CMST 29(1–4) 17–20 (2023) DOI:10.12921/cmst.2023.0000008

Numerical Determination of a Certain Mathematical Constant Related to the Möbius Function

M. Wolf

Cardinal Stefan Wyszynski University
Faculty of Mathematics and Natural Sciences
ul. Wóycickiego 1/3, PL-01-938 Warsaw, Poland
E-mail: m.wolf@uksw.edu.pl

Received: 18 April 2023; revised: 16 May 2023; accepted: 17 May 2023; published online: 7 June 2023

Abstract: We calculated numerically the value of some constant which can be regarded as an analogue of the Euler-Mascheroni constant.

Key words: Möbius function, convergent and divergent series, Stjelties constants

---

There exists a beautiful example of the series which is conditionally convergent to zero but absolutely divergent to infinity. Namely, let $\mu(n)$ denote the Möbius function:

$$\mu(n) = \begin{cases} 1, & \text{for } n=1, \\ 0, & \text{when } n \text{ is divisible by the square} \\ & \text{of some prime } p\text{: } p^2|n, \\ (-1)^r, & \text{when } n = p_1 p_2 \ldots p_r \ . \end{cases} \tag{1}$$

For example, $\mu(14) = 1, \mu(25) = 0, \mu(30) = -1$. Then we have:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0, \tag{2}$$

while the absolute series diverges to infinity

$$\lim_{x\to\infty} \sum_{n<x} \frac{|\mu(n)|}{n} = \infty. \tag{3}$$

Because

$$|\mu(n)| = \mu^2(n), \tag{4}$$

we can write the last equation as

$$\lim_{x\to\infty} \sum_{n<x} \frac{\mu^2(n)}{n} = \infty. \tag{5}$$

The definition (1) of $\mu(n)$ stems from the formula for the Dirichlet series for the reciprocal of the Riemann zeta function:

$$\frac{1}{\zeta(s)} = \prod_{p=2,3,5,7,\ldots} \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \qquad s = \sigma + \mathrm{i}t, \quad \Re[s] = \sigma > 1, \tag{6}$$

where the zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=2,3,5,7,\ldots} \frac{1}{\left(1 - \frac{1}{p^s}\right)}\ ,\ \sigma > 1. \tag{7}$$

For $s = 1$ we have divergent harmonic series and hence $1/\zeta(1) = 0$ in accordance with (2). To see that the above equality (6) really holds one needs first to recognize in the terms $1/\left(1 - \frac{1}{p^s}\right)$ the sums of the geometric series $\sum_{k=0}^{\infty} \frac{1}{p^{ks}}$. The geometrical series converges absolutely so the interchange of summation and the product are justified. Finally, the fundamental theorem of arithmetic stating that each positive integer $n > 1$ can be represented in exactly one way (up to the order of the factors) as a product of prime powers:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = \prod_{i=1}^{k} p_i^{\alpha_i}\ , \tag{8}$$

has to be invoked to obtain the series on lhs of (6). Let us note that on the rhs of (7) the product cannot start from $p = 1$ and it explains why the first prime is 2 and not 1 – physicists often think that 1 is a prime number (before the 19^{th} century 1 was indeed considered to be a prime).

The series (2) is not absolutely convergent hence its value depends on the order of summation. In fact, famous (and curious) Riemann's rearrangement theorem, see e.g. [10, Theorem 3.54], asserts that terms of a conditionally convergent infinite series can be permuted in such a way that the new series converges to *any given value*. Hence zero on rhs of (2) can be replaced by an arbitrary real number after permuting terms of the series.

It should be noted that (2) is equivalent to the Prime Number Theorem (PNT) which states that the number $\pi(x)$ of prime numbers $< x$ is given by

$$\pi(x) = (1 + o(1)) \int_2^x \frac{du}{\log(u)} \,, \tag{9}$$

for details see [1, Theorem 4.16; 2, p. 107]. Let us also note the sum [4, p. 400]

$$\sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^2} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^2} = \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2} = 1.519817754635\ldots \tag{10}$$

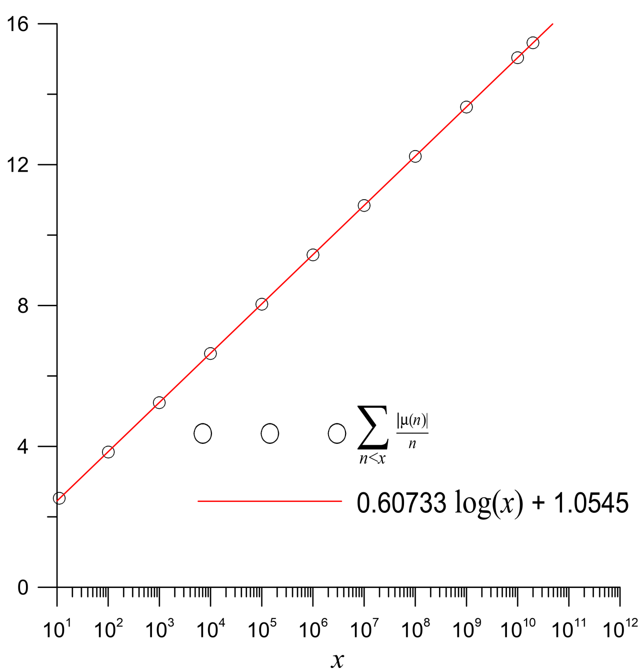

Fig. 1. The plot of values of $\sum_{n<x} \frac{|\mu(n)|}{n}$ captured at $x = 10, 100, \ldots 10^{10}, 2 \times 10^{10}$. For the fit the first two values of x were skipped

We will give heuristic arguments on how the sum (3) grows to infinity. The harmonic series diverges to infinity like [10, p. 197]

$$\lim_{x\to\infty} \left(\sum_{n<x} \frac{1}{n} - \log(x) \right) = \gamma = 0.57721566490153286\ldots, \tag{11}$$

where γ is called Euler-Mascheroni constant. It is not known whether γ is irrational, see [11]. Let $Q(x)$ denote the number of square-free integers n between 1 and x, i.e. integers n for which $\mu(n) \neq 0$. It is known that

$$Q(x) = \frac{6x}{\pi^2} + \mathcal{O}(\sqrt{x}), \tag{12}$$

see [5, Theorem 333]. Thus the probability of choosing a square-free number is roughly

$$\frac{6}{\pi^2} = 0.607927101\ldots \tag{13}$$

Because in (3) different from zero are only terms with n square-free we expect that for large x:

$$\sum_{n<x} \frac{|\mu(n)|}{n} \approx \frac{6}{\pi^2} \sum_{n<x} \frac{1}{n}. \tag{14}$$

Thus we define a constant γ_M by analogy with the Euler-Mascheroni constant by the limit

$$\gamma_M = \lim_{x\to\infty} \left(\sum_{n<x} \frac{|\mu(n)|}{n} - \frac{6}{\pi^2} \log(x) \right), \tag{15}$$

and we expect the behavior

$$\sum_{n<x} \frac{|\mu(n)|}{n} \approx \frac{6}{\pi^2} \log(x) + \frac{6}{\pi^2} \gamma. \tag{16}$$

Using the free software Pari [9] during about 30 hours on CPU with 3.9 GHz we calculated the values of $\sum_{n<x} \frac{|\mu(n)|}{n}$ up to $x = 2 \times 10^{10}$. The results are presented in Fig.1. There is a perfect straight line on the semi-logarithmic plot in accordance with the Eq. (16). However, the intercept is $1.0545\ldots$, while from the above considerations we expect that γ_M should be close to $6\gamma/\pi^2 = 0.35091\ldots$ The point is that for initial values of n, which have largest contribution, there is an inequality:

$$\sum_{n<x} \frac{|\mu(n)|}{n} > \frac{6}{\pi^2} \sum_{n<x} \frac{1}{n} \,, \tag{17}$$

see Fig.2. The surplus is about 0.662. The values of

$$\gamma_M(x) = \sum_{n<x} \frac{|\mu(n)|}{n} - \frac{6}{\pi^2} \log(x), \tag{18}$$

obtained numerically are present in the table:

x	$\gamma_M(x)$
10^6	1.0438928513
10^7	1.0438962450
10^8	1.0438943420
10^9	1.0438945405
10^{10}	1.0438945080
2×10^{10}	1.0438945070

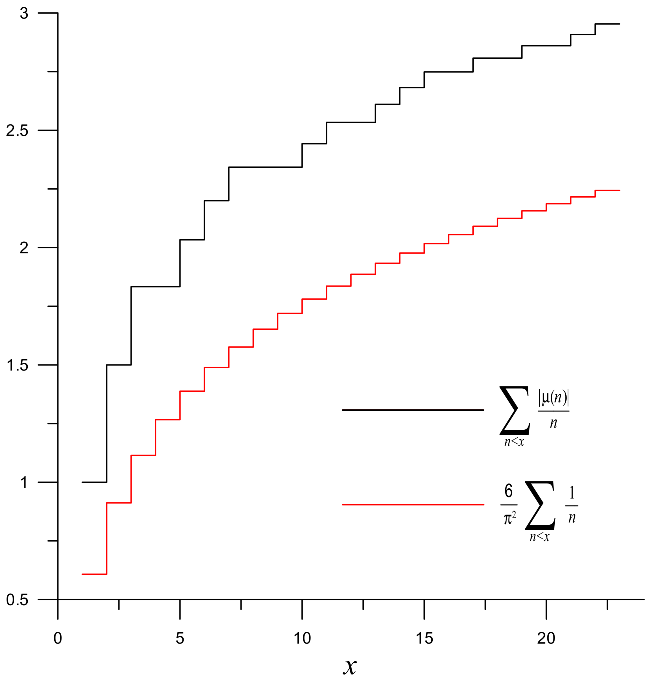

Fig. 2. The plot of values of $\sum_{n<x} \frac{|\mu(n)|}{n}$ and harmonic series or initial values $n = 1, 2, 3, \ldots, 20$

It seems that the last 7 digits for γ_M can be trusted. We searched for 1.04389 in *The On-Line Encyclopedia of Integer Sequences* at link[1] without success – it seems that this constant $1.04389\ldots$ was not determined in the past.

Finally, let us note a possible generalization. The Euler-Mascheroni constant is a zeroth element of the sequence of the Stieltjes constants defined by [6, p. 118]

$$\gamma_n = \lim_{x\to\infty}\left[\left(\sum_{k<x}\frac{(\log(k))^n}{k}\right) - \frac{(\log(x))^{n+1}}{n+1}\right]. \qquad (19)$$

(When $n = 0$ the numerator of the fraction in the first summand in (19) is formally 0^0 which is taken to be 1.) Thus we can introduce a family of constants involving the Möbius function by the formula:

$$\gamma_n^M = \lim_{x\to\infty}\left[\left(\sum_{k<x}\frac{|\mu(k)|(\log(k))^n}{k}\right) - \frac{6}{\pi^2}\frac{(\log(x))^{n+1}}{n+1}\right]. \qquad (20)$$

Note added in the Proof:

Artur Kawalec noted that it is possible to obtain a closed formula for γ_M. Namely, the original definition of the Stieltjes constants is as the coefficients of the Laurent expansion of the zeta function $\zeta(s)$ around its only simple pole at $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n\,(s-1)^n. \qquad (21)$$

From the above formula γ_n are given by the derivatives of $\zeta(s)$ or by appropriate contour integrals following from the Cauchy integral formula. However, the formula (19) can be shown to hold as an alternative expression for γ_n, see e.g. [3][2]. In [12, Eq. (1.2.7)] we found the Dirichlet series:

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s}\,, \Re(s) > 1. \qquad (22)$$

The above fraction of zetas has its own Laurent series

$$\frac{\zeta(s)}{\zeta(2s)} = \frac{6}{\pi^2(s-1)} + \gamma_0^M - \gamma_1^M(s-1) + \ldots \qquad (23)$$

Assuming that similarity to the relationship between usual Stieltjes constants the above constants $\gamma_0^M = \gamma^M$ and $\gamma_1^M, \ldots$ have the representation (20) and developing into series $\zeta(2s)$ for s close to 1

$$\zeta(2s) = \zeta(2) + 2\zeta'(2)(s-1) + \ldots, \qquad (24)$$

we obtain

$$\frac{\zeta(s)}{\zeta(2s)} = \frac{6}{\pi^2(s-1)} + \frac{6\gamma_0}{\pi^2} - \frac{12\zeta'(2)}{\pi^2\zeta(2)} + \mathcal{O}(s-1). \qquad (25)$$

Then from (23) for $s \to 1$ we obtain:

$$\gamma_0^M = \frac{6\gamma_0}{\pi^2} - \frac{72\zeta'(2)}{\pi^4}\,. \qquad (26)$$

Using for the derivative of zeta, for example, the formula (37) from [13] we obtain

$$\gamma_0^M = 1.043894515711938297404563 4\ldots \qquad (27)$$

The Eq. (26) suggests that γ_0^M can be transcendental because π is transcendental. Let us note that it is not even known whether the Euler-Mascheroni constant γ_0 is irrational, see [7, 11]. It is known that if γ_0 is rational and equal to a simple fraction p/q than $q > 10^{242080}$, see [6, p. 97]. About the irrationality of Stieltjes constants see also [8].

References

[1] T.M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York (2010). Undergraduate Texts in Mathematics.

[2] R.G. Ayoub, *An Introduction to the Analytic Theory of Numbers*, AMS (2006).

[3] W.E. Briggs, S. Chowla, *The power series coefficients of* $\zeta(s)$, The American Mathematical Monthly **62**(5), 323–325 (1955).

[4] J.B. Christopher, *The asymptotic density of some k-dimensional sets*, American Mathematical Monthly **63**, 399 (1956).

[5] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Science Publications (1980).

[1] https://oeis.org/

[2] We thank K. Maślanka for bringing this paper to our attention.

[6] J. Havil, *Gamma: Exploring Euler's Constant*, Princeton University Press, Princeton, NJ (2003).

[7] J.C. Lagarias, *Euler's constant: Euler's work and modern developments*, Bulletin of the American Mathematical Society **50**(4), 527–628 (2013).

[8] K. Maślanka, M. Wolf, *Are the Stieltjes constants irrational? Some computer experiments*, Computational Methods in Science and Technology **26**(3), 77–87 (2020).

[9] PARI/GP, version 2.3.0, 64 bits (2018). Available from http://pari.math.u-bordeaux.fr/.

[10] W. Rudin, *Principles of mathematical analysis*, McGraw-Hill Book Co., New York, 3rd ed. (1976). International Series in Pure and Applied Mathematics.

[11] J. Sondow, *Criteria for Irrationality of Euler's Constant*, Proceedings of the American Mathematical Society **131**(11), 3335–3345 (2003).

[12] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function*, The Clarendon Press Oxford University Press, New York, 2nd ed. (1986). Edited and with a preface by D.R. Heath-Brown.

[13] M. Wolf, *Some remarks on the Báez-Duarte criterion for the Riemann Hypothesis*, Computational Methods in Science and Technology **20**(2), 39–47 (2014).

Marek Wolf obtained PhD degree in Physics (1982) and habilitation (1993) at the Wroclaw University, where he was employed till 2011. Since 1984 he has been doing computer experiments in physics and mathematics. In 1991 and 1993 he was a research fellow at the Center for Polymer Studies at the Boston University. Now he is a member of the Faculty of Mathematics and Natural Sciences, College of Sciences at the Cardinal Stefan Wyszynski University in Warsaw. In April 2022 he became a grandfather. His hobby is photography, jazz and classic music.