Fundamental Solutions in the Theory of Micromorphic Thermoelastic Diffusion for Triple Porosity Materials with Microtemperature and Microconcentration Effects

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Received: 12 January 2025; revised: 2 March 2025; accepted: 4 March 2025; published online: 26 July 2025

Abstract: The main goal of this article is to derive the basic governing equations for the linearized theory of a micromorphic thermoelastic diffusion medium. This encompasses the analysis of microtemperatures, microconcentrations, and triple porosity. Moreover, the objective is to establish the fundamental solution for these equations in situations involving steady oscillations and equilibrium, presented in elementary functions.

Key words: thermoelastic diffusion, triple porosity, steady oscillations

I. Introduction

Eringen [1–5] introduced continuum theories with microstructure, which Grot [6] extended to thermodynamics of elastic bodies with varying microelement temperatures by modifying the Clausius-Duhem inequality and incorporating energy balance laws for microtemperatures. Iesan [7, 8] advanced the theory by formulating field equations for microstretch thermoelastic bodies and deriving a linear theory incorporating additional mechanical degrees of freedom. Aouadi et al. [9] developed nonlinear and linear theories for thermoelastic diffusion materials with microtemperatures and microconcentrations, proving well-posedness and studying solution behavior. Chiril and Marin [10] and Kansal [11] further refined constitutive relations and field equations for materials with microstructure, emphasizing microtemperatures and microconcentrations. Triple porosity, characterized by macro, meso, and micro levels, is pivotal in elasticity and thermoelasticity, with significant contributions by Svanadze [12], Straughan [13], and Kansal [14] in deriving governing equations. Fundamental solutions, crucial for solving boundary value problems, have been extensively explored in elasticity and thermoelasticity theories, leveraging elementary functions for practical and numerical applications [11, 12, 14–17]. This paper develops constitutive relations and field equations for anisotropic micromorphic thermoelastic diffusion materials with microtemperatures, microconcentrations, and triple porosity. It reduces the anisotropic equations to isotropic forms, derives fundamental solutions for steady oscillations using elementary functions, and constructs solutions for equilibrium conditions.

II. Basic Equations

We consider a scenario where, at time t_0, the body occupies the bounded regular region B within three-dimensional space. Our focus is specifically on the linear theory of elastic bodies. Following Iesan [7], Aouadi et al. [9] and Kansal [14], the dynamical equations are:

- Balance of linear momentum:

$$t_{ji,j} + \rho f_i = \rho \ddot{u}_i \;, \tag{1}$$

- Balance of first stress moments:

$$m_{gij,g} + t_{ji} - s_{ji} + \rho l_{ij} = \rho \dot{\sigma}_{ij} \,, \tag{2}$$

- Balance for energy:

$$\begin{aligned} \rho \dot{U} = t_{ij}\dot{u}_{j,i} + (s_{ij} - t_{ij})\dot{\phi}_{ji} + \\ + m_{gij}\dot{\phi}_{ij,g} + q_{j,j} + \Omega_{ij}\dot{\nu}_{i,j} - p_i \dot{\nu}_i \,, \end{aligned} \tag{3}$$

- Balance of first moment of energy:

$$\rho \dot{\varepsilon}_i = Q_{ji,j} + q_i - \varpi_i \,, \tag{4}$$

- Balance of first moment of mass diffusion:

$$\rho \dot{\chi}_i = \xi_{ji,j} + \eta_i - \upsilon_i \,, \tag{5}$$

where $p_i,\ i = 1,\, 2,\, 3$ satisfy the relation:

$$\begin{aligned} \Omega_{1j,j} + p_1 + \rho \Lambda_1 = \rho \kappa_1 \ddot{\nu}_1 \,, \\ \Omega_{2j,j} + p_2 + \rho \Lambda_2 = \rho \kappa_2 \ddot{\nu}_2 \,, \\ \Omega_{3j,j} + p_3 + \rho \Lambda_3 = \rho \kappa_3 \ddot{\nu}_3 \,. \end{aligned} \tag{6}$$

Here t_{ij} are the components of stress tensor, u_i are the components of displacement vector $\mathbf{u}$, ρ is the density, f_i is the body force, m_{gij} are components of first stress moment tensor, s_{ij} are the components of microstress tensor, σ_{ij} are the components of inertial spin tensor per unit mass, l_{ij} is the first body moment density, U is the internal energy density, ε_i, χ_i are the first moments of energy vector and mass diffusion, respectively, ϕ_{ij} are the components of microdeformation tensor, κ_i are the coefficients of equilibrated inertia, ν_i are the volume fraction fields corresponding to macro-, meso-, micro-pores, respectively, Ω_{ij} are the components of equilibrated stress vectors corresponding to ν_i, Λ_i are extrinsic equilibrated body forces per unit mass associated to macro-, meso-, micro-pores, respectively, q_i, η_i are the components of heat flux and mass flux vectors, Q_{ij}, ξ_{ij} are the first heat flux and mass diffusion moment tensors, ϖ_i, υ_i are the microheat flux and micromass flux averages.

Following Aoaudi [9], the local form of the principle of entropy is given by the following expression:

$$\rho \dot{S} - \left(\frac{q_j}{T} + \frac{Q_{ji}T_i}{T} \right)_{,j} + \left(\frac{P\eta_j}{T} + \frac{P\xi_{ji}T_i}{T} \right)_{,j} \geq 0 \,, \tag{7}$$

where S, P are the entropy and chemical potential per unit mass, respectively, T is the absolute temperature, and T_i is the microtemperature vector. The local form of the mass concentration law is:

$$\eta_{i,i} = \dot{C} \,, \tag{8}$$

where C is the concentration of the diffusion material in the elastic body. For each micro element, the mass conservation law becomes:

$$\dot{C} = (\eta_j + C_i \xi_{ji})_{,j} \,. \tag{9}$$

The spin inertia is given by:

$$\sigma_{ij} = \dot{\hat{n}}_{gj}\, \dot{\phi}_{iy}\, \dot{\phi}_{yg} \,, \tag{10}$$

where $\dot{\hat{n}}_{gj}$ is the microinertia tensor.

Eringen [18] introduced a special kind of micromorphic solids called microstretch solids. In this case, for all motions, we have:

$$\begin{aligned} \phi_{ij} = \phi\, \delta_{ij},\, m_{gij} = \frac{1}{3} \Re_g\, \delta_{ij}, \\ l_{ij} = \frac{1}{3}\, \tau_1 \delta_{ij},\, \dot{\hat{n}}_{ij} = \frac{1}{3} \tau_2\, \delta_{ij}, \end{aligned} \tag{11}$$

where ϕ is the dilatation function, $\Re_g$ is the microstress vector, τ_1 is the generalized external body load, and τ_2 is a given constant.

Eqs. (2) and (3) with the help of Eqs. (4), (5), (9)–(11) and inequality (7) become:

$$\Re_{g,g} - s + \rho \tau_1 = \rho \tau_2 \ddot{\phi} \,, \tag{12}$$

$$\begin{aligned} \rho\, [\dot{S}T - \dot{U} - T_i \dot{\varepsilon}_i - C_i \dot{\chi}_i] + t_{ij}\dot{e}_{ij} + \Re_g \dot{\phi}_{,g} + s\dot{\phi} + \Omega_{ij}\dot{\nu}_{i,j} - p_i \dot{\nu}_i + \frac{1}{T} q_j T_{,j} + \frac{1}{T} T_{,j} Q_{ji} T_i - Q_{ji} T_{i,j} + P\dot{C} + \\ + (q_i - \varpi_i) T_i - P C_i \xi_{ji,j} - P C_{i,j} \xi_{ji} + P_{,j} \eta_j - \frac{1}{T} P \eta_j T_{,j} + T \left(\frac{P}{T} T_i \xi_{ji} \right)_{,j} + \xi_{ji,j} C_i + (\eta_i - \upsilon_i) C_i \geq 0 \,. \end{aligned} \tag{13}$$

where $s = s_{ii} - t_{ii}$ is the intrinsic body load, and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are the components of strain tensor. Introducing the function Γ by $\Gamma = U + T_i \varepsilon_i + C_i \chi_i - TS$, the inequality (13) in the context of linear theory can be expressed as:

$$\begin{aligned} -\rho\, [\dot{\Gamma} + S\dot{T} - \dot{T}_i \varepsilon_i - \dot{C}_i \chi_i] + t_{ij}\dot{e}_{ij} + \Re_g \dot{\phi}_{,g} + s\dot{\phi} + \Omega_{ij}\dot{\nu}_{i,j} - p_i \dot{\nu}_i + \frac{1}{T} q_j T_{,j} - Q_{ji} T_{i,j} + P\dot{C} + \\ + (q_i - \varpi_i) T_i + P_{,j}\, \eta_j + \xi_{ji,j} C_i + (\eta_i - \upsilon_i) C_i \geq 0 \,. \end{aligned} \tag{14}$$

The function Γ can be written in terms of independent variables $e_{ij}, \phi, \phi_{,i}, \nu_i, \nu_{i,j}, T, T_{,i}, T_i, T_{i,j}, C, C_{,i}, C_i$ and $C_{i,j}$. Therefore, we obtain:

$$\begin{aligned}\dot{\Gamma} = \frac{\partial\Gamma}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial\Gamma}{\partial\nu_i}\dot{\nu}_i + \frac{\partial\Gamma}{\partial\nu_{i,j}}\dot{\nu}_{i,j} + \frac{\partial\Gamma}{\partial\phi}\dot{\phi} + \frac{\partial\Gamma}{\partial\phi_{,i}}\dot{\phi}_{,i} + \frac{\partial\Gamma}{\partial T}\dot{T} + \frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i} + \frac{\partial\Gamma}{\partial T_i}\dot{T}_i + \\ + \frac{\partial\Gamma}{\partial T_{i,j}}\dot{T}_{i,j} + \frac{\partial\Gamma}{\partial C}\dot{C} + \frac{\partial\Gamma}{\partial C_{,i}}\dot{C}_{,i} + \frac{\partial\Gamma}{\partial C_i}\dot{C}_i + \frac{\partial\Gamma}{\partial C_{i,j}}\dot{C}_{i,j}\ .\end{aligned} \tag{15}$$

Inequality (14) with the help of Eq. (15) becomes:

$$\begin{aligned}&\left[t_{ij} - \rho\frac{\partial\Gamma}{\partial e_{ij}}\right]\dot{e}_{ij} + \left[\Omega_{ij} - \rho\frac{\partial\Gamma}{\partial\nu_{i,j}}\right]\dot{\nu}_{i,j} - \left[p_i + \rho\frac{\partial\Gamma}{\partial\nu_i}\right]\dot{\nu}_i + \left[s - \rho\frac{\partial\Gamma}{\partial\phi}\right]\dot{\phi} + \left[\Re_g - \rho\frac{\partial\Gamma}{\partial\phi_{,g}}\right]\dot{\phi}_{,g} + \\ &+\rho\left[\varepsilon_i - \frac{\partial\Gamma}{\partial T_i}\right]\dot{T}_i + \rho\left[\chi_i - \frac{\partial\Gamma}{\partial C_i}\right]\dot{C}_i - \rho\left[S + \frac{\partial\Gamma}{\partial T}\right]\dot{T} + \left[P - \rho\frac{\partial\Gamma}{\partial C}\right]\dot{C} - \rho\frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i} - \rho\frac{\partial\Gamma}{\partial T_{i,j}}\dot{T}_{i,j} + \\ &-\rho\frac{\partial\Gamma}{\partial C_{,i}}\dot{C}_{,i} - \rho\frac{\partial\Gamma}{\partial C_{i,j}}\dot{C}_{i,j} + \frac{1}{T}q_jT_{,j} - Q_{ji}T_{i,j} + (q_i - \varpi_i)T_i + P_{,j}\eta_j + \xi_{ji,j}C_i + (\eta_i - \upsilon_i)C_i \geq 0\ .\end{aligned}$$

The inequality must hold for all rates $\dot{e}_{ij}, \dot{\phi}, \dot{\phi}_{,i}, \dot{\nu}_i, \dot{\nu}_{i,j}, \dot{T}, \dot{T}_{,i}, \dot{T}_i, \dot{T}_{i,j}, \dot{C}, \dot{C}_{,i}, \dot{C}_i$ and $\dot{C}_{i,j}$. Therefore, the coefficients of the aforementioned variables must equate to zero, indicating:

$$\begin{aligned}t_{ij} = \rho\frac{\partial\Gamma}{\partial e_{ij}},\ \Omega_{ij} = \rho\frac{\partial\Gamma}{\partial\nu_{i,j}},\ p_i = -\rho\frac{\partial\Gamma}{\partial\nu_i},\ s = \rho\frac{\partial\Gamma}{\partial\phi},\ \Re_g = \rho\frac{\partial\Gamma}{\partial\phi_{,g}},\ \varepsilon_i = \frac{\partial\Gamma}{\partial T_i},\ \chi_i = \frac{\partial\Gamma}{\partial C_i}\ , \\ S = -\frac{\partial\Gamma}{\partial T},\ P = \rho\frac{\partial\Gamma}{\partial C},\ \frac{\partial\Gamma}{\partial T_{,i}} = 0\ ,\ \frac{\partial\Gamma}{\partial T_{i,j}} = 0,\ \frac{\partial\Gamma}{\partial C_{,i}} = 0,\ \frac{\partial\Gamma}{\partial C_{i,j}} = 0\ , \\ q_jT_{,j} - TQ_{ji}T_{i,j} + T(q_i - \varpi_i)T_i + TP_{,j}\eta_j + T\xi_{ji,j}C_i + T(\eta_i - \upsilon_i)C_i \geq 0\ .\end{aligned} \tag{16}$$

Let us introduce the notations:

$$\boldsymbol{\phi} = \boldsymbol{\nu} - \boldsymbol{\nu}_0,\ \theta = T - T_0,$$

where $\boldsymbol{\phi} = (\phi_1,\ \phi_2,\ \phi_3)$, T_0 is the reference temperature of the body chosen such that $\left|\frac{\theta}{T_0}\right| << 1$, $\boldsymbol{\nu}_0$ are the volume fraction fields in reference configuration. In the linear theory of materials possessing a centre of symmetry, we can take Γ in the form:

$$\begin{aligned}2\rho\Gamma = c_{ijgy}e_{ij}e_{gy} + 2a_{ij}e_{ij}\theta + 2b_{ij}e_{ij}C + 2c_{ij}e_{ij}\phi_1 + 2d_{ij}e_{ij}\phi_2 + 2f_{ij}e_{ij}\phi_3 + 2v_{ij}e_{ij}\phi - 2l_i\phi_i\theta - 2n_i\phi_iC + \\ -\frac{\rho C_e\theta^2}{T_0} - 2a\theta C + bC^2 + \alpha_i\phi_i^2 + 2\alpha_4\phi_1\phi_2 + 2\alpha_5\phi_2\phi_3 + 2\alpha_6\phi_3\phi_1 + A_{ij}\phi_{1,i}\phi_{1,j} + B_{ij}\phi_{2,i}\phi_{2,j} + C_{ij}\phi_{3,i}\phi_{3,j} + \\ +2D_{ij}\phi_{1,i}\phi_{2,j} + 2E_{ij}\phi_{2,i}\phi_{3,j} + 2S_{ij}\phi_{3,i}\phi_{1,j} - \alpha_{ij}T_iT_j - \beta_{ij}C_iC_j - 2\gamma_{ij}T_iC_j - 2P_{ij}\phi_{1,j}T_i - 2U_{ij}\phi_{2,j}T_i - 2\lambda\!\!\!^{-}_{ij}\phi_{3,j}T_i + \\ -2M_{ij}\phi_{1,j}C_i - 2T_{ij}\phi_{2,j}C_i - 2\chi_{ij}\phi_{3,j}C_i + \alpha\phi^2 - 2c\theta\phi - 2dC\phi - 2s_i\phi_i\phi + V_{ij}\phi_{,i}\phi_{,j} - 2W_{ij}\phi_{,j}T_i - 2X_{ij}\phi_{,j}C_i.\end{aligned} \tag{17}$$

Using the above Eq. (17) in the system of Eqs. (16), the following consecutive equations are obtained:

$$\begin{aligned}t_{ij} &= c_{ijgy}e_{gy} + c_{ij}\phi_1 + d_{ij}\phi_2 + f_{ij}\phi_3 + v_{ij}\phi + a_{ij}\theta + b_{ij}C\ , \\ \Omega_{1j} &= A_{ij}\phi_{1,i} + D_{ij}\phi_{2,i} + S_{ij}\phi_{3,i} - P_{ij}T_i - M_{ij}C_i\ , \\ \Omega_{2j} &= D_{ij}\phi_{1,i} + B_{ij}\phi_{2,i} + E_{ij}\phi_{3,i} - U_{ij}T_i - T_{ij}C_i\ , \\ \Omega_{3j} &= S_{ij}\phi_{1,i} + E_{ij}\phi_{2,i} + C_{ij}\phi_{3,i} - \lambda\!\!\!^{-}_{ij}T_i - \chi_{ij}C_i\ , \\ p_1 &= -c_{ij}e_{ij} - \alpha_1\phi_1 - \alpha_4\phi_2 - \alpha_6\phi_3 + s_1\phi + l_1\theta + n_1C\ , \\ p_2 &= -d_{ij}e_{ij} - \alpha_4\phi_1 - \alpha_2\phi_2 - \alpha_5\phi_3 + s_2\phi + l_2\theta + n_2C\ , \\ p_3 &= -f_{ij}e_{ij} - \alpha_6\phi_1 - \alpha_5\phi_2 - \alpha_3\phi_3 + s_3\phi + l_3\theta + n_3C\ , \\ \rho\varepsilon_i &= -P_{ij}\phi_{1,j} - U_{ij}\phi_{2,j} - \lambda\!\!\!^{-}_{ij}\phi_{3,j} - W_{ij}\phi_{,j} - \alpha_{ij}T_j - \gamma_{ij}C_j\ , \\ \rho\chi_i &= -M_{ij}\phi_{1,j} - T_{ij}\phi_{2,j} - \chi_{ij}\phi_{3,j} - X_{ij}\phi_{,j} - \gamma_{ji}T_j - \beta_{ij}C_j\ , \\ \rho S &= -a_{ij}e_{ij} + l_i\phi_i + c\phi + \frac{\rho C_e\theta}{T_0} + aC,\ P = b_{ij}e_{ij} - n_i\phi_i - d\phi - a\theta + bC\ , \\ s &= v_{ij}e_{ij} - s_i\phi_i + \alpha\phi - c\theta - dC,\ \Re_g = V_{ig}\phi_{,i} - W_{ig}T_i - X_{ig}C_i\ .\end{aligned} \tag{18}$$

The linear expressions for q_i, Q_{ij}, ϖ_i, η_i, ξ_{ij}, υ_i are:

$$\begin{aligned} q_i = k_{ij}\theta_{,j} + \kappa_{ij}T_j,\, Q_{ij} = -m_{ijyg}T_{g,y},\, \varpi_i = (k_{ij} - K_{ij})\theta_{,j} + (\kappa_{ij} - L_{ij})T_j\ , \\ \eta_i = h_{ij}P_{,j} + m_{ij}C_j,\, \xi_{ij} = -n_{ijyg}C_{g,y},\, \upsilon_i = (h_{ij} - H_{ij})P_{,j} + (m_{ij} - O_{ij})C_j\ . \end{aligned} \tag{19}$$

The lineralized form of inequality (7) is:

$$\rho T_0 \dot{S} = q_{i,i}\ . \tag{20}$$

Considering Eqs. (18) and (19), Eqs. (1), (4)–(6), (8), (12), and (20) can be reformulated as:

$$\begin{aligned} c_{ijgy}e_{gy,j} + c_{ij}\phi_{1,j} + d_{ij}\phi_{2,j} + f_{ij}\phi_{3,j} + \upsilon_{ij}\phi_{,j} + a_{ij}\theta_{,j} + b_{ij}C_{,j} + \rho f_i = \rho\ddot{u}_i\ , \\ m_{jiyg}T_{g,yj} - \alpha_{ij}\dot{T}_j - \gamma_{ij}\dot{C}_j - P_{ij}\dot{\phi}_{1,j} - U_{ij}\dot{\phi}_{2,j} - \lambda\!\!\!^{-}_{ij}\dot{\phi}_{3,j} - W_{ij}\dot{\phi}_{,j} = K_{ij}\theta_{,j} + L_{ij}T_j\ , \\ n_{jiyg}C_{g,yj} - \gamma_{ji}\dot{T}_j - \beta_{ij}\dot{C}_j - M_{ij}\dot{\phi}_{1,j} - T_{ij}\dot{\phi}_{2,j} - \chi_{ij}\dot{\phi}_{3,j} - X_{ij}\dot{\phi}_{,j} = H_{ij}P_{,j} + O_{ij}C_j\ , \\ -c_{ij}e_{ij} - P_{ij}T_{i,j} - M_{ij}C_{i,j} + A_{ij}\phi_{1,ij} + D_{ij}\phi_{2,ij} + S_{ij}\phi_{3,ij} - \alpha_1\phi_1 - \alpha_4\phi_2 + \\ -\alpha_6\phi_3 + s_1\phi + l_1\theta + n_1C + \rho\Lambda_1 = \rho\kappa_1\ddot{\phi}_1\ , \\ -d_{ij}e_{ij} - U_{ij}T_{i,j} - T_{ij}C_{i,j} + D_{ij}\phi_{1,ij} + B_{ij}\phi_{2,ij} + E_{ij}\phi_{3,ij} - \alpha_4\phi_1 - \alpha_2\phi_2 + \\ -\alpha_5\phi_3 + s_2\phi + l_2\theta + n_2C + \rho\Lambda_2 = \rho\kappa_2\ddot{\phi}_2\ , \\ -f_{ij}e_{ij} - \lambda\!\!\!^{-}_{ij}T_{i,j} - \chi_{ij}C_{i,j} + S_{ij}\phi_{1,ij} + E_{ij}\phi_{2,ij} + C_{ij}\phi_{3,ij} - \alpha_6\phi_1 - \alpha_5\phi_2 + \\ -\alpha_3\phi_3 + s_3\phi + l_3\theta + n_3C + \rho\Lambda_3 = \rho\kappa_3\ddot{\phi}_3\ , \\ -\upsilon_{ij}e_{ij} - W_{ig}T_{i,g} - X_{ig}C_{i,g} + s_i\phi_i + V_{ig}\phi_{,ig} - \alpha\phi + c\theta + dC + \rho\tau_1 = \rho\tau_2\ddot{\phi}\ , \\ T_0[-a_{ij}\dot{e}_{ij} + l_i\dot{\phi}_i + c\dot{\phi} + a\dot{C}] + \rho C_e\dot{\theta} = k_{ij}\theta_{,ij} + \kappa_{ij}T_{j,i}\ , \\ h_{ij}[b_{zy}e_{zy} - n_g\phi_g - d\phi - a\theta + bC]_{,ij} + m_{ij}C_{j,i} = \dot{C}\ . \end{aligned} \tag{21}$$

For an isotropic and homogeneous material, the subsequent Eqs. (18) and (19) are simplified to:

$$\begin{aligned} t_{ij} = \lambda e_{yy}\delta_{ij} + 2\mu e_{ij} + [\lambda\!\!\!^{-}_g\phi_g + f\phi - \beta_1\theta - \beta_2C]\delta_{ij}\ , \\ \Omega_{1j} = A_1\phi_{1,j} + A_4\phi_{2,j} + A_6\phi_{3,j} - B_1T_j - B_4C_j\ , \\ \Omega_{2j} = A_4\phi_{1,j} + A_2\phi_{2,j} + A_5\phi_{3,j} - B_2T_j - B_5C_j\ , \\ \Omega_{3j} = A_6\phi_{1,j} + A_5\phi_{2,j} + A_3\phi_{3,j} - B_3T_j - B_6C_j\ , \\ p_1 = -\lambda\!\!\!^{-}_1e_{ii} - \alpha_1\phi_1 - \alpha_4\phi_2 - \alpha_6\phi_3 + s_1\phi + l_1\theta + n_1C\ , \\ p_2 = -\lambda\!\!\!^{-}_2e_{ii} - \alpha_4\phi_1 - \alpha_2\phi_2 - \alpha_5\phi_3 + s_2\phi + l_2\theta + n_2C\ , \\ p_3 = -\lambda\!\!\!^{-}_3e_{ii} - \alpha_6\phi_1 - \alpha_5\phi_2 - \alpha_3\phi_3 + s_3\phi + l_3\theta + n_3C\ , \\ \rho\varepsilon_i = -B_1\phi_{1,i} - B_2\phi_{2,i} - B_3\phi_{3,i} - E_1\phi_{,i} - D_1T_i - D_3C_i\ , \\ \rho\chi_i = -B_4\phi_{1,i} - B_5\phi_{2,i} - B_6\phi_{3,i} - E_2\phi_{,i} - D_3T_i - D_2C_i\ , \\ \rho S = \beta_1e_{gg} + l_i\phi_i + c\phi + \frac{\rho C_e\theta}{T_0} + aC,\, P = -\beta_2e_{gg} - n_i\phi_i - d\phi - a\theta + bC\ , \\ s = fe_{gg} - s_i\phi_i + \alpha\phi - c\theta - dC,\, \Re_g = -E_1T_g - E_2C_g + \gamma\phi_{,g}\ , \\ q_i = k\theta_{,i} + k_1T_i,\, Q_{ij} = -k_4T_{y,y}\delta_{ij} - k_5T_{i,j} - k_6T_{j,i},\, \varpi_i = (k - k_3)\theta_{,i} + (k_1 - k_2)T_i\ , \\ \eta_i = hP_{,i} + h_1C_i,\, \xi_{ij} = -h_4C_{y,y}\delta_{ij} - h_5C_{i,j} - h_6C_{j,i},\, \upsilon_i = (h - h_3)P_{,i} + (h_1 - h_2)C_i\ , \end{aligned} \tag{22}$$

where

$$\begin{aligned} c_{ijgy} = \lambda\delta_{ij}\delta_{gy} + \mu\delta_{ig}\delta_{jy} + \mu\delta_{iy}\delta_{jg},\, a_{ij} = -\beta_1\delta_{ij},\, b_{ij} = -\beta_2\delta_{ij},\, c_{ij} = \lambda\!\!\!^{-}_1\delta_{ij},\, d_{ij} = \lambda\!\!\!^{-}_2\delta_{ij},\, f_{ij} = \lambda\!\!\!^{-}_3\delta_{ij}\ , \\ \upsilon_{ij} = f\delta_{ij},\ A_{ij} = A_1\delta_{ij},\, B_{ij} = A_2\delta_{ij},\, C_{ij} = A_3\delta_{ij},\, D_{ij} = A_4\delta_{ij},\, E_{ij} = A_5\delta_{ij}, S_{ij} = A_6\delta_{ij},\, P_{ij} = B_1\delta_{ij}\ , \\ U_{ij} = B_2\delta_{ij},\, \lambda\!\!\!^{-}_{ij} = B_3\delta_{ij}, M_{ij} = B_4\delta_{ij},\, T_{ij} = B_5\delta_{ij},\, \chi_{ij} = B_6\delta_{ij},\, \alpha_{ij} = D_1\delta_{ij}, \beta_{ij} = D_2\delta_{ij}, \gamma_{ij} = D_3\delta_{ij}\ , \\ W_{ij} = E_1\delta_{ij},\, X_{ij} = E_2\delta_{ij},\, V_{ij} = \gamma\delta_{ij}, k_{ij} = k\delta_{ij},\, \kappa_{ij} = k_1\delta_{ij},\, L_{ij} = k_2\delta_{ij},\, K_{ij} = k_3\delta_{ij},\, h_{ij} = h\delta_{ij},\, m_{ij} = h_1\delta_{ij}\ , \\ H_{ij} = h_3\delta_{ij},\, m_{ijgy} = k_4\delta_{ij}\delta_{gy} + k_6\delta_{ig}\delta_{jy} + k_5\delta_{iy}\delta_{ig},\, O_{ij} = h_2\delta_{ij},\ n_{ijgy} = h_4\delta_{ij}\delta_{gy} + h_6\delta_{ig}\delta_{jy} + h_5\delta_{iy}\delta_{ig}\ . \end{aligned} \tag{23}$$

Here δ_{ij} is Kronecker's delta and λ, μ, f, β_1, β_2, $\lambdabar_1$, $\lambdabar_2$, $\lambdabar_3$, $A_1, \ldots, A_6$, $B_1, \ldots, B_6$, D_1, D_2, D_3, E_1, E_2, γ, k, h, $k_1, \ldots, k_6$ and $h_1, \ldots, h_6$ are material constants. Using Eq. (23) in Eq. (21), we obtain the governing equations for homogeneous isotropic micromorphic thermoelastic diffusion materials with microtemperature, microconcentration and triple porosity effects in the absence of heat and mass diffusive sources:

$$
\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \lambdabar_i\nabla\phi_i + f\nabla\phi - \beta_1\nabla\theta - \beta_2\nabla C = \rho\ddot{\mathbf{u}}\,,\\
&(k_6\Delta - k_2)\mathbf{v} + (k_4+k_5)\nabla(\nabla\cdot\mathbf{v}) - k_3\nabla\theta = D_1\dot{\mathbf{v}} + D_3\dot{\mathbf{w}} + B_i\nabla\dot{\phi}_i + E_1\nabla\dot{\phi}\,,\\
&(h_6\Delta - h_2)\mathbf{w} + (h_4+h_5)\nabla(\nabla\cdot\mathbf{w}) - h_3\nabla P = D_3\dot{\mathbf{v}} + D_2\dot{\mathbf{w}} + B_{i+3}\nabla\dot{\phi}_i + E_2\nabla\dot{\phi}\,,\\
&-\lambdabar_1(\nabla\cdot\mathbf{u}) - B_1(\nabla\cdot\mathbf{v}) - B_4(\nabla\cdot\mathbf{w}) + (A_1\Delta-\alpha_1)\phi_1 + (A_4\Delta-\alpha_4)\phi_2 + (A_6\Delta-\alpha_6)\phi_3 +\\
&\quad + s_1\phi + l_1\theta + n_1 C = \rho\kappa_1\ddot{\phi}_1\,,\\
&-\lambdabar_2(\nabla\cdot\mathbf{u}) - B_2(\nabla\cdot\mathbf{v}) - B_5(\nabla\cdot\mathbf{w}) + (A_4\Delta-\alpha_4)\phi_1 + (A_2\Delta-\alpha_2)\phi_2 + (A_5\Delta-\alpha_5)\phi_3 +\\
&\quad + s_2\phi + l_2\theta + n_2 C = \rho\kappa_2\ddot{\phi}_2\,,\\
&-\lambdabar_3(\nabla\cdot\mathbf{u}) - B_3(\nabla\cdot\mathbf{v}) - B_6(\nabla\cdot\mathbf{w}) + (A_6\Delta-\alpha_6)\phi_1 + (A_5\Delta-\alpha_5)\phi_2 + (A_3\Delta-\alpha_3)\phi_3 +\\
&\quad + s_3\phi + l_3\theta + n_3 C = \rho\kappa_3\ddot{\phi}_3\,,\\
&-f(\nabla\cdot\mathbf{u}) - E_1(\nabla\cdot\mathbf{v}) - E_2(\nabla\cdot\mathbf{w}) + s_i\phi_i + (\gamma\Delta-\alpha)\phi + c\theta + dC = \rho\tau_2\ddot{\phi}\,,\\
&T_0[\beta_1(\nabla\cdot\dot{\mathbf{u}}) + l_i\dot{\phi}_i + c\dot{\phi} + a\dot{C}] + \rho C_e\dot{\theta} = k\,\Delta\theta + k_1(\nabla\cdot\mathbf{v})\,,\\
&h\Delta\left[-\beta_2(\nabla\cdot\mathbf{u}) - n_i\phi_i - d\phi - a\theta + bC\right] + h_1(\nabla\cdot\mathbf{w}) = \dot{C}\,,
\end{aligned}
\tag{24}
$$

where Δ, ∇ are Laplacian and Del operators, respectively.

In the following sections, the chemical potential has been adopted as a state variable rather than concentration. For this purpose, the eleventh equation of (22) can be rewritten as:

$$
C = \frac{1}{b}\left[\beta_2 e_{gg} + n_i\phi_i + d\phi + a\theta + P\right]. \tag{25}
$$

Hence, the system of Eqs. (24), with the assistance Eq. (25) is transformed into:

$$
\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda'+\mu)\nabla(\nabla\cdot\mathbf{u}) + \tilde{\lambdabar}_i\nabla\phi_i + \wp\nabla\phi - \vartheta_1\nabla\theta - \vartheta_2\nabla P = \rho\ddot{\mathbf{u}}\,,\\
&(k_6\Delta - k_2)\mathbf{v} + (k_4+k_5)\nabla(\nabla\cdot\mathbf{v}) - k_3\nabla\theta = D_1\dot{\mathbf{v}} + D_3\dot{\mathbf{w}} + B_i\nabla\dot{\phi}_i + E_1\nabla\dot{\phi}\,,\\
&(h_6\Delta - h_2)\mathbf{w} + (h_4+h_5)\nabla(\nabla\cdot\mathbf{w}) - h_3\nabla P = D_3\dot{\mathbf{v}} + D_2\dot{\mathbf{w}} + B_{i+3}\nabla\dot{\phi}_i + E_2\nabla\dot{\phi}\,,\\
&-\tilde{\lambdabar}_1(\nabla\cdot\mathbf{u}) - B_1(\nabla\cdot\mathbf{v}) - B_4(\nabla\cdot\mathbf{w}) + (A_1\Delta-\zeta_1)\phi_1 + (A_4\Delta-\zeta_4)\phi_2 + (A_6\Delta-\zeta_6)\phi_3 +\\
&\quad + \delta_1\phi + \xi_1\theta + v_1 P = \rho\kappa_1\ddot{\phi}_1\,,\\
&-\tilde{\lambdabar}_2(\nabla\cdot\mathbf{u}) - B_2(\nabla\cdot\mathbf{v}) - B_5(\nabla\cdot\mathbf{w}) + (A_4\Delta-\zeta_4)\phi_1 + (A_2\Delta-\zeta_2)\phi_2 + (A_5\Delta-\zeta_5)\phi_3 +\\
&\quad + \delta_2\phi + \xi_2\theta + v_2 P = \rho\kappa_2\ddot{\phi}_2\,,\\
&-\tilde{\lambdabar}_3(\nabla\cdot\mathbf{u}) - B_3(\nabla\cdot\mathbf{v}) - B_6(\nabla\cdot\mathbf{w}) + (A_6\Delta-\zeta_6)\phi_1 + (A_5\Delta-\zeta_5)\phi_2 + (A_3\Delta-\zeta_3)\phi_3 +\\
&\quad + \delta_3\phi + \xi_3\theta + v_3 P = \rho\kappa_3\ddot{\phi}_3\,,\\
&-\wp(\nabla\cdot\mathbf{u}) - E_1(\nabla\cdot\mathbf{v}) - E_2(\nabla\cdot\mathbf{w}) + \delta_i\phi_i + (\gamma\Delta-\beta)\phi + n\theta + \xi P = \rho\tau_2\ddot{\phi}\,,\\
&-T_0[\vartheta_1(\nabla\cdot\dot{\mathbf{u}}) + \xi_i\dot{\phi}_i + n\dot{\phi} + \varsigma\dot{P} + \eta\dot{\theta}] + k_1(\nabla\cdot\mathbf{v}) + k\,\Delta\theta = 0\,,\\
&-\left[\vartheta_2(\nabla\cdot\dot{\mathbf{u}}) + v_i\dot{\phi}_i + \xi\dot{\phi} + \varsigma\dot{\theta} + \varpi\dot{P}\right] + h_1(\nabla\cdot\mathbf{w}) + h\Delta P = 0\,,
\end{aligned}
\tag{26}
$$

where

$$
\begin{gathered}
\varpi = b^{-1},\ \vartheta_2 = \varpi\beta_2,\ \vartheta_1 = \beta_1 + a\vartheta_2,\ \tilde{\lambdabar}_i = \lambdabar_i - n_i\vartheta_2,\ \lambda' = \lambda - \vartheta_2\beta_2,\ \varsigma = a\varpi,\ v_i = n_i\varpi,\\
\zeta_i = \alpha_i - n_i v_i,\ \zeta_4 = \alpha_4 - n_1 v_2,\ \zeta_5 = \alpha_5 - n_2 v_3,\ \zeta_6 = \alpha_6 - n_3 v_1,\ \xi_i = l_i + n_i\varsigma,\\
\wp = f - \vartheta_2 d,\ \delta_i = s_i + v_i d,\ \xi = d\varpi,\ \beta = \alpha - d\xi,\ n = c + d\varsigma,\ \eta = \frac{\rho C_e}{T_0} + a\varsigma.
\end{gathered}
$$

III. Steady Oscillations

The displacement vector, microtemperature, microconcentaration, volume fraction fields, microstretch, temperature change and chemical potential functions are presumed as

$$[\mathbf{u}(\mathbf{x},t),\ \mathbf{v}(\mathbf{x},t),\ \mathbf{w}(\mathbf{x},t),\boldsymbol{\phi}(\mathbf{x},t),\ \phi(\mathbf{x},t),\theta(\mathbf{x},t),\ P(\mathbf{x},t)] = \mathrm{Re}\,[(\mathbf{u}^*,\ \mathbf{v}^*,\ \mathbf{w}^*,\boldsymbol{\phi}^*,\ \phi^*,\ \theta^*,\ P^*)\,e^{-\iota\omega t}], \tag{27}$$

where, ω is oscillation frequency. By employing Eq. (27) in the system of Eqs. (26) and omitting asterisks (*) for simplicity, we derive the system of equations for steady oscillations as:

$$\begin{gathered}
[\mu\Delta+\rho\omega^2]\mathbf{u}+(\lambda'+\mu)\nabla(\nabla\cdot\mathbf{u})+\tilde{\lambda}_i\nabla\phi_i+\wp\nabla\phi-\vartheta_1\nabla\theta-\vartheta_2\nabla P=\mathbf{0},\\
[k_6\Delta-k_2+\iota\omega D_1]\mathbf{v}+(k_4+k_5)\nabla(\nabla\cdot\mathbf{v})+\iota\omega D_3\mathbf{w}+\iota\omega B_i\nabla\phi_i+\iota\omega E_1\nabla\phi-k_3\nabla\theta=\mathbf{0},\\
\iota\omega D_3\mathbf{v}+[h_6\Delta-h_2+\iota\omega D_2]\mathbf{w}+(h_4+h_5)\nabla(\nabla\cdot\mathbf{w})+\iota\omega B_{i+3}\nabla\phi_i+\iota\omega E_2\nabla\phi-h_3\nabla P=\mathbf{0},\\
-\tilde{\lambda}_1(\nabla\cdot\mathbf{u})-B_1(\nabla\cdot\mathbf{v})-B_4(\nabla\cdot\mathbf{w})+(A_1\Delta-\gamma_1)\phi_1+(A_4\Delta-\zeta_4)\phi_2+(A_6\Delta-\zeta_6)\phi_3+\\
+\delta_1\phi+\xi_1\theta+v_1P=0,\\
-\tilde{\lambda}_2(\nabla\cdot\mathbf{u})-B_2(\nabla\cdot\mathbf{v})-B_5(\nabla\cdot\mathbf{w})+(A_4\Delta-\zeta_4)\phi_1+(A_2\Delta-\gamma_2)\phi_2+(A_5\Delta-\zeta_5)\phi_3+\\
+\delta_2\phi+\xi_2\theta+v_2P=0,\\
-\tilde{\lambda}_3(\nabla\cdot\mathbf{u})-B_3(\nabla\cdot\mathbf{v})-B_6(\nabla\cdot\mathbf{w})+(A_6\Delta-\zeta_6)\phi_1+(A_5\Delta-\zeta_5)\phi_2+(A_3\Delta-\gamma_3)\phi_3+\\
+\delta_3\phi+\xi_3\theta+v_3P=0,\\
-\wp(\nabla\cdot\mathbf{u})-E_1(\nabla\cdot\mathbf{v})-E_2(\nabla\cdot\mathbf{w})+\delta_i\phi_i+(\gamma\Delta-\tilde{\beta})\phi+n\theta+\xi P=0,\\
\iota\omega T_0[\vartheta_1(\nabla\cdot\mathbf{u})+\xi_i\phi_i+n\phi+\varsigma P]+k_1(\nabla\cdot\mathbf{v})+[k\Delta+\iota\omega\eta T_0]\,\theta=0,\\
\iota\omega\ [\vartheta_2(\nabla\cdot\mathbf{u})+v_i\phi_i+\xi\phi+\varsigma\theta]+h_1(\nabla\cdot\mathbf{w})+[h\Delta+\iota\omega\varpi]P=0,
\end{gathered}\tag{28}$$

where, $\gamma_i=\zeta_i-\rho\kappa_i\omega^2$, $\tilde{\beta}=\beta-\rho\omega^2\tau_2$, $i=1,2,3$. We introduce the second order matrix differential operators with constant coefficients:

$$\mathbf{F}(\mathbf{D}_\mathbf{x})=(F_{gz}(\mathbf{D}_\mathbf{x}))_{15\times15}\,,$$

where

$$\begin{gathered}
F_{ij}(\mathbf{D}_\mathbf{x})=[\mu\Delta+\rho\omega^2]\delta_{ij}+(\lambda'+\mu)\frac{\partial^2}{\partial x_i\partial x_j},\ F_{i;j+3}(\mathbf{D}_\mathbf{x})=\ F_{i;j+6}(\mathbf{D}_\mathbf{x})=0,\ F_{i;j+9}(\mathbf{D}_\mathbf{x})=\ \tilde{\lambda}_j\frac{\partial}{\partial x_i},\\
F_{i;13}(\mathbf{D}_\mathbf{x})=\ \wp\frac{\partial}{\partial x_i},\ F_{i;14}(\mathbf{D}_\mathbf{x})=-\vartheta_1\frac{\partial}{\partial x_i},\ F_{i;15}(\mathbf{D}_\mathbf{x})=-\vartheta_2\frac{\partial}{\partial x_i},\ F_{i+3;j}(\mathbf{D}_\mathbf{x})=0,\\
F_{i+3;j+3}(\mathbf{D}_\mathbf{x})=[k_6\Delta+k_8]\delta_{ij}+(k_4+k_5)\frac{\partial^2}{\partial x_i\partial x_j},\ F_{i+3;j+6}(\mathbf{D}_\mathbf{x})=\iota\omega D_3\delta_{ij},F_{i+3;j+9}(\mathbf{D}_\mathbf{x})=\iota\omega B_j\frac{\partial}{\partial x_i},\\
F_{i+3;13}(\mathbf{D}_\mathbf{x})=\iota\omega E_1\frac{\partial}{\partial x_i},\ F_{i+3;14}(\mathbf{D}_\mathbf{x})=-k_3\frac{\partial}{\partial x_i},\ F_{i+3;15}(\mathbf{D}_\mathbf{x})=0,\ F_{i+6;j}(\mathbf{D}_\mathbf{x})=0,\ F_{i+6;j+3}(\mathbf{D}_\mathbf{x})=\iota\omega D_3\delta_{ij},\\
F_{i+6;j+6}(\mathbf{D}_\mathbf{x})=[h_6\Delta+h_8]\delta_{ij}+(h_4+h_5)\frac{\partial^2}{\partial x_i\partial x_j},F_{i+6;j+9}(\mathbf{D}_\mathbf{x})=\iota\omega B_{j+3}\frac{\partial}{\partial x_i},\ F_{i+6;13}(\mathbf{D}_\mathbf{x})=\iota\omega E_2\frac{\partial}{\partial x_i},\\
F_{i+6;14}(\mathbf{D}_\mathbf{x})=0,\ F_{i+6;15}(\mathbf{D}_\mathbf{x})=-h_3\frac{\partial}{\partial x_i},\ F_{i+9;j}(\mathbf{D}_\mathbf{x})=\ -\tilde{\lambda}_i\frac{\partial}{\partial x_j},\ F_{i+9;j+3}(\mathbf{D}_\mathbf{x})=-B_i\frac{\partial}{\partial x_j},\\
F_{i+9;j+6}(\mathbf{D}_\mathbf{x})=-B_{i+3}\frac{\partial}{\partial x_j},\ F_{i+9;i+9}(\mathbf{D}_\mathbf{x})=A_i\Delta-\gamma_i,\ F_{10;11}(\mathbf{D}_\mathbf{x})=F_{11;10}(\mathbf{D}_\mathbf{x})=A_4\Delta-\zeta_4,\\
F_{10;12}(\mathbf{D}_\mathbf{x})=F_{12;10}(\mathbf{D}_\mathbf{x})=A_6\Delta-\zeta_6,\ F_{11;12}(\mathbf{D}_\mathbf{x})=F_{12;11}(\mathbf{D}_\mathbf{x})=A_5\Delta-\zeta_5,\ F_{i+9;13}(\mathbf{D}_\mathbf{x})=\ \delta_i,\ F_{i+9;14}(\mathbf{D}_\mathbf{x})=\xi_i,\\
F_{i+9;15}(\mathbf{D}_\mathbf{x})=v_i,\ F_{13;j}(\mathbf{D}_\mathbf{x})=-\wp\frac{\partial}{\partial x_j},\ F_{13;j+3}(\mathbf{D}_\mathbf{x})=-E_1\frac{\partial}{\partial x_j},\ F_{13;j+6}(\mathbf{D}_\mathbf{x})=-E_2\frac{\partial}{\partial x_j},\ F_{13;j+9}(\mathbf{D}_\mathbf{x})=\delta_j,\\
F_{13;13}(\mathbf{D}_\mathbf{x})=\gamma\Delta-\tilde{\beta},\ F_{13;14}(\mathbf{D}_\mathbf{x})=n,\ F_{13;15}(\mathbf{D}_\mathbf{x})=\xi,\ F_{14;j}(\mathbf{D}_\mathbf{x})=\iota\omega\vartheta_1T_0\frac{\partial}{\partial x_j},\ F_{14;j+3}(\mathbf{D}_\mathbf{x})=k_1\frac{\partial}{\partial x_j},
\end{gathered}$$

$$F_{14;j+6}(\mathbf{D_x}) = 0,\, F_{14;j+9}(\mathbf{D_x}) = \iota\omega\xi_j T_0,\, F_{14;13}(\mathbf{D_x}) = \iota\omega n T_0,\, F_{14;14}(\mathbf{D_x}) = k\Delta + \iota\omega\eta T_0,$$
$$F_{14;15}(\mathbf{D_x}) = \iota\omega\varsigma T_0,\, F_{15;j}(\mathbf{D_x}) = \iota\omega\vartheta_2 \frac{\partial}{\partial x_j},\, F_{15;j+3}(\mathbf{D_x}) = 0,\, F_{15;j+6}(\mathbf{D_x}) = h_1 \frac{\partial}{\partial x_j},\, F_{15;j+9}(\mathbf{D_x}) = \iota\omega\upsilon_j,$$
$$F_{15;13}(\mathbf{D_x}) = \iota\omega\xi, F_{15;14}(\mathbf{D_x}) = \iota\omega\varsigma,\, F_{15;15}(\mathbf{D_x}) = h\Delta + \iota\omega\varpi \;\; ; \; i,j = 1,2,3\,,$$

where $k_8 = \iota\omega D_1 - k_2,\; h_8 = \iota\omega D_2 - h_2$ and

$$\tilde{\mathbf{F}}(\mathbf{D_x}) = \left(\tilde{F}_{gz}(\mathbf{D_x})\right)_{15\times 15},$$

where

$$\tilde{F}_{ij}(\mathbf{D_x}) = \mu\Delta\,\delta_{ij} + (\lambda' + \mu)\frac{\partial^2}{\partial x_i \partial x_j},\, \tilde{F}_{i+3;j+3}(\mathbf{D_x}) = k_6\Delta\,\delta_{ij} + (k_4 + k_5)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$\tilde{F}_{i+6;j+6}(\mathbf{D_x}) = h_6\Delta\,\delta_{ij} + (h_4 + h_5)\frac{\partial^2}{\partial x_i \partial x_j},\, \tilde{F}_{i+6;i+6}(\mathbf{D_x}) = A_i\Delta,$$
$$\tilde{F}_{10;11}(\mathbf{D_x}) = \tilde{F}_{11;10}(\mathbf{D_x}) = A_4\Delta,\; \tilde{F}_{10;12}(\mathbf{D_x}) = \tilde{F}_{12;10}(\mathbf{D_x}) = A_6\Delta,\, \tilde{F}_{11;12}(\mathbf{D_x}) = \tilde{F}_{12;11}(\mathbf{D_x}) = A_5\Delta,$$
$$\tilde{F}_{13;13}(\mathbf{D_x}) = \gamma\Delta,\; \tilde{F}_{14;14}(\mathbf{D_x}) = k\Delta,\, \tilde{F}_{15;15}(\mathbf{D_x}) = h\Delta\;;\; i,j = 1,2,3\,,$$

and remaining elements of matrix $\tilde{\mathbf{F}}(\mathbf{D_x})$ are zero. The system of Eqs. (28) can be represented as:

$$\mathbf{F}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{v},\, \mathbf{w}, \boldsymbol{\phi}, \phi,\, \theta, P)$ is a fifteen-component vector function for E^3. The matrix $\tilde{\mathbf{F}}(\mathbf{D_x})$ is called the principal part of the operator $\mathbf{F}(\mathbf{D_x})$.

Definition 1. The operator $\mathbf{F}(\mathbf{D_x})$ is said to be elliptic if $\left|\tilde{\mathbf{F}}(\mathbf{m})\right| \neq 0$, where $\mathbf{m} = (m_1, m_2, m_3)$. Since $\left|\tilde{\mathbf{F}}(\mathbf{m})\right| = \mu^2\tilde{\lambda}kk_6k_7hh_6h_7\gamma\vartheta\,|m|^{30}$, $\tilde{\lambda} = \lambda' + 2\mu,\; k_7 = k_4 + k_5 + k_6,\; h_7 = h_4 + h_5 + h_6$, and:

$$\vartheta = \begin{vmatrix} A_1 & A_4 & A_6 \\ A_4 & A_2 & A_5 \\ A_6 & A_5 & A_3 \end{vmatrix},$$

therefore operator $\mathbf{F}(\mathbf{D_x})$ is an elliptic differential operator iff

$$\mu\tilde{\lambda}kk_6k_7hh_6h_7\gamma\vartheta \neq 0\,. \tag{29}$$

Definition 2. The fundamental solution of the system of Eqs. (28) (the fundamental matrix of operator $\mathbf{F}$) is the matrix $\mathbf{G}(\mathbf{x}) = (G_{gz}(\mathbf{x}))_{15\times 15}$ satisfying condition

$$\mathbf{F}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}), \tag{30}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{I}(\mathbf{x}) = (\delta_{gz})_{15\times 15}$ is the unit matrix and $\mathbf{x} \in E^3$.

Now, we construct $\mathbf{G}(\mathbf{x})$ in terms of elementary functions.

IV. Construction of $\mathbf{G}(\mathbf{x})$ in Terms of Elementary Functions

Let us consider the system of non-homogeneous equations

$$(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda' + \mu)\nabla(\nabla\cdot\,\mathbf{u}) - \tilde{\lambda}_i\nabla\phi_i - \wp\nabla\phi + \iota\omega T_0\vartheta_1\nabla\theta + \iota\omega\vartheta_2\nabla P = \mathbf{H}, \tag{31}$$
$$(k_6\Delta + k_8)\mathbf{v} + (k_4 + k_5)\nabla(\nabla\cdot\,\mathbf{v}) + \iota\omega D_3\mathbf{w} - B_i\nabla\phi_i - E_1\nabla\phi + k_1\nabla\theta = \mathbf{V}, \tag{32}$$
$$\iota\omega D_3\mathbf{v} + (h_6\Delta + h_8)\mathbf{w} + (h_4 + h_5)\nabla(\nabla\cdot\,\mathbf{w}) - B_{i+3}\nabla\phi_i - E_2\nabla\phi + h_1\nabla P = \mathbf{W}, \tag{33}$$

$$\tilde{\lambda}_1(\nabla\cdot\mathbf{u})+\iota\omega B_1(\nabla\cdot\mathbf{v})+\iota\omega B_4(\nabla\cdot\mathbf{w})+(A_1\Delta-\gamma_1)\phi_1+(A_4\Delta-\zeta_4)\phi_2+(A_6\Delta-\zeta_6)\phi_3+ \\ +\delta_1\phi+\iota\omega\xi_1T_0\theta+\iota\omega v_1P=X_1, \tag{34}$$

$$\tilde{\lambda}_2(\nabla\cdot\mathbf{u})+\iota\omega B_2(\nabla\cdot\mathbf{v})+\iota\omega B_5(\nabla\cdot\mathbf{w})+(A_4\Delta-\zeta_4)\phi_1+(A_2\Delta-\gamma_2)\phi_2+(A_5\Delta-\zeta_5)\phi_3+ \\ +\delta_2\phi+\iota\omega\xi_2T_0\theta+\iota\omega v_2P=X_2, \tag{35}$$

$$\tilde{\lambda}_3(\nabla\cdot\mathbf{u})+\iota\omega B_3(\nabla\cdot\mathbf{v})+\iota\omega B_6(\nabla\cdot\mathbf{w})+(A_6\Delta-\zeta_6)\phi_1+(A_5\Delta-\zeta_5)\phi_2+(A_3\Delta-\gamma_3)\phi_3+ \\ +\delta_3\phi+\iota\omega\xi_3T_0\theta++\iota\omega v_3P=X_3, \tag{36}$$

$$\wp(\nabla\cdot\mathbf{u})+\iota\omega E_1(\nabla\cdot\mathbf{v})+\iota\omega E_2(\nabla\cdot\mathbf{w})+\delta_i\phi_i+(\gamma\Delta-\tilde{\beta})\phi+\iota\omega nT_0\theta++\iota\omega\xi P=L, \tag{37}$$

$$-\vartheta_1(\nabla\cdot\mathbf{u})-k_3(\nabla\cdot\mathbf{v})+\xi_i\phi_i+n\phi+[k\Delta+\iota\omega T_0\eta]\,\theta+\iota\omega\varsigma P=Y, \tag{38}$$

$$-\vartheta_2(\nabla\cdot\mathbf{u})-h_3(\nabla\cdot\mathbf{w})+v_i\phi_i+\xi\phi+\iota\omega T_0\varsigma\theta+[h\Delta+\iota\omega\varpi]\,P=Z, \tag{39}$$

where $\mathbf{H}$, $\mathbf{V}$, $\mathbf{W}$ are three-component vector functions on E^3; X_i, L, Y and Z are scalar functions on E^3. The system of Eqs. (31)–(39) may be written in the form:

$$\mathbf{F}^{\mathrm{tr}}(\mathbf{D_x})\mathbf{U}(\mathbf{x})=\mathbf{Q}(\mathbf{x}), \tag{40}$$

where $\mathbf{F}^{\mathrm{tr}}$ is the transpose of matrix $\mathbf{F}$, $\mathbf{Q}=(\mathbf{H},\mathbf{V},\mathbf{W},X_i,L,Y,Z),\mathbf{x}\in E^3$. Applying operator $(\nabla\cdot)$ to the Eqs. (31)–(33), we obtain:

$$[\tilde{\lambda}\Delta+\rho\omega^2](\nabla\cdot\mathbf{u})-\tilde{\lambda}_i\Delta\phi_i-\wp\Delta\phi+\iota\omega T_0\vartheta_1\Delta\theta+\iota\omega\vartheta_2\Delta P=\nabla\cdot\mathbf{H}, \tag{41}$$

$$[k_7\Delta+k_8](\nabla\cdot\mathbf{v})+\iota\omega D_3(\nabla\cdot\mathbf{w})-B_i\Delta\phi_i-E_1\Delta\phi+k_1\Delta\theta=\nabla\cdot\mathbf{V}, \tag{42}$$

$$\iota\omega D_3(\nabla\cdot\mathbf{v})+[h_7\Delta+h_8](\nabla\cdot\mathbf{w})-B_{i+3}\Delta\phi_i-E_2\Delta\phi+h_1\Delta P=\nabla\cdot\mathbf{W}. \tag{43}$$

The Eqs. (34)–(39) and (41)–(43) may be expressed in the form:

$$\mathbf{N}(\Delta)\,\mathbf{S}=\tilde{\mathbf{Q}}, \tag{44}$$

where $\mathbf{S}=(\nabla\cdot\mathbf{u},\,\nabla\cdot\mathbf{v},\,\nabla\cdot\mathbf{w},\,\boldsymbol{\phi},\,\phi,\,\theta,\,P)$, $\tilde{\mathbf{Q}}=(w_1,\dots,w_9)=(\nabla\cdot\mathbf{H},\nabla\cdot\mathbf{V},\nabla\cdot\mathbf{W},\,X_i,L,Y,Z)$ and

$$\mathbf{N}(\Delta)=(N_{gz}(\Delta))_{9\times9}=$$

$$=\begin{pmatrix}
\tilde{\lambda}\Delta+\rho\omega^2 & 0 & 0 & -\tilde{\lambda}_1\Delta & -\tilde{\lambda}_2\Delta & -\tilde{\lambda}_3\Delta & -\wp\Delta & \iota\omega T_0\vartheta_1\Delta & \iota\omega\vartheta_2\Delta\\
0 & k_7\Delta+k_8 & \iota\omega D_3 & -B_1\Delta & -B_2\Delta & -B_3\Delta & -E_1\Delta & k_1\Delta & 0\\
0 & \iota\omega D_3 & h_7\Delta+h_8 & -B_4\Delta & -B_5\Delta & -B_6\Delta & -E_2\Delta & 0 & h_1\Delta\\
\tilde{\lambda}_1 & \iota\omega B_1 & \iota\omega B_4 & A_1\Delta-\gamma_1 & A_4\Delta-\zeta_4 & A_6\Delta-\zeta_6 & \delta_1 & \iota\omega T_0\xi_1 & \iota\omega v_1\\
\tilde{\lambda}_2 & \iota\omega B_2 & \iota\omega B_5 & A_4\Delta-\zeta_4 & A_2\Delta-\gamma_2 & A_5\Delta-\zeta_5 & \delta_2 & \iota\omega T_0\xi_2 & \iota\omega v_2\\
\tilde{\lambda}_3 & \iota\omega B_3 & \iota\omega B_6 & A_6\Delta-\zeta_6 & A_5\Delta-\zeta_5 & A_3\Delta-\gamma_3 & \delta_3 & \iota\omega T_0\xi_3 & \iota\omega v_3\\
\wp & \iota\omega E_1 & \iota\omega E_2 & \delta_1 & \delta_2 & \delta_3 & \gamma\Delta-\tilde{\beta} & \iota\omega T_0 n & \iota\omega\xi\\
-\vartheta_1 & -k_3 & 0 & \xi_1 & \xi_2 & \xi_3 & n & k\Delta+\iota\omega T_0\eta & \iota\omega\varsigma\\
-\vartheta_2 & 0 & -h_3 & v_1 & v_2 & v_3 & \xi & \iota\omega T_0\varsigma & h\Delta+\iota\omega\varpi
\end{pmatrix}_{9\times9}$$

Eq. (44) may also be written in determinant form as

$$\Gamma_1(\Delta)\,\mathbf{S}=\boldsymbol{\Psi}, \tag{45}$$

where $\boldsymbol{\Psi}=(\Psi_1,\dots,\Psi_9)$, $\Psi_j=\frac{1}{\tilde{A}}\sum_{i=1}^{9}N^*_{ij}w_i,\Gamma_1(\Delta)=\frac{|\mathbf{N}(\Delta)|}{\tilde{A}}$, $\tilde{A}=\tilde{\lambda}kk_7hh_7\gamma\vartheta$; $j=1,\dots,9$ and N^*_{ij} is the cofactor of the element N_{ij} of the matrix $\mathbf{N}$. On expanding $\Gamma_1(\Delta)$, we see that

$$\Gamma_1(\Delta)=\prod_{i=1}^{9}(\Delta+\lambda_i^2),$$

where $\lambda_i^2,\ i=1,\dots,9$ are the roots of the equation $\Gamma_1(-m)=0$ (with respect to m). Applying operator $\Gamma_1(\Delta)$ to Eq. (31) with the assistance of Eq. (45), we obtain:

$$\Gamma_1(\Delta)(\Delta+\lambda_{10}^2)\mathbf{u}=\boldsymbol{\Psi}', \tag{46}$$

where $\lambda_{10}^2 = \frac{\rho\omega^2}{\mu}$, $\mathbf{\Psi}' = \frac{1}{\mu}\left\{\Gamma_1(\Delta)\mathbf{H} - \nabla[(\lambda'+\mu)\psi_1 - \tilde{\lambda}_i\psi_{i+3} - \wp\psi_7 + \iota\omega\vartheta_1 T_0\psi_8 + \iota\omega\vartheta_2\psi_9]\right\}$. Multiplying Eqs. (32) and (33) by $h_6\Delta + h_8$ and $\iota\omega D_3$, respectively, we obtain

$$\begin{gathered}(h_6\Delta+h_8)[(k_6\Delta+k_8)\mathbf{v}+(k_4+k_5)\nabla(\nabla\cdot\mathbf{v})]+(h_6\Delta+h_8)\iota\omega D_3\mathbf{w}=\\=(h_6\Delta+h_8)[\mathbf{V}+B_i\nabla\phi_i+E_1\nabla\phi-k_1\nabla\theta],\end{gathered} \tag{47}$$

$$(\iota\omega D_3)^2\mathbf{v}+\iota\omega D_3[(h_6\Delta+h_8)\mathbf{w}+(h_4+h_5)\nabla(\nabla\cdot\mathbf{w})]=\iota\omega D_3[\mathbf{W}+B_{i+3}\nabla\phi_i+E_2\nabla\phi-h_1\nabla P]. \tag{48}$$

Using Eq. (48) in (47) and then applying $\Gamma_1(\Delta)$ to the resulting equation with the assistance of Eq. (45), we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{v} = \mathbf{\Psi}'', \tag{49}$$

where $\Gamma_2(\Delta) = \frac{1}{\tilde{B}}\begin{vmatrix} k_6\Delta+k_8 & \iota\omega D_3 \\ \iota\omega D_3 & h_6\Delta+h_8\end{vmatrix}$, $\tilde{B} = k_6h_6$ and

$$\mathbf{\Psi}'' = \frac{1}{\tilde{B}}\left\{\begin{array}{l}(h_6\Delta+h_8)[\Gamma_1(\Delta)\mathbf{V}-(k_4+k_5)\nabla\psi_2+B_i\nabla\psi_{i+3}+E_1\nabla\psi_7-k_1\nabla\psi_8]\\ -\iota\omega D_3[\Gamma_1(\Delta)\mathbf{W}-(h_4+h_5)\nabla\psi_3+B_{i+3}\nabla\psi_{i+3}+E_2\nabla\psi_7-h_1\nabla\psi_9]\end{array}\right\}.$$

It can be seen that

$$\Gamma_2(\Delta) = (\Delta+\lambda_{11}^2)(\Delta+\lambda_{12}^2).$$

Multiplying Eqs. (32) and (33) by $\iota\omega D_3$ and $k_6\Delta + k_8$, respectively, we obtain

$$\iota\omega D_3[(k_6\Delta+k_8)\mathbf{v}+(k_4+k_5)\nabla(\nabla\cdot\mathbf{v})]+(\iota\omega D_3)^2\mathbf{w}=\iota\omega D_3[\mathbf{V}+B_i\nabla\phi_i+E_1\nabla\phi-k_1\nabla\theta], \tag{50}$$

$$\begin{gathered}\iota\omega D_3(k_6\Delta+k_8)\mathbf{v}+(k_6\Delta+k_8)[(h_6\Delta+h_8)\mathbf{w}+(h_4+h_5)\nabla(\nabla\cdot\mathbf{w})]=\\=(k_6\Delta+k_8)[\mathbf{W}+B_{i+3}\nabla\phi_i+E_2\nabla\phi-h_1\nabla P].\end{gathered} \tag{51}$$

Using Eq. (50) in (51) and applying $\Gamma_1(\Delta)$ to the resulting equation with the help of Eq. (45), we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{w} = \mathbf{\Psi}''', \tag{52}$$

where

$$\mathbf{\Psi}''' = \frac{1}{\tilde{B}}\left\{\begin{array}{l}(k_6\Delta+k_8)[\Gamma_1(\Delta)\mathbf{W}-(h_4+h_5)\nabla\psi_3+B_{i+3}\nabla\psi_{i+3}+E_2\nabla\psi_7-h_1\nabla\psi_9]\\ -\iota\omega D_3[\Gamma_1(\Delta)\mathbf{V}-(k_4+k_5)\nabla\psi_2+B_i\nabla\psi_{i+3}+E_1\nabla\psi_7-k_1\nabla\psi_8]\end{array}\right\}.$$

From Eqs. (45), (46), (49) and (52), we have

$$\mathbf{\Theta}(\Delta)\mathbf{U}(\mathbf{x}) = \hat{\mathbf{\Psi}}(\mathbf{x}), \tag{53}$$

where

$$\hat{\mathbf{\Psi}}(\mathbf{x}) = (\mathbf{\Psi}',\ \mathbf{\Psi}'',\ \mathbf{\Psi}''',\ \Psi_4,\ \ldots,\Psi_9) \text{ and } \mathbf{\Theta}(\Delta) = (\Theta_{gz}(\Delta))_{15\times 15},$$

$$\Theta_{ii}(\Delta) = \Gamma_1(\Delta)(\Delta+\lambda_{10}^2) = \prod_{y=1}^{10}(\Delta+\lambda_y^2),\ \Theta_{i+3;i+3}(\Delta) = \Theta_{i+6;i+6}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{y=1}^{9,11,12}(\Delta+\lambda_y^2),$$

$$\Theta_{jj}(\Delta) = \Gamma_1(\Delta) = \prod_{y=1}^{9}(\Delta+\lambda_y^2),\ \Theta_{gz}(\Delta) = 0\ ;\ i=1,2,3\ ;\ j=10,\ldots,15\ ;\ g,z=1,\ldots,15,\ g\neq z.$$

The expressions for $\mathbf{\Psi}'$, $\mathbf{\Psi}''$, $\mathbf{\Psi}'''$, $\Psi_j, j = 4,\ldots,9$ can be rewritten in the form

$$\mathbf{\Psi}' = \frac{1}{\mu}\Gamma_1(\Delta)\mathbf{J}\mathbf{H} + w_{11}(\Delta)\nabla(\nabla\cdot\mathbf{H}) + w_{21}(\Delta)\nabla(\nabla\cdot\mathbf{V}) + w_{31}(\Delta)\nabla(\nabla\cdot\mathbf{W}) + \sum_{i=4}^{9} w_{i1}(\Delta)\nabla\mathrm{w}_{\mathrm{i}},$$

$$\begin{gathered}\mathbf{\Psi}'' = w_{12}(\Delta)\nabla(\nabla\cdot\mathbf{H}) + \frac{1}{\tilde{B}}(h_6\Delta+h_8)\Gamma_1(\Delta)\mathbf{J}\mathbf{V} + w_{22}(\Delta)\nabla(\nabla\cdot\mathbf{V}) + \\ -\frac{1}{\tilde{B}}\iota\omega D_3\Gamma_1(\Delta)\mathbf{J}\mathbf{W} + w_{32}(\Delta)\nabla(\nabla\cdot\mathbf{W}) + \sum_{i=4}^{9} w_{i2}(\Delta)\nabla\mathrm{w}_{\mathrm{i}},\end{gathered}$$

$$\Psi''' = w_{13}(\Delta)\,\nabla(\nabla\cdot\,\mathbf{H}) - \frac{1}{\tilde{B}}\,\iota\omega D_3\Gamma_1(\Delta)\mathbf{J}\mathbf{V} + w_{23}(\Delta)\,\nabla(\nabla\cdot\,\mathbf{V}) +$$
$$+\frac{1}{\tilde{B}}\,(k_6\Delta + k_8)\Gamma_1(\Delta)\mathbf{J}\mathbf{W} + w_{33}(\Delta)\,\nabla(\nabla\cdot\,\mathbf{W}) + \sum_{i=4}^{9} w_{i3}(\Delta)\,\nabla\mathbf{w}_{\mathrm{i}}, \tag{54}$$
$$\Psi_j = w_{1j}(\Delta)\nabla\cdot\,\mathbf{H} + w_{2j}(\Delta)\nabla\cdot\mathbf{V} + w_{3j}(\Delta)\,\nabla\cdot\,\mathbf{W} + \sum_{i=4}^{9} w_{ij}(\Delta)\,w_i,$$

where $\mathbf{J} = (\delta_{ij})_{3\times 3}$ is the unit matrix and

$$w_{i1}(\Delta) = -\frac{1}{\tilde{A}\mu}\left[(\lambda' + \mu)\,N^*_{i1}(\Delta) - [\tilde{\lambda}_j N^*_{i;j+3}(\Delta) + \wp N^*_{i7}(\Delta) - \iota\omega\vartheta_1 T_0 N^*_{i8}(\Delta) - \iota\omega\vartheta_2 N^*_{i9}(\Delta)]\right],$$
$$[7pt] w_{i2}(\Delta) = -\frac{1}{\tilde{A}\tilde{B}}\left\{\begin{array}{l}(h_6\Delta + h_8)[(k_4 + k_5)N^*_{i2}(\Delta) - B_j N^*_{i;j+3}(\Delta) - E_1 N^*_{i7}(\Delta) + k_1 N^*_{i8}(\Delta)] + \\ -\iota\omega D_3[(h_4 + h_5)N^*_{i3}(\Delta) - B_{j+3}N^*_{i;j+3}(\Delta) - E_2 N^*_{i7}(\Delta) + h_1 N^*_{i9}(\Delta)]\end{array}\right\},$$
$$w_{i3}(\Delta) = -\frac{1}{\tilde{A}\tilde{B}}\left\{\begin{array}{l}-\iota\omega D_3[\,(k_4 + k_5)N^*_{i2}(\Delta) - B_j N^*_{i;j+3}(\Delta) - E_1 N^*_{i7}(\Delta) + k_1 N^*_{i8}(\Delta)] + \\ +(k_6\Delta + k_8)[(h_4 + h_5)N^*_{i3}(\Delta) - B_{j+3}N^*_{i;j+3}(\Delta) - E_2 N^*_{i7}(\Delta) + h_1 N^*_{i9}(\Delta)]\end{array}\right\},$$
$$w_{ig}(\Delta) = \frac{N^*_{ig}(\Delta)}{\tilde{A}}\;;\; i = 1,\ldots,9\;;\; g = 4,\ldots,9.$$

From Eq. (54), we have

$$\hat{\Psi}(\mathbf{x}) = \mathbf{R}^{\mathrm{tr}}(\mathbf{D_x})\,\mathbf{Q}(\mathbf{x}), \tag{55}$$

where

$$\mathbf{R}(\mathbf{D_x}) = (R_{gz}(\mathbf{D_x}))_{15\times 15}\,,$$
$$R_{ij}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma_1(\Delta)\delta_{ij} + w_{11}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},\; R_{i;j+3}(\mathbf{D_x}) = w_{12}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$
$$R_{i;j+6}(\mathbf{D_x}) = w_{13}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},\; R_{i;y+6}(\mathbf{D_x}) = w_{1y}(\Delta)\frac{\partial}{\partial x_i},$$
$$R_{i+3;j}(\mathbf{D_x}) = w_{21}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j}, R_{i+3;j+3}(\mathbf{D_x}) = \frac{1}{\tilde{B}}(h_6\Delta + h_8)\Gamma_1(\Delta)\delta_{ij} + w_{22}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$
$$R_{i+3;j+6}(\mathbf{D_x}) = -\frac{1}{\tilde{B}}\iota\omega D_3\Gamma_1(\Delta)\delta_{ij} + w_{23}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},\; R_{i+3;y+6}(\mathbf{D_x}) = w_{2y}(\Delta)\frac{\partial}{\partial x_i},$$
$$R_{i+6;j}(\mathbf{D_x}) = w_{31}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},\; R_{i+6;j+3}(\mathbf{D_x}) = -\frac{1}{\tilde{B}}\iota\omega D_3\Gamma_1(\Delta)\delta_{ij} + w_{32}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$
$$R_{i+6;j+6}(\mathbf{D_x}) = \frac{1}{\tilde{B}}(k_6\Delta + k_8)\Gamma_1(\Delta)\delta_{ij} + w_{33}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},\; R_{i+6;y+6}(\mathbf{D_x}) = w_{3y}(\Delta)\frac{\partial}{\partial x_i},$$
$$R_{y+6;i}(\mathbf{D_x}) = w_{y1}(\Delta)\frac{\partial}{\partial x_i},\; R_{y+6;i+3}(\mathbf{D_x}) = w_{y2}(\Delta)\frac{\partial}{\partial x_i},\; R_{y+6;i+6}(\mathbf{D_x}) = w_{y3}(\Delta)\frac{\partial}{\partial x_i},$$
$$R_{y+6;\hbar+6}(\mathbf{D_x}) = w_{y\hbar}(\Delta)\;;\; i,j = 1,2,3\;;\; y,\hbar = 4,\ldots,9.$$

From Eqs. (40), (53) and (55), we obtain

$$\mathbf{F}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x}) = \mathbf{\Theta}(\Delta). \tag{56}$$

We assume that $\lambda_i^2 \neq \lambda_j^2 \neq 0$ and $i, j = 1, \dots, 12$. Let

$$\mathbf{Y}(\mathbf{x}) = (Y_{ij}(\mathbf{x}))_{15\times 15}\,,\; Y_{zz}(\mathbf{x}) = \sum_{g=1}^{10} r_{1g}\varsigma_g(\mathbf{x}),$$
$$Y_{z+3;z+3}(\mathbf{x}) = Y_{z+6;z+6}(\mathbf{x}) = \sum_{g=1}^{9,11,12} r_{2g}\varsigma_g(\mathbf{x}),\; Y_{\ell\ell}(\mathbf{x}) = \sum_{g=1}^{9} r_{3g}\varsigma_g(\mathbf{x}),$$
$$Y_{ij}(\mathbf{x}) = 0\,;\; z = 1,2,3\,;\; \ell = 10,\dots,15\,;\; i,j = 1,\dots,15\,;\; i \neq j$$

where

$$\begin{gathered}\varsigma_g(\mathbf{x}) = -\frac{e^{\iota\lambda_g|\mathbf{x}|}}{4\pi\,|\mathbf{x}|},\; r_{1y} = \prod_{i=1,i\neq y}^{10}(\lambda_i^2-\lambda_y^2)^{-1}, r_{2\ell} = \prod_{i=1,i\neq \ell}^{9,11,12}(\lambda_i^2-\lambda_\ell^2)^{-1},\\ r_{3z} = \prod_{i=1,i\neq z}^{9}(\lambda_i^2-\lambda_z^2)^{-1}\,;\; g=1,\dots,12\,;\; y=1,\dots,10\,;\\ \ell = 1,\dots,9,11,12\,;\; z = 1,\dots,9.\end{gathered} \tag{57}$$

Lemma 1. The matrix $\mathbf{Y}$ is the fundamental matrix of the operator $\mathbf{\Theta}(\Delta)$ i.e.

$$\mathbf{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}). \tag{58}$$

Proof: To establish the lemma, it is adequate to prove that

$$\Gamma_1(\Delta)(\Delta+\lambda_{10}^2)Y_{11}(\mathbf{x}) = \delta(\mathbf{x}), \tag{59}$$
$$\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{44}(\mathbf{x}) = \delta(\mathbf{x}), \tag{60}$$
$$\Gamma_1(\Delta)Y_{10;10}(\mathbf{x}) = \delta(\mathbf{x}). \tag{61}$$

Consider:

$$\sum_{i=1}^{9} r_{3i} = \frac{1}{z_{10}}\sum_{j=1}^{9}(-1)^{j+1}z_j,$$

where

$$z_1 = \prod_{i=3}^{9}(\lambda_2^2-\lambda_i^2)\prod_{j=4}^{9}(\lambda_3^2-\lambda_j^2)\prod_{z=5}^{9}(\lambda_4^2-\lambda_z^2)\prod_{y=6}^{9}(\lambda_5^2-\lambda_y^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{\ell=8}^{9}(\lambda_7^2-\lambda_\ell^2)(\lambda_8^2-\lambda_9^2),$$
$$z_2 = \prod_{i=3}^{9}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{9}(\lambda_3^2-\lambda_j^2)\prod_{z=5}^{9}(\lambda_4^2-\lambda_z^2)\prod_{y=6}^{9}(\lambda_5^2-\lambda_y^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{\ell=8}^{9}(\lambda_7^2-\lambda_\ell^2)(\lambda_8^2-\lambda_9^2),$$
$$z_3 = \prod_{\substack{i=2\\ i\neq 3}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{9}(\lambda_2^2-\lambda_j^2)\prod_{z=5}^{9}(\lambda_4^2-\lambda_z^2)\prod_{y=6}^{9}(\lambda_5^2-\lambda_y^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{\ell=8}^{9}(\lambda_7^2-\lambda_\ell^2)(\lambda_8^2-\lambda_9^2),$$
$$z_4 = \prod_{\substack{i=2\\ i\neq 4}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{\substack{j=3\\ j\neq 4}}^{9}(\lambda_2^2-\lambda_j^2)\prod_{z=5}^{9}(\lambda_3^2-\lambda_z^2)\prod_{y=6}^{9}(\lambda_5^2-\lambda_y^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{\ell=8}^{9}(\lambda_7^2-\lambda_\ell^2)(\lambda_8^2-\lambda_9^2),$$
$$z_5 = \prod_{\substack{i=2\\ i\neq 5}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{\substack{j=3\\ j\neq 5}}^{9}(\lambda_2^2-\lambda_j^2)\prod_{\substack{z=4\\ z\neq 5}}^{9}(\lambda_3^2-\lambda_z^2)\prod_{y=6}^{9}(\lambda_4^2-\lambda_y^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{\ell=8}^{9}(\lambda_7^2-\lambda_\ell^2)(\lambda_8^2-\lambda_9^2),$$
$$z_6 = \prod_{\substack{i=2\\ i\neq 6}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{\substack{j=3\\ j\neq 6}}^{9}(\lambda_2^2-\lambda_j^2)\prod_{\substack{z=4\\ z\neq 6}}^{9}(\lambda_3^2-\lambda_z^2)\prod_{\substack{y=5\\ y\neq 6}}^{9}(\lambda_4^2-\lambda_y^2)\prod_{g=7}^{9}(\lambda_5^2-\lambda_g^2)\prod_{\ell=8}^{9}(\lambda_7^2-\lambda_\ell^2)(\lambda_8^2-\lambda_9^2),$$

$$z_7 = \prod_{\substack{i=2\\ i\neq 7}}^{9} (\lambda_1^2-\lambda_i^2) \prod_{\substack{j=3\\ j\neq 7}}^{9} (\lambda_2^2-\lambda_j^2) \prod_{\substack{z=4\\ z\neq 7}}^{9} (\lambda_3^2-\lambda_z^2) \prod_{\substack{y=5\\ y\neq 7}}^{9} (\lambda_4^2-\lambda_y^2) \prod_{\substack{g=6\\ g\neq 7}}^{9} (\lambda_5^2-\lambda_g^2) \prod_{\ell=8}^{9}(\lambda_6^2-\lambda_\ell^2)(\lambda_8^2-\lambda_9^2),$$

$$z_8 = \prod_{\substack{i=2\\ i\neq 8}}^{9} (\lambda_1^2-\lambda_i^2) \prod_{\substack{j=3\\ j\neq 8}}^{9} (\lambda_2^2-\lambda_j^2) \prod_{\substack{z=4\\ z\neq 8}}^{9} (\lambda_3^2-\lambda_z^2) \prod_{\substack{y=5\\ y\neq 8}}^{9} (\lambda_4^2-\lambda_y^2) \prod_{\substack{g=6\\ g\neq 8}}^{9} (\lambda_5^2-\lambda_g^2) \prod_{\substack{\ell=7\\ \ell\neq 8}}^{9} (\lambda_6^2-\lambda_\ell^2)(\lambda_7^2-\lambda_9^2),$$

$$z_9 = \prod_{i=2}^{8}(\lambda_1^2-\lambda_i^2) \prod_{j=3}^{8}(\lambda_2^2-\lambda_j^2) \prod_{z=4}^{8}(\lambda_3^2-\lambda_z^2) \prod_{y=5}^{8}(\lambda_4^2-\lambda_y^2) \prod_{g=6}^{8}(\lambda_5^2-\lambda_g^2) \prod_{\ell=7}^{8}(\lambda_6^2-\lambda_\ell^2)(\lambda_7^2-\lambda_8^2),$$

$$z_{10} = \prod_{i=2}^{9}(\lambda_1^2-\lambda_i^2) \prod_{j=3}^{9}(\lambda_2^2-\lambda_j^2) \prod_{z=4}^{9}(\lambda_3^2-\lambda_z^2) \prod_{y=5}^{9}(\lambda_4^2-\lambda_y^2) \prod_{g=6}^{9}(\lambda_5^2-\lambda_g^2) \prod_{\ell=7}^{9}(\lambda_6^2-\lambda_\ell^2) \prod_{\hbar=8}^{9}(\lambda_7^2-\lambda_\hbar^2)(\lambda_8^2-\lambda_9^2).$$

Upon simplifying the R.H.S. of the above relation, we obtain

$$\sum_{i=1}^{9} r_{3i} = 0. \tag{62}$$

Similarly, we find that

$$\begin{gathered} \sum_{i=2}^{9} r_{3i}(\lambda_1^2-\lambda_i^2) = 0, \sum_{i=3}^{9} r_{3i}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_i^2)\right] = 0, \sum_{i=4}^{9} r_{3i}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_i^2)\right] = 0, \\ \sum_{i=5}^{9} r_{3i}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_i^2)\right] = 0, \sum_{i=6}^{9} r_{3i}\left[\prod_{j=1}^{5}(\lambda_j^2-\lambda_i^2)\right] = 0, \sum_{i=7}^{9} r_{3i}\left[\prod_{j=1}^{6}(\lambda_j^2-\lambda_i^2)\right] = 0, \\ \sum_{i=8}^{9} r_{3i}\left[\prod_{j=1}^{7}(\lambda_j^2-\lambda_i^2)\right] = 0, r_{39}\left[\prod_{i=1}^{8}(\lambda_i^2-\lambda_9^2)\right] = 1. \end{gathered} \tag{63}$$

Also, we have

$$(\Delta+\lambda_y^2)\varsigma_g(\mathbf{x}) = \delta(\mathbf{x}) + (\lambda_y^2-\lambda_g^2)\varsigma_g(\mathbf{x})\ ;\ y,g = 1,\ldots,12. \tag{64}$$

Now, let us consider

$$\Gamma_1(\Delta)Y_{10;10}(\mathbf{x}) = \prod_{i=1}^{9}(\Delta+\lambda_i^2)\sum_{g=1}^{9} r_{3g}\varsigma_g(\mathbf{x}) = \prod_{i=2}^{9}(\Delta+\lambda_i^2)\sum_{g=1}^{9} r_{3g}\ \left[\delta(\mathbf{x}) +\ (\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right].$$

Using Eqs. (62)–(64) in the above relation, we obtain

$$\begin{aligned} &\Gamma_1(\Delta)Y_{10;10}(\mathbf{x}) = \\ &= \prod_{i=2}^{9}(\Delta+\lambda_i^2)\sum_{g=2}^{9} r_{3g}(\lambda_1^2-\lambda_g^2)\,\varsigma_g(\mathbf{x}) = \prod_{i=3}^{9}(\Delta+\lambda_i^2)\sum_{g=2}^{9} r_{3g}(\lambda_1^2-\lambda_g^2)\ \left[\delta(\mathbf{x}) +\ (\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] = \\ &= \prod_{i=3}^{9}(\Delta+\lambda_i^2)\sum_{g=3}^{9} r_{3g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x}) = \prod_{i=4}^{9}(\Delta+\lambda_i^2)\sum_{g=3}^{9} r_{3g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x}) +\ (\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] = \\ &= \prod_{i=4}^{9}(\Delta+\lambda_i^2)\sum_{g=4}^{9} r_{3g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x}) = \prod_{i=5}^{9}(\Delta+\lambda_i^2)\sum_{g=4}^{9} r_{3g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x}) +\ (\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] = \\ &= \prod_{i=5}^{9}(\Delta+\lambda_i^2)\sum_{g=5}^{9} r_{3g}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x}) = \prod_{i=6}^{9}(\Delta+\lambda_i^2)\sum_{g=5}^{9} r_{3g}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x}) +\ (\lambda_5^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] = \\ &= \prod_{i=6}^{9}(\Delta+\lambda_i^2)\sum_{g=6}^{9} r_{3g}\left[\prod_{j=1}^{5}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x}) = \prod_{i=7}^{9}(\Delta+\lambda_i^2)\sum_{g=6}^{9} r_{3g}\left[\prod_{j=1}^{5}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x}) +\ (\lambda_6^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] = \end{aligned}$$

$$= \prod_{i=7}^{9}(\Delta+\lambda_i^2)\sum_{g=7}^{9} r_{3g}\left[\prod_{j=1}^{6}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x}) = \prod_{i=8}^{9}(\Delta+\lambda_i^2)\sum_{g=7}^{9} r_{3g}\left[\prod_{j=1}^{6}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+\,(\lambda_7^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] =$$

$$= \prod_{i=8}^{9}(\Delta+\lambda_i^2)\sum_{g=8}^{9} r_{3g}\left[\prod_{j=1}^{7}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x}) = (\Delta+\lambda_9^2)\sum_{g=8}^{9} r_{3g}\left[\prod_{j=1}^{7}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+\,(\lambda_8^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] =$$

$$= (\Delta+\lambda_9^2)\,\varsigma_9(\mathbf{x}) = \delta(\mathbf{x}).$$

Eqs. (60) and (61) can be proved in the similar way.

We introduce the matrix

$$\mathbf{G}(\mathbf{x}) = \mathbf{R}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}). \tag{65}$$

From Eqs. (56), (58) and (65), we obtain

$$\mathbf{F}(\mathbf{D_x})\,\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{D_x})\,\mathbf{R}(\mathbf{D_x})\,\mathbf{Y}(\mathbf{x}) = \mathbf{\Theta}(\Delta)\,\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Therefore, $\mathbf{G}(\mathbf{x})$ is a solution to Eq. (30).

Theorem 1. If the condition (29) is satisfied, then the matrix $\mathbf{G}(\mathbf{x})$ defined by the Eq. (65) is the fundamental solution of the system of Eqs. (28) and each element of the matrix $\mathbf{G}(\mathbf{x})$ is represented in the following form:

$$G_{gz}(\mathbf{x}) = R_{gz}(\mathbf{D_x})Y_{11}(\mathbf{x}),\; G_{gy}(\mathbf{x}) = R_{gy}(\mathbf{D_x})Y_{44}(\mathbf{x}),$$

$$G_{gj}(\mathbf{x}) = R_{gj}(\mathbf{D_x})Y_{10;10}(\mathbf{x})\;;\; g=1,\ldots,15\;;\; z=1,2,3\;;\; y=4,\ldots,9\;;\; j=10,\ldots,15.$$

V. Basic Properties of Matrix $\mathbf{G}(\mathbf{x})$

Theorem 2. Each column of the matrix $\mathbf{G}(\mathbf{x})$ is a solution of a system of Eqs. (28) at every point $\mathbf{x} \in E^3$ except at the origin.

Theorem 3. If the condition (29) is satisfied, then the fundamental solution of the system

$$\tilde{\mathbf{F}}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0},$$

is the matrix $\tilde{\mathbf{G}}(\mathbf{x}) = \left(\tilde{G}_{gz}(\mathbf{x})\right)_{15\times 15}$, where

$$\tilde{G}_{ij}(\mathbf{x}) = \frac{1}{\tilde{\lambda}}\nabla\left(\nabla\cdot\varsigma_2^*(\mathbf{x})\right) - \frac{1}{\mu}\nabla\times\;(\nabla\times\,\varsigma_2^*(\mathbf{x})),\, \tilde{G}_{i+3;j+3}(\mathbf{x}) = \frac{1}{k_7}\nabla\left(\nabla\cdot\varsigma_2^*(\mathbf{x})\right) - \frac{1}{k_6}\nabla\times\;(\nabla\times\,\varsigma_2^*(\mathbf{x})),$$

$$\tilde{G}_{i+6;j+6}(\mathbf{x}) = \frac{1}{h_7}\nabla\left(\nabla\cdot\varsigma_2^*(\mathbf{x})\right) - \frac{1}{h_6}\nabla\times\;(\nabla\times\,\varsigma_2^*(\mathbf{x})),\, \tilde{G}_{10;10}(\mathbf{x}) = \frac{A_2A_3-A_5^2}{\vartheta}\varsigma_1^*(\mathbf{x}),$$

$$\tilde{G}_{10;11}(\mathbf{x}) = \tilde{G}_{11;10}(\mathbf{x}) = \frac{A_5A_6-A_4A_3}{\vartheta}\varsigma_1^*(\mathbf{x}),\, \tilde{G}_{10;12}(\mathbf{x}) = \tilde{G}_{12;10}(\mathbf{x}) = \frac{A_4A_5-A_2A_6}{\vartheta}\varsigma_1^*(\mathbf{x}),$$

$$\tilde{G}_{11;11}(\mathbf{x}) = \frac{A_1A_3-A_6^2}{\vartheta}\varsigma_1^*(\mathbf{x}),\, \tilde{G}_{11;12}(\mathbf{x}) = \tilde{G}_{12;11}(\mathbf{x}) = \frac{A_4A_6-A_1A_5}{\vartheta}\varsigma_1^*(\mathbf{x}),$$

$$\tilde{G}_{12;12}(\mathbf{x}) = \frac{A_1A_2-A_4^2}{\vartheta}\varsigma_1^*(\mathbf{x}),\, \tilde{G}_{13;13}(\mathbf{x}) = \frac{\varsigma_1^*(\mathbf{x})}{\gamma},\, \tilde{G}_{14;14}(\mathbf{x}) = \frac{\varsigma_1^*(\mathbf{x})}{k},\, \tilde{G}_{15;15}(\mathbf{x}) = \frac{\varsigma_1^*(\mathbf{x})}{h},$$

$$\varsigma_1^*(\mathbf{x}) = -\frac{1}{4\pi\,|\mathbf{x}|},\, \varsigma_2^*(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}\;;\; i,j=1,2,3,$$

and the remaining elements of matrix $\tilde{\mathbf{G}}(\mathbf{x})$ are zero.

VI. Fundamental Solution of System of Equations in Equilibrium Theory

If we substitute $\omega = 0$ into the system of Eqs. (28), we derive the system of equations for the equilibrium theory of micromorphic thermoelastic diffusion with microtemperatures, microconcentrations, and triple porosity as:

$$\begin{gathered}\mu\Delta\mathbf{u} + (\lambda' + \mu)\nabla(\nabla\cdot\mathbf{u}) + \tilde{\lambda}_i\nabla\phi_i + \wp\nabla\phi - \vartheta_1\nabla\theta - \vartheta_2\nabla P = \mathbf{0},\\ (k_6\Delta - k_2)\mathbf{v} + (k_4 + k_5)\nabla(\nabla\cdot\mathbf{v}) - k_3\nabla\theta = \mathbf{0},\\ (h_6\Delta - h_2)\mathbf{w} + (h_4 + h_5)\nabla(\nabla\cdot\mathbf{w}) - h_3\nabla P = \mathbf{0},\\ -\tilde{\lambda}_1(\nabla\cdot\mathbf{u}) - B_1(\nabla\cdot\mathbf{v}) - B_4(\nabla\cdot\mathbf{w}) + (A_1\Delta - \zeta_1)\phi_1 + (A_4\Delta - \zeta_4)\phi_2 + (A_6\Delta - \zeta_6)\phi_3 +\\ +\delta_1\phi + \xi_1\theta + v_1 P = 0,\\ -\tilde{\lambda}_2(\nabla\cdot\mathbf{u}) - B_2(\nabla\cdot\mathbf{v}) - B_5(\nabla\cdot\mathbf{w}) + (A_4\Delta - \zeta_4)\phi_1 + (A_2\Delta - \zeta_2)\phi_2 + (A_5\Delta - \zeta_5)\phi_3 +\\ +\delta_2\phi + \xi_2\theta + v_2 P = 0,\\ -\tilde{\lambda}_3(\nabla\cdot\mathbf{u}) - B_3(\nabla\cdot\mathbf{v}) - B_6(\nabla\cdot\mathbf{w}) + (A_6\Delta - \zeta_6)\phi_1 + (A_5\Delta - \zeta_5)\phi_2 + (A_3\Delta - \zeta_3)\phi_3 +\\ +\delta_3\phi + \xi_3\theta + v_3 P = 0,\\ -\wp(\nabla\cdot\mathbf{u}) - E_1(\nabla\cdot\mathbf{v}) - E_2(\nabla\cdot\mathbf{w}) + \delta_i\phi_i + (\gamma\Delta - \beta)\phi + n\theta + \xi P = 0,\\ k_1(\nabla\cdot\mathbf{v}) + k\Delta\theta = 0,\\ h_1(\nabla\cdot\mathbf{w}) + h\Delta P = 0.\end{gathered} \tag{66}$$

We introduce the second order matrix differential operators with constant coefficients

$$\mathbf{E}(\mathbf{D_x}) = (E_{gz}(\mathbf{D_x}))_{15\times 15},$$

where matrix $\mathbf{E}(\mathbf{D_x})$ can be obtained from $\mathbf{F}(\mathbf{D_x})$ by taking $\omega = 0$. The system of Eqs. (66) can be represented as

$$\mathbf{E}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{0}.$$

Definition 3. The operator $\mathbf{E}(\mathbf{D_x})$ is an elliptic differential operator iff Eq. (29) is satisfied.

Definition 4. The fundamental solution of the system of Eqs. (66) is the matrix $\mathbf{G}'(\mathbf{x}) = \left(G'_{gz}(\mathbf{x})\right)_{15\times 15}$ satisfying condition

$$\mathbf{E}(\mathbf{D_x})\mathbf{G}'(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}). \tag{67}$$

We consider the system of non-homogeneous equations

$$\mu\Delta\mathbf{u} + (\lambda' + \mu)\nabla(\nabla\cdot\mathbf{u}) - \tilde{\lambda}_i\nabla\phi_i - \wp\nabla\phi = \mathbf{H}', \tag{68}$$

$$(k_6\Delta - k_2)\mathbf{v} + (k_4 + k_5)\nabla(\nabla\cdot\mathbf{v}) - B_i\nabla\phi_i - E_1\nabla\phi + k_1\nabla\theta = \mathbf{V}', \tag{69}$$

$$(h_6\Delta - h_2)\mathbf{w} + (h_4 + h_5)\nabla(\nabla\cdot\mathbf{w}) - B_{i+3}\nabla\phi_i - E_2\nabla\phi + h_1 P = \mathbf{W}', \tag{70}$$

$$\tilde{\lambda}_1(\nabla\cdot\mathbf{u}) + (A_1\Delta - \zeta_1)\phi_1 + (A_4\Delta - \zeta_4)\phi_2 + (A_6\Delta - \zeta_6)\phi_3 + \delta_1\phi = X'_1, \tag{71}$$

$$\tilde{\lambda}_2(\nabla\cdot\mathbf{u}) + (A_4\Delta - \zeta_4)\phi_1 + (A_2\Delta - \zeta_2)\phi_2 + (A_5\Delta - \zeta_5)\phi_3 + \delta_2\phi = X'_2, \tag{72}$$

$$\tilde{\lambda}_3(\nabla\cdot\mathbf{u}) + (A_6\Delta - \zeta_6)\phi_1 + (A_5\Delta - \zeta_5)\phi_2 + (A_3\Delta - \zeta_3)\phi_3 + \delta_3\phi = X'_3, \tag{73}$$

$$\wp(\nabla\cdot\mathbf{u}) + \delta_i\phi_i + (\gamma\Delta - \beta)\phi = L', \tag{74}$$

$$-\vartheta_1(\nabla\cdot\mathbf{u}) - k_3(\nabla\cdot\mathbf{v}) + \xi_i\phi_i + n\phi + k\Delta\theta = Y', \tag{75}$$

$$-\vartheta_2(\nabla\cdot\mathbf{u}) - h_3(\nabla\cdot\mathbf{w}) + v_i\phi_i + \xi\phi + h\Delta P = Z', \tag{76}$$

where $\mathbf{H}'$, $\mathbf{V}'$, $\mathbf{W}'$ are three-component vector functions on E^3; X'_i, L', Y', Z' are scalar functions on E^3. The Eqs. (68)–(76) can also be expressed in the following form:

$$\mathbf{E}^{\mathrm{tr}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{Q}'(\mathbf{x}), \tag{77}$$

where $\mathbf{E}^{\mathrm{tr}}$ is the transpose of matrix $\mathbf{E}$, $\mathbf{Q}' = (\mathbf{H}', \mathbf{V}', \mathbf{W}', X'_{\mathrm{i}}, L', Y', Z')$, $\mathbf{x} \in E^3$. Applying operator $(\nabla\cdot)$ to the Eqs. (68)–(70), we obtain

$$\tilde{\lambda}\Delta(\nabla\cdot\mathbf{u}) - \tilde{\lambda}_i\Delta\phi_i - \wp\Delta\phi = \nabla\cdot\mathbf{H}', \tag{78}$$

$$[k_7\Delta - k_2](\nabla\cdot\mathbf{v}) - B_i\Delta\phi_i - E_1\Delta\phi + k_1\Delta\theta = \nabla\cdot\mathbf{V}', \tag{79}$$

$$[h_7\Delta - h_2](\nabla\cdot\mathbf{w}) - B_{i+3}\Delta\phi_i - E_2\Delta\phi + h_1\Delta P = \nabla\cdot\mathbf{W}'. \tag{80}$$

The Eqs. (71)–(74) and (78) may be expressed in the form

$$\mathbf{N}'(\Delta)\,\mathbf{S}' = \tilde{\mathbf{Q}}', \tag{81}$$

where $\mathbf{S}' = (\nabla \cdot\, \mathbf{u},\, \boldsymbol{\phi}, \phi)$, $\tilde{\mathbf{Q}} = (w_1',\, w_2',\, w_3',\, w_4', w_5') = (\nabla \cdot\, \mathbf{H}',\, X_i',\, L')$ and

$$\widehat{\mathbf{N}}\,(\Delta) = \left(\widehat{N}_{gz}\,(\Delta)\right)_{5\times 5} = \begin{pmatrix} \tilde{\lambda} & -\tilde{\lambda}_1 & -\tilde{\lambda}_2 & -\tilde{\lambda}_3 & -\wp \\ \tilde{\lambda}_1 & A_1\Delta - \zeta_1 & A_4\Delta - \zeta_4 & A_6\Delta - \zeta_6 & \delta_1 \\ \tilde{\lambda}_2 & A_4\Delta - \zeta_4 & A_2\Delta - \zeta_2 & A_5\Delta - \zeta_5 & \delta_2 \\ \tilde{\lambda}_3 & A_6\Delta - \zeta_6 & A_5\Delta - \zeta_5 & A_3\Delta - \zeta_3 & \delta_3 \\ \wp & \delta_1 & \delta_2 & \delta_3 & \gamma\Delta - \beta \end{pmatrix}_{5\times 5},$$

$$\mathbf{N}'(\Delta) = \left(N_{gz}'(\Delta)\right)_{5\times 5} = \Delta\left(\widehat{N}_{gz}\,(\Delta)\right)_{5\times 5}.$$

Eq. (81) can also be expressed in determinant form as:

$$\Gamma_3(\Delta)\,\mathbf{S}' = \boldsymbol{\Phi}, \tag{82}$$

where $\boldsymbol{\Phi} = (\Phi_1, \ldots,\, \Phi_5)$, $\Phi_j = \frac{1}{\tilde{C}}\sum_{i=1}^{5} M_{ij}^* w_i$, $\Gamma_3(\Delta) = \frac{|\mathbf{N}'(\Delta)|}{\tilde{C}}$, $\tilde{C} = \tilde{\lambda}\vartheta\gamma$, $j = 1, \ldots, 5$ and M_{ij}^* is the cofactor of the element N_{ij}' of the matrix $\mathbf{N}'$. On expanding $\Gamma_3(\Delta)$, we see that

$$\Gamma_3(\Delta) = \Delta \prod_{i=1}^{4}(\Delta + \tilde{\mu}_i^2),$$

where $\tilde{\mu}_i^2,\ i = 1, 2, 3, 4$ are the roots of the equation $\left|\widehat{\mathbf{N}}\,(-m)\right| = 0$ (with respect to m). From Eq. (75), we get $\Delta\theta = \frac{1}{k}[Y' + \vartheta_1(\nabla \cdot\, \mathbf{u}) + k_3(\nabla \cdot\, \mathbf{v}) - \xi_i\phi_i - n\phi]$. Using the above equation in Eq. (79) and then applying operator $\Gamma_3(\Delta)$ to the resulting equation, we get

$$\Gamma_3(\Delta)(\Delta + \tilde{\mu}_5^2)\nabla \cdot \mathbf{v} = \Phi_6,\ \tilde{\mu}_5^2 = \frac{k_1k_3 - k_2k}{kk_7}, \tag{83}$$

where $\Phi_6 = \frac{1}{kk_7}\left\{k\left[\left(B_i\Delta + \frac{\xi_i k_1}{k}\right)\Phi_{i+1} + \left(E_1\Delta + \frac{nk_1}{k}\right)\Phi_5 + \Gamma_3(\Delta)\nabla \cdot \mathbf{V}'\right] - k_1[\Gamma_3(\Delta)Y' + \vartheta_1\Phi_1]\right\}$. From Eq. (76), we get: $\Delta P = \frac{1}{h}[Z' + \vartheta_2(\nabla \cdot\, \mathbf{u}) + h_3(\nabla \cdot\, \mathbf{w}) - v_i\phi_i - \xi\phi]$. Using the above equation in Eq. (80) and then applying operator $\Gamma_3(\Delta)$ to the resulting equation, we get

$$\Gamma_3(\Delta)(\Delta + \tilde{\mu}_6^2)\nabla \cdot \mathbf{w} = \Phi_7,\ \tilde{\mu}_6^2 = \frac{h_1h_3 - h_2h}{hh_7}, \tag{84}$$

where $\Phi_7 = \frac{1}{hh_7}\left\{h\left[\left(B_{i+3}\Delta + \frac{v_ih_1}{h}\right)\Phi_{i+1} + \left(E_2\Delta + \frac{\xi h_1}{k}\right)\Phi_5 + \Gamma_3(\Delta)\nabla \cdot \mathbf{W}'\right] - h_1[\Gamma_3(\Delta)Z' + \vartheta_2\Phi_1]\right\}$. Applying operators $\Gamma_3(\Delta)(\Delta + \tilde{\mu}_5^2)$ and $\Gamma_3(\Delta)(\Delta + \tilde{\mu}_6^2)$ to Eqs. (75) and (76) and using Eqs. (83) and (84), we get

$$\Delta^2 \prod_{i=1}^{5}(\Delta + \tilde{\mu}_i^2)\theta = \Phi_8, \tag{85}$$

$$\Delta^2 \prod_{i=1}^{4,6}(\Delta + \tilde{\mu}_i^2)P = \Phi_9, \tag{86}$$

where

$$\Phi_8 = \frac{1}{k}\left[(\Delta + \tilde{\mu}_5^2)[\Gamma_3(\Delta)Y' + \vartheta_1\Phi_1 - \xi_i\Phi_{i+1} - n\Phi_5] + k_3\Phi_6\right],$$

$$\Phi_9 = \frac{1}{h}\left[(\Delta + \tilde{\mu}_6^2)[\Gamma_3(\Delta)Z' + \vartheta_2\Phi_1 - v_i\Phi_{i+1} - \xi\Phi_5] + h_3\Phi_7\right].$$

Applying operators $\Gamma_3(\Delta)$, $\Delta^2 \prod_{i=1}^{5}(\Delta + \tilde{\mu}_i^2)$ and $\Delta^2 \prod_{i=1}^{4,6}(\Delta + \tilde{\mu}_i^2)$ to Eqs. (68)–(70), respectively, and using (82)–(86), we obtain

$$\Delta\Gamma_3(\Delta)\mathbf{u} = \mathbf{\Phi}',$$
$$\Delta^2 \prod_{i=1}^{5,7}(\Delta + \tilde{\mu}_i^2)\mathbf{v} = \mathbf{\Phi}'', \ \tilde{\mu}_7^2 = -\frac{k_2}{k_6}\,,$$
$$\Delta^2 \prod_{i=1}^{4,6,8}(\Delta + \tilde{\mu}_i^2)\mathbf{w} = \mathbf{\Phi}''', \tilde{\mu}_8^2 = -\frac{h_2}{h_6}\,. \tag{87}$$

where

$$\mathbf{\Phi}' = \frac{1}{\mu}\Big[\Gamma_3(\Delta)\mathbf{H}' - (\lambda' + \mu)\nabla\Phi_1 + \tilde{\lambda}_i\nabla\Phi_{i+1} + \wp\nabla\Phi_5\Big],$$
$$\mathbf{\Phi}'' = \frac{1}{k_6}\left[\Delta^2 \prod_{i=1}^{5}(\Delta + \tilde{\mu}_i^2)\mathbf{V}' - \Delta(k_4 + k_5)\nabla\Phi_6 + \Delta(\Delta + \tilde{\mu}_5^2)(B_i\nabla\Phi_{i+1} + E_1\nabla\Phi_5) - k_1\nabla\Phi_8\right],$$
$$\mathbf{\Phi}''' = \frac{1}{h_6}\left[\Delta^2 \prod_{i=1}^{4,6}(\Delta + \tilde{\mu}_i^2)\mathbf{W}' - \Delta(h_4 + h_5)\nabla\Phi_7 + \Delta(\Delta + \tilde{\mu}_6^2)(B_{i+3}\nabla\Phi_{i+1} + E_2\nabla\Phi_5) - h_1\nabla\Phi_9\right].$$

From Eqs. (82) and (85)–(87), we get

$$\mathbf{\Lambda}(\Delta)\mathbf{U}(\mathbf{x}) = \widehat{\mathbf{\Phi}}\,(\mathbf{x}), \tag{88}$$

where

$$\widehat{\mathbf{\Phi}}\,(\mathbf{x}) = (\mathbf{\Phi}',\, \mathbf{\Phi}'', \mathbf{\Phi}''',\, \Phi_2, \Phi_3, \Phi_4,\, \Phi_5, \Phi_8,\, \Phi_9) \text{ and } \mathbf{\Lambda}(\Delta) = (\Lambda_{gz}(\Delta))_{15\times 15}\,,$$
$$\Lambda_{yy}(\Delta) = \Delta\Gamma_3(\Delta) = \Delta^2 \prod_{i=1}^{4}(\Delta + \tilde{\mu}_i^2), \ \Lambda_{y+3;y+3}(\Delta) = \Delta^2 \prod_{i=1}^{5,7}(\Delta + \tilde{\mu}_i^2),$$
$$\Lambda_{y+6;y+6}(\Delta) = \Delta^2 \prod_{i=1}^{4,6,8}(\Delta + \tilde{\mu}_i^2), \Lambda_{\ell\ell}(\Delta) = \Gamma_3(\Delta) = \Delta \prod_{i=1}^{4}(\Delta + \tilde{\mu}_i^2),$$
$$\Lambda_{14;14}(\Delta) = \Delta^2 \prod_{i=1}^{5}(\Delta + \tilde{\mu}_i^2),\ \Lambda_{15;15}(\Delta) = \Delta^2 \prod_{i=1}^{4,6}(\Delta + \tilde{\mu}_i^2),$$
$$\Lambda_{gz}(\Delta) = 0\ ;\ y = 1,2,3\ ;\ \ell = 10,\ldots,13\ ;\ g,z = 1,\ldots,15\ ;\ g \neq z.$$

The expressions for $\mathbf{\Phi}'$, $\mathbf{\Phi}''$, $\mathbf{\Phi}'''$, $\Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_8,\ \Phi_9$ can be rewritten as

$$\hat{\mathbf{\Phi}}(\mathbf{x}) = \mathbf{Z}^{\mathrm{tr}}(\mathbf{D_x})\mathbf{Q}'(\mathbf{x}), \tag{89}$$

where

$$\mathbf{Z}(\mathbf{D_x}) = (Z_{g\ell}(\mathbf{D_x}))_{15\times 15}\,,$$
$$Z_{ij}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma_3(\Delta)\delta_{ij} + m_{11}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},\ Z_{i;j+3}(\mathbf{D_x}) = m_{12}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$Z_{i;j+6}(\mathbf{D_x}) = m_{13}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},\ Z_{i;\hbar+8}(\mathbf{D_x}) = m_{1;\hbar+2}(\Delta)\frac{\partial}{\partial x_i},$$
$$Z_{i+3;j}(\mathbf{D_x}) = 0,\ Z_{i+3;j+3}(\mathbf{D_x}) = \frac{1}{k_6}\Delta^2 \prod_{i=1}^{5}(\Delta + \tilde{\mu}_i^2)\delta_{ij} + m_{22}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$Z_{i+3;j+6}(\mathbf{D_x}) = Z_{i+3;q}(\mathbf{D_x}) = 0,\ Z_{i+3;14}(\mathbf{D_x}) = m_{28}(\Delta)\frac{\partial}{\partial x_i},\ Z_{i+6;j}(\mathbf{D_x}) = 0,$$
$$Z_{i+6;j+3}(\mathbf{D_x}) = 0, Z_{i+6;j+6}(\mathbf{D_x}) = \frac{1}{h_6}\Delta^2 \prod_{i=1}^{4,6}(\Delta + \tilde{\mu}_i^2)\delta_{ij} + m_{33}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$Z_{i+6;e}(\mathbf{D_x}) = 0,\ Z_{i+6;15}(\mathbf{D_x}) = m_{39}(\Delta)\frac{\partial}{\partial x_i},\ Z_{y+8;i}(\mathbf{D_x}) = m_{y+2;1}(\Delta)\frac{\partial}{\partial x_i},$$

$$Z_{y+8;i+3}(\mathbf{D_x}) = m_{y+2;2}(\Delta)\frac{\partial}{\partial x_i},\ Z_{y+8;i+6}(\mathbf{D_x}) = m_{y+2;3}(\Delta)\frac{\partial}{\partial x_i},$$

$$Z_{y+8;\hbar+8}(\mathbf{D_x}) = m_{y+2;\hbar+2}(\Delta),\ Z_{14;i}(\mathbf{D_x}) = Z_{14;i+6}(\mathbf{D_x}) = Z_{14;q}(\mathbf{D_x}) = 0,\ Z_{14;i+3}(\mathbf{D_x}) = m_{82}(\Delta)\frac{\partial}{\partial x_i},$$

$$Z_{14;14}(\mathbf{D_x}) = m_{88}(\Delta), Z_{15;i}(\mathbf{D_x}) = Z_{15;i+3}(\mathbf{D_x}) = Z_{15;e}(\mathbf{D_x}) = 0, Z_{15;i+6}(\mathbf{D_x}) = m_{93}(\Delta)\frac{\partial}{\partial x_i},$$

$$Z_{15;15}(\mathbf{D_x}) = m_{99}(\Delta), m_{11}(\Delta) = -\frac{1}{\mu\tilde{C}}\left\{(\lambda'+\mu)M_{11}^*(\Delta) - \tilde{\lambda}_i M_{1;i+1}^*(\Delta) - \wp M_{15}^*(\Delta)\right\},$$

$$m_{12}(\Delta) = \frac{\Delta+\tilde{\mu}_7^2}{kk_7\tilde{C}}[-k_1\vartheta_1 M_{11}^*(\Delta) + (B_i k\Delta + k_1\xi_i)M_{1;i+1}^*(\Delta) + (E_1 k\Delta + k_1 n)M_{15}^*(\Delta)],$$

$$m_{13}(\Delta) = \frac{\Delta+\tilde{\mu}_8^2}{hh_7\tilde{C}}[-h_1\vartheta_2 M_{11}^*(\Delta) + (B_{i+3} h\Delta + h_1 v_i)M_{1;i+1}^*(\Delta) + (E_2 h\Delta + h_1\xi)M_{15}^*(\Delta)],$$

$$m_{1;y+2}(\Delta) = \frac{M_{1y}^*(\Delta)}{\tilde{C}},\ m_{18}(\Delta) = \frac{1}{kk_7\tilde{C}}\left[\begin{array}{l}(k_7\Delta - k_2)(\vartheta_1 M_{11}^*(\Delta) - \xi_i M_{1;i+1}^*(\Delta) - nM_{15}^*(\Delta)) + \\ +k_3\Delta(B_i M_{1;i+1}^*(\Delta) + E_1 M_{15}^*(\Delta))\end{array}\right],$$

$$m_{19}(\Delta) = \frac{1}{hh_7\tilde{C}}\left[\begin{array}{l}(h_7\Delta - h_2)(\vartheta_2 M_{11}^*(\Delta) - v_i M_{1;i+1}^*(\Delta) - \xi M_{15}^*(\Delta)) + \\ +h_3\Delta(B_{i+3} M_{1;i+1}^*(\Delta) + E_2 M_{15}^*(\Delta))\end{array}\right],$$

$$m_{2z}(\Delta) = 0,\ m_{22}(\Delta) = -\frac{\Gamma_3(\Delta)}{kk_6k_7}[(k_4+k_5)k\Delta + k_1k_3],\ m_{28}(\Delta) = \frac{k_3}{kk_7}\Gamma_3(\Delta),$$

$$m_{3\lambda}(\Delta) = 0,\ m_{33}(\Delta) = -\frac{\Gamma_3(\Delta)}{hh_6h_7}[(h_4+h_5)h\Delta + h_1h_3],\ m_{39}(\Delta) = \frac{h_3}{hh_7}\Gamma_3(\Delta),$$

$$m_{y+2;1}(\Delta) = -\frac{1}{\mu\tilde{C}}\left\{(\lambda'+\mu)M_{y1}^*(\Delta) - \tilde{\lambda}_i M_{y;i+1}^*(\Delta) - \wp M_{y5}^*(\Delta)\right\},$$

$$m_{y+2;2}(\Delta) = \frac{\Delta+\tilde{\mu}_7^2}{kk_7\tilde{C}}[-k_1\vartheta_1 M_{y1}^*(\Delta) + (B_i k\Delta + k_1\xi_i)M_{y;i+1}^*(\Delta) + (E_1 k\Delta + k_1 n)M_{y5}^*(\Delta)],$$

$$m_{y+2;3}(\Delta) = \frac{\Delta+\tilde{\mu}_8^2}{hh_7\tilde{C}}[-h_1\vartheta_2 M_{y1}^*(\Delta) + (B_{i+3} h\Delta + h_1 v_i)M_{y;i+1}^*(\Delta) + (E_2 h\Delta + h_1\xi)M_{y5}^*(\Delta)],$$

$$m_{y+2;r+2}(\Delta) = \frac{M_{yr}^*(\Delta)}{\tilde{C}},\ m_{y+2;8}(\Delta) = \frac{1}{kk_7\tilde{C}}\left[\begin{array}{l}(k_7\Delta - k_2)[\vartheta_1 M_{y1}^*(\Delta) - \xi_i M_{y;i+1}^*(\Delta) - nM_{y5}^*(\Delta)] + \\ +k_3\Delta(B_i M_{y;i+1}^*(\Delta) + E_1 M_{y5}^*(\Delta))\end{array}\right],$$

$$m_{y+2;9}(\Delta) = \frac{1}{hh_7\tilde{C}}\left[\begin{array}{l}(h_7\Delta - h_2)[\vartheta_2 M_{y1}^*(\Delta) - v_i M_{y;i+1}^*(\Delta) - \xi M_{y5}^*(\Delta)] + \\ +h_3\Delta(B_{i+3} M_{y;i+1}^*(\Delta) + E_2 M_{y5}^*(\Delta))\end{array}\right],$$

$$m_{8z}(\Delta) = m_{9\lambda}(\Delta) = 0,\ m_{82}(\Delta) = -\frac{k_1\Gamma_3(\Delta)(\Delta+\tilde{\mu}_7^2)}{kk_7},\ m_{88}(\Delta) = \frac{\Gamma_3(\Delta)(k_7\Delta - k_2)}{kk_7},$$

$$m_{93}(\Delta) = -\frac{h_1\Gamma_3(\Delta)(\Delta+\tilde{\mu}_8^2)}{hh_7},\ m_{99}(\Delta) = \frac{\Gamma_3(\Delta)(h_7\Delta - h_2)}{hh_7}\ ;\ i,j = 1,2,3\ ;$$

$$\hbar = 2,\dots,7\ ;\ q = 10,\dots,13,15\ ;\ e = 10,\dots,14\ ;\ y,r = 2,\dots,5\ ;\ z = 1,3,\dots,7,9\ ;\ \lambda = 1,2,4,\dots,8.$$

From Eqs. (77), (88) and (89), we get

$$\mathbf{E}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta). \tag{90}$$

Let

$$\mathbf{Y}'(\mathbf{x}) = \left(Y'_{ij}(\mathbf{x})\right)_{15\times 15},\ Y'_{zz}(\mathbf{x}) = r'_{11}\varsigma_1^*(\mathbf{x}) + r'_{12}\varsigma_2^*(\mathbf{x}) + \sum_{g=1}^{4} r'_{1;g+2}\ \widehat{\varsigma}_g\ (\mathbf{x}),$$

$$Y'_{z+3;z+3}(\mathbf{x}) = r'_{21}\varsigma_1^*(\mathbf{x}) + r'_{22}\varsigma_2^*(\mathbf{x}) + \sum_{g=1}^{5,7} r'_{2;g+2}\ \widehat{\varsigma}_g\ (\mathbf{x}),$$

$$Y'_{z+6;z+6}(\mathbf{x}) = r'_{31}\varsigma_1^*(\mathbf{x}) + r'_{32}\varsigma_2^*(\mathbf{x}) + \sum_{g=1}^{4,6,8} r'_{3;g+2}\ \widehat{\varsigma}_g\ (\mathbf{x}),$$

$$Y'_{\ell\ell}(\mathbf{x}) = r'_{41}\varsigma_1^*(\mathbf{x}) + \sum_{g=1}^{4} r'_{4;g+1}\ \widehat{\varsigma}_g\ (\mathbf{x}),$$

$$Y'_{14;14}(\mathbf{x}) = r'_{51}\varsigma_1^*(\mathbf{x}) + r'_{52}\varsigma_2^*(\mathbf{x}) + \sum_{g=1}^{5} r'_{5;g+2}\ \widehat{\varsigma}_g\ (\mathbf{x}),$$

$$Y'_{15;15}(\mathbf{x}) = r'_{61}\varsigma_1^*(\mathbf{x}) + r'_{62}\varsigma_2^*(\mathbf{x}) + \sum_{g=1}^{4,6} r'_{6;g+2}\ \widehat{\varsigma}_g\ (\mathbf{x}),$$

$$Y'_{ij}(\mathbf{x}) = 0\ ;\ z = 1,2,3\ ;\ i,j = 1,\ldots,15\ ;\ i \neq j\ ;\ \ell = 10,\ldots,13,$$

where

$$\widehat{\varsigma}_g\ (\mathbf{x}) = -\frac{e^{\iota\tilde{\mu}_g|\mathbf{x}|}}{4\pi\,|\mathbf{x}|}\ ;\ g = 1,\ldots,8,$$

$$r'_{11} = -\sum_{g=1}^{4}\left(\prod_{j=1,j\neq g}^{4}\tilde{\mu}_j^2\right)\prod_{i=1}^{4}\tilde{\mu}_i^{-4},\ r'_{12} = \prod_{i=1}^{4}\tilde{\mu}_i^{-2},\ r'_{1;y+2} = \tilde{\mu}_y^{-4}\prod_{i=1,i\neq y}^{4}\left(\tilde{\mu}_i^2 - \tilde{\mu}_y^2\right)^{-1},$$

$$r'_{21} = -\sum_{g=1}^{5,7}\left(\prod_{j=1,j\neq g}^{5,7}\tilde{\mu}_j^2\right)\prod_{i=1}^{5,7}\tilde{\mu}_i^{-4},\ r'_{22} = \prod_{i=1}^{5,7}\tilde{\mu}_i^{-2},\ r'_{2;\ell+2} = \tilde{\mu}_\ell^{-4}\prod_{i=1,i\neq \ell}^{5,7}\left(\tilde{\mu}_i^2 - \tilde{\mu}_\ell^2\right)^{-1},$$

$$r'_{31} = -\sum_{g=1}^{4,6,8}\left(\prod_{j=1,j\neq g}^{4,6,8}\tilde{\mu}_j^2\right)\prod_{i=1}^{4,6,8}\tilde{\mu}_i^{-4},\ r'_{32} = \prod_{i=1}^{4,6,8}\tilde{\mu}_i^{-2},\ r'_{3;z+2} = \tilde{\mu}_z^{-4}\prod_{i=1,i\neq z}^{4,6,8}\left(\tilde{\mu}_i^2 - \tilde{\mu}_z^2\right)^{-1},$$

$$r'_{41} = \prod_{i=1}^{4}\tilde{\mu}_i^{-2},\ r'_{4;y+1} = -\tilde{\mu}_y^{-2}\prod_{i=1,i\neq y}^{4}\left(\tilde{\mu}_i^2 - \tilde{\mu}_y^2\right)^{-1},$$

$$r'_{51} = -\sum_{g=1}^{5}\left(\prod_{j=1,j\neq g}^{5}\tilde{\mu}_j^2\right)\prod_{i=1}^{5}\tilde{\mu}_i^{-4},\ r'_{52} = \prod_{i=1}^{5}\tilde{\mu}_i^{-2},\ r'_{5;\hbar+2} = \tilde{\mu}_\hbar^{-4}\prod_{i=1,i\neq \hbar}^{5}\left(\tilde{\mu}_i^2 - \tilde{\mu}_\hbar^2\right)^{-1},$$

$$r'_{61} = -\sum_{g=1}^{4,6}\left(\prod_{j=1,j\neq g}^{4,6}\tilde{\mu}_j^2\right)\prod_{i=1}^{4,6}\tilde{\mu}_i^{-4},\ r'_{62} = \prod_{i=1}^{4,6}\tilde{\mu}_i^{-2},\ r'_{6;\lambda+2} = \tilde{\mu}_\lambda^{-4}\prod_{i=1,i\neq \lambda}^{4,6}\left(\tilde{\mu}_i^2 - \tilde{\mu}_\lambda^2\right)^{-1},$$

$$y = 1,\ldots,4\ ;\ \ell = 1,\ldots,5,7\ ;\ z = 1,\ldots,6,8\ ;\ \hbar = 1,\ldots,5\ ;\ \lambda = 1,\ldots,4,6.$$

Lemma 2. The matrix $\mathbf{Y}'$ is the fundamental matrix of the operator $\mathbf{\Lambda}(\Delta)$, i.e.

$$\mathbf{\Lambda}(\Delta)\mathbf{Y}'(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}). \tag{91}$$

Proof: To prove the lemma, it is sufficient to prove that

$$\begin{aligned}
\Delta\Gamma_3(\Delta)Y'_{11}(\mathbf{x}) &= \delta(\mathbf{x}),\\
\Delta\Gamma_3(\Delta)(\Delta+\tilde{\mu}_5^2)(\Delta+\tilde{\mu}_7^2)Y'_{44}(\mathbf{x}) &= \delta(\mathbf{x}),\\
\Delta\Gamma_3(\Delta)(\Delta+\tilde{\mu}_6^2)(\Delta+\tilde{\mu}_8^2)Y'_{77}(\mathbf{x}) &= \delta(\mathbf{x}),\\
\Gamma_3(\Delta)Y'_{10;10}(\mathbf{x}) &= \delta(\mathbf{x}),\\
\Delta\Gamma_3(\Delta)(\Delta+\tilde{\mu}_5^2)Y'_{14;14}(\mathbf{x}) &= \delta(\mathbf{x}),\\
\Delta\Gamma_3(\Delta)(\Delta+\tilde{\mu}_6^2)Y'_{15;15}(\mathbf{x}) &= \delta(\mathbf{x}).
\end{aligned} \tag{92}$$

It is much easier to prove the system of Eqs. (92). It has been left for the reader.

We introduce the matrix

$$\mathbf{G}'(\mathbf{x}) = \mathbf{Z}(\mathbf{D_x})\mathbf{Y}'(\mathbf{x}), \tag{93}$$

From Eqs. (90), (91) and (93), we obtain: $\mathbf{E}(\mathbf{D_x})\,\mathbf{G}'(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x})$. Hence, $\mathbf{G}(\mathbf{x})$ is a solution to Eq. (67).

Theorem 4. If the condition (29) is satisfied, then the matrix $\mathbf{G}'(\mathbf{x})$ defined by the Eq. (93) is the fundamental solution of the system of equations (66).

VII. Conclusions

This manuscript presents the development of a linear theory for micromorphic thermoelastic diffusion materials, incorporating advanced features such as microtemperatures, microconcentrations, and triple porosity. Fundamental solutions have been derived for two key systems of equations: one for steady oscillations (Eq. (28)), expressed in terms of elementary functions, and another for equilibrium conditions (Eq. (66)). These solutions enable:

1. Comprehensive analysis of three-dimensional boundary value problems within this theoretical framework, utilizing the potential method.
2. Development of integral representation expressions for regular solutions of the governing equations.
3. Numerical resolution of boundary value problems through the boundary element method.
4. Construction of Green's functions for specific cases in 3D domains.

By incorporating triple porosity, microtemperatures, and microconcentrations, the proposed theory offers a deeper understanding of thermoelastic behavior in complex materials. This holistic approach equips researchers and engineers with tools to model and analyze materials with greater precision. The theoretical advancements provide practical insights for designing and optimizing materials, facilitating their application in diverse engineering and scientific fields.

In conclusion, the contributions of this work establish a robust foundation for advancing the field of thermoelasticity, offering a more comprehensive theoretical framework and practical methodologies for analyzing materials with intricate thermoelastic properties.

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 DOI:10.12921/cmst.2025.0000001