\mathbf{E}(\mathbf{D}_x) \mathbf{G}'(\mathbf{x}) = \delta(\mathbf{x}) \mathbf{I}(\mathbf{x}). \tag{67}$$

We consider the system of non-homogeneous equations

$$\mu \Delta \mathbf{u} + (\lambda' + \mu) \nabla(\nabla \cdot \mathbf{u}) - \tilde{\lambda}_i \nabla \phi_i - \wp \nabla \phi = \mathbf{H}', \tag{68}$$

$$(k_6 \Delta - k_2) \mathbf{v} + (k_4 + k_5) \nabla(\nabla \cdot \mathbf{v}) - B_i \nabla \phi_i - E_1 \nabla \phi + k_1 \nabla \theta = \mathbf{V}', \tag{69}$$

$$(h_6 \Delta - h_2) \mathbf{w} + (h_4 + h_5) \nabla(\nabla \cdot \mathbf{w}) - B_{i+3} \nabla \phi_i - E_2 \nabla \phi + h_1 P = \mathbf{W}', \tag{70}$$

$$\tilde{\lambda}_1(\nabla \cdot \mathbf{u}) + (A_1 \Delta - \zeta_1) \phi_1 + (A_4 \Delta - \zeta_4) \phi_2 + (A_6 \Delta - \zeta_6) \phi_3 + \delta_1 \phi = X'_1, \tag{71}$$

$$\tilde{\lambda}_2(\nabla \cdot \mathbf{u}) + (A_4 \Delta - \zeta_4) \phi_1 + (A_2 \Delta - \zeta_2) \phi_2 + (A_5 \Delta - \zeta_5) \phi_3 + \delta_2 \phi = X'_2, \tag{72}$$

$$\tilde{\lambda}_3(\nabla \cdot \mathbf{u}) + (A_6 \Delta - \zeta_6) \phi_1 + (A_5 \Delta - \zeta_5) \phi_2 + (A_3 \Delta - \zeta_3) \phi_3 + \delta_3 \phi = X'_3, \tag{73}$$

$$\wp(\nabla \cdot \mathbf{u}) + \delta_i \phi_i + (\gamma \Delta - \beta) \phi = L', \tag{74}$$

$$-\mathfrak{g}_1(\nabla \cdot \mathbf{u}) - k_3(\nabla \cdot \mathbf{v}) + \xi_i \phi_i + n\phi + k\Delta\theta = Y', \quad (75)$$

$$-\mathfrak{g}_2(\nabla \cdot \mathbf{u}) - h_3(\nabla \cdot \mathbf{w}) + v_i \phi_i + \xi\phi + h\Delta P = Z', \quad (76)$$

where $\mathbf{H}', \mathbf{V}', \mathbf{W}'$ are three-component vector functions on E^3 ; X'_i, L', Y', Z' are scalar functions on E^3 .

The equations (68)-(76) can also be expressed in the following form:

$$\mathbf{E}^{\text{tr}}(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{Q}'(\mathbf{x}), \quad (77)$$

where $\mathbf{E}^{\text{tr}}$ is the transpose of matrix $\mathbf{E}$, $\mathbf{Q}' = (\mathbf{H}', \mathbf{V}', \mathbf{W}', X'_i, L', Y', Z')$, $\mathbf{x} \in E^3$.

Applying operator $(\nabla \cdot)$ to the equations (68)-(70), we obtain

$$\tilde{\lambda}\Delta(\nabla \cdot \mathbf{u}) - \tilde{\lambda}_i\Delta\phi_i - \wp\Delta\phi = \nabla \cdot \mathbf{H}', \quad (78)$$

$$[k_7\Delta - k_2](\nabla \cdot \mathbf{v}) - B_i\Delta\phi_i - E_1\Delta\phi + k_1\Delta\theta = \nabla \cdot \mathbf{V}', \quad (79)$$

$$[h_7\Delta - h_2](\nabla \cdot \mathbf{w}) - B_{i+3}\Delta\phi_i - E_2\Delta\phi + h_1\Delta P = \nabla \cdot \mathbf{W}'. \quad (80)$$

The equations (71)-(74) and (78) may be expressed in the form

$$\mathbf{N}'(\Delta)\mathbf{S}' = \tilde{\mathbf{Q}}', \quad (81)$$

where $\mathbf{S}' = (\nabla \cdot \mathbf{u}, \phi, \phi)$, $\tilde{\mathbf{Q}} = (w'_1, w'_2, w'_3, w'_4, w'_5) = (\nabla \cdot \mathbf{H}', X'_i, L')$ and

$$\hat{\mathbf{N}}(\Delta) = \left(\hat{\mathbf{N}}_{\text{gz}}(\Delta) \right)_{5 \times 5} = \begin{pmatrix} \tilde{\lambda} & -\tilde{\lambda}_1 & -\tilde{\lambda}_2 & -\tilde{\lambda}_3 & -\wp \\ \tilde{\lambda}_1 & A_1\Delta - \zeta_1 & A_4\Delta - \zeta_4 & A_6\Delta - \zeta_6 & \delta_1 \\ \tilde{\lambda}_2 & A_4\Delta - \zeta_4 & A_2\Delta - \zeta_2 & A_5\Delta - \zeta_5 & \delta_2 \\ \tilde{\lambda}_3 & A_6\Delta - \zeta_6 & A_5\Delta - \zeta_5 & A_3\Delta - \zeta_3 & \delta_3 \\ \wp & \delta_1 & \delta_2 & \delta_3 & \gamma\Delta - \beta \end{pmatrix}_{5 \times 5},$$

$$\mathbf{N}'(\Delta) = \left(\mathbf{N}'_{\text{gz}}(\Delta) \right)_{5 \times 5} = \Delta \left(\hat{\mathbf{N}}_{\text{gz}}(\Delta) \right)_{5 \times 5}.$$

Equation (81) can also be expressed in determinant form as:

$$\Gamma_3(\Delta)\mathbf{S}' = \Phi, \quad (82)$$

where $\Phi = (\Phi_1, \dots, \Phi_5)$, $\Phi_j = \frac{1}{\tilde{C}} \sum_{i=1}^5 M_{ij}^* w_i$, $\Gamma_3(\Delta) = \frac{|\mathbf{N}'(\Delta)|}{\tilde{C}}$, $\tilde{C} = \tilde{\lambda} \wp \gamma$ $j = 1, \dots, 5$

and M_{ij}^* is the cofactor of the element N'_{ij} of the matrix $\mathbf{N}'$.

On expanding $\Gamma_3(\Delta)$, we see that

$$\Gamma_3(\Delta) = \Delta \prod_{i=1}^4 (\Delta + \tilde{\mu}_i^2),$$

where $\tilde{\mu}_i^2$, $i = 1, 2, 3, 4$ are the roots of the equation $|\hat{\mathbf{N}}(-m)| = 0$ (with respect to m).

From equation (75), we get $\Delta\theta = \frac{1}{k}[Y' + \vartheta_1(\nabla \cdot \mathbf{u}) + k_3(\nabla \cdot \mathbf{v}) - \xi_i\phi_i - n\phi]$.

Using the above equation in equation (79) and then applying operator $\Gamma_3(\Delta)$ to the resulting equation, we get

$$\Gamma_3(\Delta)(\Delta + \tilde{\mu}_5^2)\nabla \cdot \mathbf{v} = \Phi_6, \tilde{\mu}_5^2 = \frac{k_1k_3 - k_2k}{kk_7}, \quad (83)$$

$$\text{where } \Phi_6 = \frac{1}{kk_7} \left\{ k \left[\left(B_i\Delta + \frac{\xi_i k_1}{k} \right) \Phi_{i+1} + \left(E_1\Delta + \frac{nk_1}{k} \right) \Phi_5 + \Gamma_3(\Delta)\nabla \cdot \mathbf{V}' \right] - k_1[\Gamma_3(\Delta)Y' + \vartheta_1\Phi_1] \right\}.$$

From equation (76), we get $\Delta P = \frac{1}{h}[Z' + \vartheta_2(\nabla \cdot \mathbf{u}) + h_3(\nabla \cdot \mathbf{w}) - v_i\phi_i - \xi\phi]$.

Using the above equation in equation (80) and then applying operator $\Gamma_3(\Delta)$ to the resulting equation, we get

$$\Gamma_3(\Delta)(\Delta + \tilde{\mu}_6^2)\nabla \cdot \mathbf{w} = \Phi_7, \tilde{\mu}_6^2 = \frac{h_1h_3 - h_2h}{hh_7}, \quad (84)$$

where

$$\Phi_7 = \frac{1}{hh_7} \left\{ h \left[\left(B_{i+3}\Delta + \frac{v_i h_1}{h} \right) \Phi_{i+1} + \left(E_2\Delta + \frac{\xi h_1}{k} \right) \Phi_5 + \Gamma_3(\Delta)\nabla \cdot \mathbf{W}' \right] - h_1[\Gamma_3(\Delta)Z' + \vartheta_2\Phi_1] \right\}.$$

Applying operators $\Gamma_3(\Delta)(\Delta + \tilde{\mu}_5^2)$ and $\Gamma_3(\Delta)(\Delta + \tilde{\mu}_6^2)$ to equations (75) and (76) and using equations (83) and (84), we get

$$\Delta^2 \prod_{i=1}^5 (\Delta + \tilde{\mu}_i^2) \theta = \Phi_8, \quad (85)$$

$$\Delta^2 \prod_{i=1}^{4,6} (\Delta + \tilde{\mu}_i^2) P = \Phi_9, \quad (86)$$

where

$$\Phi_8 = \frac{1}{k} [(\Delta + \tilde{\mu}_5^2)[\Gamma_3(\Delta)Y' + \vartheta_1\Phi_1 - \xi_i\Phi_{i+1} - n\Phi_5] + k_3\Phi_6],$$

$$\Phi_9 = \frac{1}{h} [(\Delta + \tilde{\mu}_6^2)[\Gamma_3(\Delta)Z' + \vartheta_2\Phi_1 - v_i\Phi_{i+1} - \xi\Phi_5] + h_3\Phi_7].$$

Applying operators $\Gamma_3(\Delta), \Delta^2 \prod_{i=1}^5 (\Delta + \tilde{\mu}_i^2)$ and $\Delta^2 \prod_{i=1}^{4,6} (\Delta + \tilde{\mu}_i^2)$ to equations (68)-(70) respectively and using (82)-(86), we obtain

$$\Delta\Gamma_3(\Delta)\mathbf{u} = \Phi',$$

$$\Delta^2 \prod_{i=1}^{5,7} (\Delta + \tilde{\mu}_i^2) \mathbf{v} = \Phi'', \tilde{\mu}_7^2 = -\frac{k_2}{k_6} \quad (87)$$

$$\Delta^2 \prod_{i=1}^{4,6,8} (\Delta + \tilde{\mu}_i^2) \mathbf{w} = \Phi''', \tilde{\mu}_8^2 = -\frac{h_2}{h_6}.$$

where

$$\begin{aligned}
\Phi' &= \frac{1}{\mu} \left[\Gamma_3(\Delta) \mathbf{H}' - (\lambda' + \mu) \nabla \Phi_1 + \tilde{\lambda}_i \nabla \Phi_{i+1} + \wp \nabla \Phi_5 \right] \\
\Phi'' &= \frac{1}{k_6} \left[\Delta^2 \prod_{i=1}^5 (\Delta + \tilde{\mu}_i^2) \mathbf{V}' - \Delta(k_4 + k_5) \nabla \Phi_6 + \Delta(\Delta + \tilde{\mu}_5^2) (\mathbf{B}_i \nabla \Phi_{i+1} + \mathbf{E}_1 \nabla \Phi_5) - k_1 \nabla \Phi_8 \right], \\
\Phi''' &= \frac{1}{h_6} \left[\Delta^2 \prod_{i=1}^{4,6} (\Delta + \tilde{\mu}_i^2) \mathbf{W}' - \Delta(h_4 + h_5) \nabla \Phi_7 + \Delta(\Delta + \tilde{\mu}_6^2) (\mathbf{B}_{i+3} \nabla \Phi_{i+1} + \mathbf{E}_2 \nabla \Phi_5) - h_1 \nabla \Phi_9 \right].
\end{aligned}$$

From equations (82) and (85)-(87), we get

$$\Lambda(\Delta) \mathbf{U}(\mathbf{x}) = \hat{\Phi}(\mathbf{x}), \quad (88)$$

where

$$\begin{aligned}
\hat{\Phi}(\mathbf{x}) &= (\Phi', \Phi'', \Phi''', \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_8, \Phi_9) \text{ and } \Lambda(\Delta) = (\Lambda_{gz}(\Delta))_{15 \times 15}, \\
\Lambda_{yy}(\Delta) &= \Delta \Gamma_3(\Delta) = \Delta^2 \prod_{i=1}^4 (\Delta + \tilde{\mu}_i^2), \quad \Lambda_{y+3;y+3}(\Delta) = \Delta^2 \prod_{i=1}^{5,7} (\Delta + \tilde{\mu}_i^2), \\
\Lambda_{y+6;y+6}(\Delta) &= \Delta^2 \prod_{i=1}^{4,6,8} (\Delta + \tilde{\mu}_i^2), \quad \Lambda_{\ell\ell}(\Delta) = \Gamma_3(\Delta) = \Delta \prod_{i=1}^4 (\Delta + \tilde{\mu}_i^2), \\
\Lambda_{14;14}(\Delta) &= \Delta^2 \prod_{i=1}^5 (\Delta + \tilde{\mu}_i^2), \quad \Lambda_{15;15}(\Delta) = \Delta^2 \prod_{i=1}^{4,6} (\Delta + \tilde{\mu}_i^2), \\
\Lambda_{gz}(\Delta) &= 0 \quad y = 1, 2, 3 \quad \ell = 10, \dots, 13 \quad g, z = 1, \dots, 15 \quad g \neq z.
\end{aligned}$$

The expressions for $\Phi', \Phi'', \Phi''', \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_8, \Phi_9$ can be rewritten as

$$\hat{\Phi}(\mathbf{x}) = \mathbf{Z}^{\text{tr}}(\mathbf{D}_x) \mathbf{Q}'(\mathbf{x}), \quad (89)$$

where

$$\begin{aligned}
\mathbf{Z}(\mathbf{D}_x) &= (Z_{g\ell}(\mathbf{D}_x))_{15 \times 15}, \\
Z_{ij}(\mathbf{D}_x) &= \frac{1}{\mu} \Gamma_3(\Delta) \delta_{ij} + m_{11}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, \quad Z_{i;j+3}(\mathbf{D}_x) = m_{12}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, \\
Z_{i;j+6}(\mathbf{D}_x) &= m_{13}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, \quad Z_{i;\hbar+8}(\mathbf{D}_x) = m_{1;\hbar+2}(\Delta) \frac{\partial}{\partial x_i}, \\
Z_{i+3;j}(\mathbf{D}_x) &= 0, \quad Z_{i+3;j+3}(\mathbf{D}_x) = \frac{1}{k_6} \Delta^2 \prod_{i=1}^5 (\Delta + \tilde{\mu}_i^2) \delta_{ij} + m_{22}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, \\
Z_{i+3;j+6}(\mathbf{D}_x) &= Z_{i+3;q}(\mathbf{D}_x) = 0, \quad Z_{i+3;14}(\mathbf{D}_x) = m_{28}(\Delta) \frac{\partial}{\partial x_i}, \quad Z_{i+6;j}(\mathbf{D}_x) = 0, \\
Z_{i+6;j+3}(\mathbf{D}_x) &= 0, \quad Z_{i+6;j+6}(\mathbf{D}_x) = \frac{1}{h_6} \Delta^2 \prod_{i=1}^{4,6} (\Delta + \tilde{\mu}_i^2) \delta_{ij} + m_{33}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j}, \\
Z_{i+6;e}(\mathbf{D}_x) &= 0, \quad Z_{i+6;15}(\mathbf{D}_x) = m_{39}(\Delta) \frac{\partial}{\partial x_i}, \quad Z_{y+8;i}(\mathbf{D}_x) = m_{y+2;1}(\Delta) \frac{\partial}{\partial x_i},
\end{aligned}$$

$$\begin{aligned}
Z_{y+8;i+3}(\mathbf{D}_x) &= m_{y+2;2}(\Delta) \frac{\partial}{\partial \mathbf{x}_i}, Z_{y+8;i+6}(\mathbf{D}_x) = m_{y+2;3}(\Delta) \frac{\partial}{\partial \mathbf{x}_i}, \\
Z_{y+8;\hbar+8}(\mathbf{D}_x) &= m_{y+2;\hbar+2}(\Delta), Z_{14;i}(\mathbf{D}_x) = Z_{14;i+6}(\mathbf{D}_x) = Z_{14;q}(\mathbf{D}_x) = 0, Z_{14;i+3}(\mathbf{D}_x) = m_{82}(\Delta) \frac{\partial}{\partial \mathbf{x}_i}, \\
Z_{14;14}(\mathbf{D}_x) &= m_{88}(\Delta), Z_{15;i}(\mathbf{D}_x) = Z_{15;i+3}(\mathbf{D}_x) = Z_{15;e}(\mathbf{D}_x) = 0, Z_{15;i+6}(\mathbf{D}_x) = m_{93}(\Delta) \frac{\partial}{\partial \mathbf{x}_i}, \\
Z_{15;15}(\mathbf{D}_x) &= m_{99}(\Delta), m_{11}(\Delta) = -\frac{1}{\mu \tilde{C}} \left\{ (\lambda' + \mu) M_{11}^*(\Delta) - \tilde{\lambda}_i M_{1,i+1}^*(\Delta) - \wp M_{15}^*(\Delta) \right\}, \\
m_{12}(\Delta) &= \frac{\Delta + \tilde{\mu}_7^2}{kk_7 \tilde{C}} [-k_1 \wp_1 M_{11}^*(\Delta) + (B_1 k \Delta + k_1 \xi_i) M_{1,i+1}^*(\Delta) + (E_1 k \Delta + k_1 n) M_{15}^*(\Delta)], \\
m_{13}(\Delta) &= \frac{\Delta + \tilde{\mu}_8^2}{hh_7 \tilde{C}} [-h_1 \wp_2 M_{11}^*(\Delta) + (B_{i+3} h \Delta + h_1 v_i) M_{1,i+1}^*(\Delta) + (E_2 h \Delta + h_1 \xi) M_{15}^*(\Delta)], \\
m_{1;y+2}(\Delta) &= \frac{M_{1y}^*(\Delta)}{\tilde{C}}, m_{18}(\Delta) = \frac{1}{kk_7 \tilde{C}} \left[(k_7 \Delta - k_2) (\wp_1 M_{11}^*(\Delta) - \xi_i M_{1,i+1}^*(\Delta) - n M_{15}^*(\Delta)) \right. \\
&\quad \left. + k_3 \Delta (B_i M_{1,i+1}^*(\Delta) + E_1 M_{15}^*(\Delta)) \right], \\
m_{19}(\Delta) &= \frac{1}{hh_7 \tilde{C}} \left[(h_7 \Delta - h_2) (\wp_2 M_{11}^*(\Delta) - v_i M_{1,i+1}^*(\Delta) - \xi M_{15}^*(\Delta)) \right. \\
&\quad \left. + h_3 \Delta (B_{i+3} M_{1,i+1}^*(\Delta) + E_2 M_{15}^*(\Delta)) \right], \\
m_{2z}(\Delta) &= 0, m_{22}(\Delta) = -\frac{\Gamma_3(\Delta)}{kk_6 k_7} [(k_4 + k_5) k \Delta + k_1 k_3], m_{28}(\Delta) = \frac{k_3}{kk_7} \Gamma_3(\Delta), \\
m_{3\lambda}(\Delta) &= 0, m_{33}(\Delta) = -\frac{\Gamma_3(\Delta)}{hh_6 h_7} [(h_4 + h_5) h \Delta + h_1 h_3], m_{39}(\Delta) = \frac{h_3}{hh_7} \Gamma_3(\Delta), \\
m_{y+2;1}(\Delta) &= -\frac{1}{\mu \tilde{C}} \left\{ (\lambda' + \mu) M_{y1}^*(\Delta) - \tilde{\lambda}_i M_{y,i+1}^*(\Delta) - \wp M_{y5}^*(\Delta) \right\}, \\
m_{y+2;2}(\Delta) &= \frac{\Delta + \tilde{\mu}_7^2}{kk_7 \tilde{C}} [-k_1 \wp_1 M_{y1}^*(\Delta) + (B_i k \Delta + k_1 \xi_i) M_{y,i+1}^*(\Delta) + (E_1 k \Delta + k_1 n) M_{y5}^*(\Delta)], \\
m_{y+2;3}(\Delta) &= \frac{\Delta + \tilde{\mu}_8^2}{hh_7 \tilde{C}} [-h_1 \wp_2 M_{y1}^*(\Delta) + (B_{i+3} h \Delta + h_1 v_i) M_{y,i+1}^*(\Delta) + (E_2 h \Delta + h_1 \xi) M_{y5}^*(\Delta)], \\
m_{y+2;r+2}(\Delta) &= \frac{M_{yr}^*(\Delta)}{\tilde{C}}, m_{y+2;8}(\Delta) = \frac{1}{kk_7 \tilde{C}} \left[(k_7 \Delta - k_2) [\wp_1 M_{y1}^*(\Delta) - \xi_i M_{y,i+1}^*(\Delta) - n M_{y5}^*(\Delta)] + \right. \\
&\quad \left. k_3 \Delta (B_i M_{y,i+1}^*(\Delta) + E_1 M_{y5}^*(\Delta)) \right], \\
m_{y+2;9}(\Delta) &= \frac{1}{hh_7 \tilde{C}} \left[(h_7 \Delta - h_2) [\wp_2 M_{y1}^*(\Delta) - v_i M_{y,i+1}^*(\Delta) - \xi M_{y5}^*(\Delta)] + \right. \\
&\quad \left. h_3 \Delta (B_{i+3} M_{y,i+1}^*(\Delta) + E_2 M_{y5}^*(\Delta)) \right], \\
m_{8z}(\Delta) &= m_{9\lambda}(\Delta) = 0, m_{82}(\Delta) = -\frac{k_1 \Gamma_3(\Delta) (\Delta + \tilde{\mu}_7^2)}{kk_7}, m_{88}(\Delta) = \frac{\Gamma_3(\Delta) (k_7 \Delta - k_2)}{kk_7}, \\
m_{93}(\Delta) &= -\frac{h_1 \Gamma_3(\Delta) (\Delta + \tilde{\mu}_8^2)}{hh_7}, m_{99}(\Delta) = \frac{\Gamma_3(\Delta) (h_7 \Delta - h_2)}{hh_7} \quad i, j = 1, 2, 3 \\
\hbar &= 2, \dots, 7 \quad q = 10, \dots, 13, 15 \quad e = 10, \dots, 14 \quad y, r = 2, \dots, 5 \quad z = 1, 3, \dots, 7, 9 \quad \tilde{\lambda} = 1, 2, 4, \dots, 8.
\end{aligned}$$

From equations (77), (88) and (89), we get $\mathbf{E}(\mathbf{D}_x)\mathbf{R}(\mathbf{D}_x) = \Lambda(\Delta)$. (90)

Let

$$\mathbf{Y}'(\mathbf{x}) = (\mathbf{Y}'_{ij}(\mathbf{x}))_{15 \times 15}, \mathbf{Y}'_{zz}(\mathbf{x}) = r'_{11}\zeta_1^*(\mathbf{x}) + r'_{12}\zeta_2^*(\mathbf{x}) + \sum_{g=1}^4 r'_{1;g+2}\widehat{\zeta}_g(\mathbf{x}),$$

$$\mathbf{Y}'_{z+3;z+3}(\mathbf{x}) = r'_{21}\zeta_1^*(\mathbf{x}) + r'_{22}\zeta_2^*(\mathbf{x}) + \sum_{g=1}^{5,7} r'_{2;g+2}\widehat{\zeta}_g(\mathbf{x}),$$

$$\mathbf{Y}'_{z+6;z+6}(\mathbf{x}) = r'_{31}\zeta_1^*(\mathbf{x}) + r'_{32}\zeta_2^*(\mathbf{x}) + \sum_{g=1}^{4,6,8} r'_{3;g+2}\widehat{\zeta}_g(\mathbf{x}),$$

$$\mathbf{Y}'_{\ell\ell}(\mathbf{x}) = r'_{41}\zeta_1^*(\mathbf{x}) + \sum_{g=1}^4 r'_{4;g+1}\widehat{\zeta}_g(\mathbf{x}),$$

$$\mathbf{Y}'_{14;14}(\mathbf{x}) = r'_{51}\zeta_1^*(\mathbf{x}) + r'_{52}\zeta_2^*(\mathbf{x}) + \sum_{g=1}^5 r'_{5;g+2}\widehat{\zeta}_g(\mathbf{x}),$$

$$\mathbf{Y}'_{15;15}(\mathbf{x}) = r'_{61}\zeta_1^*(\mathbf{x}) + r'_{62}\zeta_2^*(\mathbf{x}) + \sum_{g=1}^{4,6} r'_{6;g+2}\widehat{\zeta}_g(\mathbf{x}),$$

$$\mathbf{Y}'_{ij}(\mathbf{x}) = 0 \quad z = 1, 2, 3 \quad i, j = 1, \dots, 15 \quad i \neq j \quad \ell = 10, \dots, 13$$

where

$$\widehat{\zeta}_g(\mathbf{x}) = -\frac{e^{i\tilde{\mu}_g|\mathbf{x}|}}{4\pi|\mathbf{x}|}, g = 1, \dots, 8,$$

$$r'_{11} = -\sum_{g=1}^4 \left(\prod_{j=1, j \neq g}^4 \tilde{\mu}_j^2 \right) \prod_{i=1}^4 \tilde{\mu}_i^{-4}, r'_{12} = \prod_{i=1}^4 \tilde{\mu}_i^{-2}, r'_{1;y+2} = \tilde{\mu}_y^{-4} \prod_{i=1, i \neq y}^4 (\tilde{\mu}_i^2 - \tilde{\mu}_y^2)^{-1},$$

$$r'_{21} = -\sum_{g=1}^{5,7} \left(\prod_{j=1, j \neq g}^{5,7} \tilde{\mu}_j^2 \right) \prod_{i=1}^{5,7} \tilde{\mu}_i^{-4}, r'_{22} = \prod_{i=1}^{5,7} \tilde{\mu}_i^{-2}, r'_{2;\ell+2} = \tilde{\mu}_\ell^{-4} \prod_{i=1, i \neq \ell}^{5,7} (\tilde{\mu}_i^2 - \tilde{\mu}_\ell^2)^{-1},$$

$$r'_{31} = -\sum_{g=1}^{4,6,8} \left(\prod_{j=1, j \neq g}^{4,6,8} \tilde{\mu}_j^2 \right) \prod_{i=1}^{4,6,8} \tilde{\mu}_i^{-4}, r'_{32} = \prod_{i=1}^{4,6,8} \tilde{\mu}_i^{-2}, r'_{3;z+2} = \tilde{\mu}_z^{-4} \prod_{i=1, i \neq z}^{4,6,8} (\tilde{\mu}_i^2 - \tilde{\mu}_z^2)^{-1},$$

$$r'_{41} = \prod_{i=1}^4 \tilde{\mu}_i^{-2}, r'_{4;y+1} = -\tilde{\mu}_y^{-2} \prod_{i=1, i \neq y}^4 (\tilde{\mu}_i^2 - \tilde{\mu}_y^2)^{-1},$$

$$r'_{51} = -\sum_{g=1}^5 \left(\prod_{j=1, j \neq g}^5 \tilde{\mu}_j^2 \right) \prod_{i=1}^5 \tilde{\mu}_i^{-4}, r'_{52} = \prod_{i=1}^5 \tilde{\mu}_i^{-2}, r'_{5;h+2} = \tilde{\mu}_h^{-4} \prod_{i=1, i \neq h}^5 (\tilde{\mu}_i^2 - \tilde{\mu}_h^2)^{-1},$$

$$r'_{61} = -\sum_{g=1}^{4,6} \left(\prod_{j=1, j \neq g}^{4,6} \tilde{\mu}_j^2 \right) \prod_{i=1}^{4,6} \tilde{\mu}_i^{-4}, r'_{62} = \prod_{i=1}^{4,6} \tilde{\mu}_i^{-2}, r'_{6;\lambda+2} = \tilde{\mu}_\lambda^{-4} \prod_{i=1, i \neq \lambda}^{4,6} (\tilde{\mu}_i^2 - \tilde{\mu}_\lambda^2)^{-1},$$

$$y = 1, \dots, 4 \quad \ell = 1, \dots, 5, 7 \quad z = 1, \dots, 6, 8 \quad h = 1, \dots, 5 \quad \lambda = 1, \dots, 4, 6.$$

LEMMA 2: The matrix $\mathbf{Y}'$ is the fundamental matrix of the operator $\Lambda(\Delta)$ i.e.

$$\Lambda(\Delta)\mathbf{Y}'(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}). \quad (91)$$

PROOF: To prove the lemma, it is sufficient to prove that

$$\begin{aligned}
\Delta\Gamma_3(\Delta)Y'_{11}(\mathbf{x}) &= \delta(\mathbf{x}), \\
\Delta\Gamma_3(\Delta)(\Delta + \tilde{\mu}_5^2)(\Delta + \tilde{\mu}_7^2)Y'_{44}(\mathbf{x}) &= \delta(\mathbf{x}), \\
\Delta\Gamma_3(\Delta)(\Delta + \tilde{\mu}_6^2)(\Delta + \tilde{\mu}_8^2)Y'_{77}(\mathbf{x}) &= \delta(\mathbf{x}), \\
\Gamma_3(\Delta)Y'_{10;10}(\mathbf{x}) &= \delta(\mathbf{x}), \\
\Delta\Gamma_3(\Delta)(\Delta + \tilde{\mu}_5^2)Y'_{14;14}(\mathbf{x}) &= \delta(\mathbf{x}), \\
\Delta\Gamma_3(\Delta)(\Delta + \tilde{\mu}_6^2)Y'_{15;15}(\mathbf{x}) &= \delta(\mathbf{x}).
\end{aligned} \tag{92}$$

It is much easier to prove the system of equations (92). It has been left for the reader.

We introduce the matrix $\mathbf{G}'(\mathbf{x}) = \mathbf{Z}(\mathbf{D}_x)\mathbf{Y}'(\mathbf{x})$, (93)

From equations (90), (91) and (93), we obtain $\mathbf{E}(\mathbf{D}_x)\mathbf{G}'(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x})$.

Hence, $\mathbf{G}(\mathbf{x})$ is a solution to equation (67).

Theorem 4: If the condition (29) is satisfied, then the matrix $\mathbf{G}'(\mathbf{x})$ defined by the equation (93) is the fundamental solution of the system of equations (66).

7. CONCLUSIONS

This manuscript presents the development of a linear theory for micromorphic thermoelastic diffusion materials, incorporating advanced features such as microtemperatures, microconcentrations, and triple porosity. Fundamental solutions have been derived for two key systems of equations: one for steady oscillations (equation 28), expressed in terms of elementary functions, and another for equilibrium conditions (equation 66).

These solutions enable:

1. Comprehensive analysis of three-dimensional boundary value problems within this theoretical framework, utilizing the potential method.
2. Development of integral representation expressions for regular solutions of the governing equations.
3. Numerical resolution of boundary value problems through the boundary element method.
4. Construction of Green's functions for specific cases in 3D domains.

By incorporating triple porosity, microtemperatures, and microconcentrations, the proposed theory offers a deeper understanding of thermoelastic behavior in complex materials. This holistic approach equips researchers and engineers with tools to model and analyze materials with greater precision. The theoretical advancements provide practical insights for designing and optimizing materials, facilitating their application in diverse engineering and scientific fields.

In conclusion, the contributions of this work establish a robust foundation for advancing the field of thermoelasticity, offering a more comprehensive theoretical framework and practical methodologies for analyzing materials with intricate thermoelastic properties.

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