THE THEORY OF THERMOELASTICITY WITH DOUBLE POROSITY AND MICROTEMPERATURES

Tarun Kansal

Department of Mathematics, M.N.College, Shahabad(M.)-136135, India

Abstract

The aim of the paper is to establish the basic governing equations for anisotropic thermoelastic medium with double porosity and microtemperatures and to construct the fundamental solution of system of equations in cases of steady, pseudo-, quasi-static oscillations and equilibrium.

Keywords: Thermoelasticity; Double Porosity; Steady oscillations.

INTRODUCTION

Grot[1] extended the theory of thermodynamics of elastic bodies with microstructure with the assumption that the microelements have different temperatures. He modified Clausius-Duhem inequality to include microtemperatures and added first-order moment of energy equations to the basic balance laws for determining the microtemperatures of a continuum. Iesan and Quintanilla[2] constructed a linear theory for elastic materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures. They proved an existence theorem for initial boundary value problems via the semigroup theory and established the continuous dependence of solutions of the initial data and body loads. Iesan[3] established the field equations of a theory of microstretch thermoelastic bodies with microtemperatures. He proved a uniqueness theorem in the dynamic theory of anisotropic materials. Iesan[4] derived a linear theory of microstretch elastic solids with microtemperatures in which a microelement of a continuum is equipped with the mechanical degrees of freedom for rigid rotations and microdilatation in addition to the classical translation degrees of freedom. He also established a uniqueness result in the dynamical theory of anisotropic bodies.

E-mail address: tarun1_kansal@yahoo.co.in

The double porosity model represents a double porous structure, one is macro porosity which is connected to pores and other is micro porosity which is connected to fissures. Wilson and Aifantis [5] developed the theory for deformable materials with double porosity. Iesan and Quintanilla[6] derived a non-linear theory of thermoelastic solids with double porosity structure. They also linearized the above theory and formulated the basic initial-boundary-value problems. Kansal[7-9] developed linear generalized theories of thermoelastic diffusion and micropolar thermoelastic diffusion with double porosity and constructed fundamental solutions of system of equations in case of steady oscillations. Svandaze and his co-workers[10-15] have also constructed fundamental solutions in the theory of thermoelasticity with double porosity as well as microtemperatures. Recently Kansal[16] developed fundamental solutions of a system of equations of isotropic micromorphic thermoelastic diffusion materials with microtemperatures and microconcentrations in case of steady oscillations in terms of elementary functions.

In the section 2, the constitutive relations, field equations for anisotropic thermoelastic bodies with double porosity and microtemperatures are derived. The system of linearlized equations of steady, peudo-, quasi-static oscillations and equilibrium in the theory of thermoelastic solids with double porosity and microtemperatures are obtained in section 3. In sections 4 and 5, in terms of elementary functions, the fundamental solutions of basic governing equations in cases of steady, peudo-, quasi-static oscillations and equilibrium are constructed. Finally some basic properties of fundamental matrix in case of steady oscillations are discussed in section 6.

1. BASIC EQUATIONS

Following[2,6,7], the balance of linear momentum, the balance of energy and the balance of first moment of energy are given by

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i, \tag{1}$$

$$\rho \dot{U} = \sigma_{ji}\dot{e}_{ji} + \Omega_i \dot{\nu}_{1,i} + \chi_i \dot{\nu}_{2,i} - q_{i,i} - \xi \dot{\nu}_1 - \zeta \dot{\nu}_2, \tag{2}$$

$$\rho \dot{\varepsilon}_i = -Q_{ji,j} - q_i + Q_i, \tag{3}$$

where ξ and ζ satisfy the relations

$$\Omega_{i,i} + \xi + \rho \tilde{g} = \rho k_1 \ddot{\nu}_1, \tag{4}$$

$$\chi_{i,i} + \zeta + \rho \tilde{l} = \rho k_2 \ddot{\nu}_2. \tag{5}$$

Here U is the internal energy per unit mass, ρ is the density, q_i is the heat flux vector, Q_{ij} is the first heat flux moment tensor, Q_i is the micro heat flux average vector, u_i are the components of the displacement vector $\boldsymbol{u}$, F_i are the components of the external forces per unit mass, ε_i is the first moment of energy vector, $\sigma_{ij}(=\sigma_{ji})$ are the components of stress tensor, $e_{ij}(=e_{ji})=\frac{1}{2}(u_{i,j}+u_{j,i})$ are components of strain tensor, ν_1 and ν_2 are the volume fraction fields corresponding to pores and fissures respectively, k_1 and k_2 are coefficients of equilibrated inertia, $\tilde{g}$ and $\tilde{l}$ are, respectively, extrinsic equilibrated body forces per unit mass associated to macro pores and fissures, Ω_i, χ_i are respectively the components of equilibrated stress vectors corresponding to ν_1, ν_2.

The local form of the principle of entropy can be expressed in the form of an inequality called Clausius-Duhem inequality

$$\rho\dot{S}+\frac{q_{i,i}}{T}-\frac{q_i}{T^2}T_{,i}+\frac{Q_{ij,i}}{T}T_j-\frac{Q_{ij}}{T^2}T_{,i}T_j+\frac{Q_{ij}}{T}T_{j,i}\ \geq\ 0. \tag{6}$$

where S is entropy per unit mass, T is absolute temperature, T_i is the microtemperature vector.

In view of equations (3) and (6), the balance of energy (2) reduces to

$$\rho[T\dot{S}-\dot{U}-T_i\dot{\varepsilon}_i]+\sigma_{ij}\dot{e}_{ij}+\Omega_i\dot{\nu}_{1,i}+\chi_i\dot{\nu}_{2,i}-\xi\dot{\nu}_1-\zeta\dot{\nu}_2-\frac{q_i}{T}T_{,i}-\frac{Q_{ij}}{T}T_{,i}T_j+Q_{ij}T_{j,i}-(q_i-Q_i)T_i\ \geq\ 0. \tag{7}$$

If we introduce Helmholtz free energy function Γ as

$$\Gamma=U+T_i\varepsilon_i-TS, \tag{8}$$

Then relation (7) becomes

$$-\rho[\dot{\Gamma}+\dot{T}S-\dot{T}_i\varepsilon_i]+\sigma_{ij}\dot{e}_{ij}+\Omega_i\dot{\nu}_{1,i}+\chi_i\dot{\nu}_{2,i}-\xi\dot{\nu}_1-\zeta\dot{\nu}_2-\frac{q_i}{T}T_{,i}-\frac{Q_{ij}}{T}T_{,i}T_j+Q_{ij}T_{j,i}-(q_i-Q_i)T_i\ \geq\ 0. \tag{9}$$

The function Γ can be expressed in terms of independent variables $e_{ij},\nu_1,\nu_{1,i},\nu_2,\nu_{2,i},T,T_{,i},T_i$ and $T_{i,j}$. Therefore, we have

$$\begin{aligned}\dot{\Gamma}=\frac{\partial\Gamma}{\partial e_{ij}}\dot{e}_{ij}+\frac{\partial\Gamma}{\partial\nu_1}\dot{\nu}_1+\frac{\partial\Gamma}{\partial\nu_{1,i}}\dot{\nu}_{1,i}+\frac{\partial\Gamma}{\partial\nu_2}\dot{\nu}_2+\\ \frac{\partial\Gamma}{\partial\nu_{2,i}}\dot{\nu}_{2,i}+\frac{\partial\Gamma}{\partial T}\dot{T}+\frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i}+\frac{\partial\Gamma}{\partial T_i}\dot{T}_i+\frac{\partial\Gamma}{\partial T_{i,j}}\dot{T}_{i,j}.\end{aligned} \tag{10}$$

Equation (9) with the help of equation (10) becomes

$$[\sigma_{ij}-\rho\frac{\partial\Gamma}{\partial e_{ij}}]\dot{e}_{ij}+[\Omega_i-\rho\frac{\partial\Gamma}{\partial\nu_{1,i}}]\dot{\nu}_{1,i}+[\chi_i-\rho\frac{\partial\Gamma}{\partial\nu_{2,i}}]\dot{\nu}_{2,i}-[\xi+\rho\frac{\partial\Gamma}{\partial\nu_1}]\dot{\nu}_1$$

$$-[\zeta+\rho\frac{\partial\Gamma}{\partial\nu_2}]\dot{\nu}_2-\rho[S+\frac{\partial\Gamma}{\partial T}]\dot{T}+\rho[\varepsilon_i-\frac{\partial\Gamma}{\partial T_i}]\dot{T}_i-\rho\frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i}-\rho\frac{\partial\Gamma}{\partial T_{i,j}}\dot{T}_{i,j}$$

$$-\frac{q_i}{T}T_{,i}-\frac{Q_{ij}}{T}T_{,i}T_j+Q_{ij}T_{j,i}-(q_i-Q_i)T_i\ \geq\ 0.$$

The inequality should be convinced for all rates $\dot{e}_{ij},\dot{\nu}_1,\dot{\nu}_{1,i},\dot{\nu}_2,\dot{\nu}_{2,i},\dot{T},\dot{T}_{,i},\dot{T}_i$ and $\dot{T}_{i,j}$. Hence the coefficients of above variables must vanish, that is

$$\sigma_{ij}=\rho\frac{\partial\Gamma}{\partial e_{ij}},\Omega_i=\rho\frac{\partial\Gamma}{\partial\nu_{1,i}},\chi_i=\rho\frac{\partial\Gamma}{\partial\nu_{2,i}},\xi=-\rho\frac{\partial\Gamma}{\partial\nu_1},$$

$$\zeta=-\rho\frac{\partial\Gamma}{\partial\nu_2},S=-\frac{\partial\Gamma}{\partial T},\varepsilon_i=\frac{\partial\Gamma}{\partial T_i},\tag{11}$$

$$\frac{\partial\Gamma}{\partial T_{,i}}=\frac{\partial\Gamma}{\partial T_{i,j}}=0,\tag{12}$$

$$-q_iT_{,i}-Q_{ij}T_{,i}T_j+TQ_{ij}T_{j,i}-T(q_i-Q_i)T_i\ \geq\ 0.\tag{13}$$

Let us introduce the notations

$$\phi=\nu_1-(\nu_1)_0,\quad\psi=\nu_2-(\nu_2)_0,\quad\theta=T-T_0,\tag{14}$$

where T_0 is the reference temperature of the body chosen such that $|\frac{\theta}{T_0}|\ll 1$, $(\nu_1)_0$ and $(\nu_2)_0$ are the volume fraction fields in reference configuration.

In the linear theory of materials possessing a centre of symmetry, we can take Γ in the form

$$2\rho\Gamma=c_{ijpn}e_{ij}e_{pn}+d^*\phi^2+f\psi^2-\frac{a\theta^2}{T_0}+q_{ij}\phi_{,i}\phi_{,j}+f_{ij}\psi_{,i}\psi_{,j}-s_{ij}T_iT_j$$

$$+2p_{ij}e_{ij}\phi+2\gamma_{ij}e_{ij}\psi-2a_{ij}e_{ij}\theta+2\alpha_{ij}\phi_{,i}\psi_{,j}-r_{ij}\phi_{,i}T_j-d_{ij}\psi_{,i}T_j$$

$$+2\alpha_1\phi\psi-2\gamma_1\phi\theta-2\gamma_2\psi\theta.\tag{15}$$

From equation (11), it follows that

$$\sigma_{ij}=c_{ijpn}e_{pn}+p_{ij}\phi+\gamma_{ij}\psi-a_{ij}\theta,$$

$$\Omega_i=q_{ij}\phi_{,j}+\alpha_{ij}\psi_{,j}-r_{ij}T_j,$$

$$\chi_i=\alpha_{ij}\phi_{,j}+f_{ij}\psi_{,j}-d_{ij}T_j,$$

$$\xi=-p_{ij}e_{ij}-d^*\phi-\alpha_1\psi+\gamma_1\theta,$$

$$\zeta=-\gamma_{ij}e_{ij}-\alpha_1\phi-f\psi+\gamma_2\theta,$$

$$\rho S = a_{ij}e_{ij} + \gamma_1\phi + \gamma_2\psi + \frac{a\theta}{T_0},$$
$$\rho\varepsilon_i = -s_{ij}T_j - r_{ij}\phi_{,j} - d_{ij}\psi_{,j}. \tag{16}$$

The linear expressions for q_i, Q_i and Q_{ij} are

$$q_i = -[k_{ij}\theta_{,j} + \kappa_{ij}T_j],$$
$$Q_i = (K_{ij} - k_{ij})\theta_{,j} + (-\kappa_{ij} + L_{ij})T_j,$$
$$Q_{ij} = m_{ijpn}T_{n,p}\ . \tag{17}$$

The linearized form of equation (6) is

$$\rho T_0\dot{S} = -q_{i,i}. \tag{18}$$

In view of equations (16) and (17), equations (1), (3)-(5) and (18) become

$$c_{ijpn}e_{pn,j} + p_{ij}\phi_{,j} + \gamma_{ij}\psi_{,j} - a_{ij}\theta_{,j} + \rho F_i = \rho\ddot{u}_i,$$
$$-p_{ij}e_{ij} + q_{ij}\phi_{,ij} - d^*\phi + \alpha_{ij}\psi_{,ij} - \alpha_1\psi + \gamma_1\theta - r_{ij}T_{j,i} + \rho\tilde{g} = \rho k_1\ddot{\phi},$$
$$-\gamma_{ij}e_{ij} + \alpha_{ij}\phi_{,ij} - \alpha_1\phi + f_{ij}\psi_{,ij} - f\psi + \gamma_2\theta - d_{ij}T_{j,i} + \rho\tilde{l} = \rho k_2\ddot{\psi}.$$
$$T_0[a_{ij}\dot{e}_{ij} + \gamma_1\dot{\phi} + \gamma_2\dot{\psi}] + a\dot{\theta} = k_{ij}\theta_{,ij} + \kappa_{ij}T_{j,i}$$
$$m_{jipn}T_{n,pj} - s_{ij}\dot{T}_j - r_{ij}\dot{\phi}_{,j} - d_{ij}\dot{\psi}_{,j} = K_{ij}\theta_{,j} + L_{ij}T_j\ . \tag{19}$$

In the case of an isotropic and homogeneous material, the constitutive equations become

$$\sigma_{ij} = \lambda e_{pp}\delta_{ij} + 2\mu e_{ij} - \beta\theta\delta_{ij} + p_1\phi\delta_{ij} + p_2\psi\delta_{ij},$$
$$\Omega_i = t_1\phi_{,i} + r_1\psi_{,i} - r_2T_i, \chi_i = r_1\phi_{,i} + t_2\psi_{,i} - r_3T_i,$$
$$\xi = -p_1e_{pp} - d^*\phi - \alpha_1\psi + \gamma_1\theta, \zeta = -p_2e_{pp} - \alpha_1\phi - f\psi + \gamma_2\theta,$$
$$\rho S = \beta e_{pp} + \gamma_1\phi + \gamma_2\psi + \frac{a\theta}{T_0}, \rho\varepsilon_i = -\alpha T_i - r_2\phi_{,i} - r_3\psi_{,i},$$
$$q_i = -[k\theta_{,i} + \kappa_1T_i], Q_i = (\kappa_3 - k)\theta_{,i} + (-\kappa_1 + \kappa_2)T_i,$$
$$Q_{ij} = \kappa_4T_{p,p}\delta_{ij} + \kappa_5T_{i,j} + \kappa_6T_{j,i}, \tag{20}$$

where

$$c_{ijpn} = \lambda\delta_{ij}\delta_{pn} + \mu\delta_{ip}\delta_{jn} + \mu\delta_{in}\delta_{jp}, a_{ij} = \beta\delta_{ij}, p_{ij} = p_1\delta_{ij}, \gamma_{ij} = p_2\delta_{ij},$$
$$q_{ij} = t_1\delta_{ij}, \alpha_{ij} = r_1\delta_{ij}, r_{ij} = r_2\delta_{ij}, f_{ij} = t_2\delta_{ij}, d_{ij} = r_3\delta_{ij}, s_{ij} = \alpha\delta_{ij},$$
$$m_{ijpn} = \kappa_4\delta_{ij}\delta_{pn} + \kappa_6\delta_{ip}\delta_{jn} + \kappa_5\delta_{in}\delta_{jp}, k_{ij} = k\delta_{ij}, \kappa_{ij} = \kappa_1\delta_{ij}, L_{ij} = \kappa_2\delta_{ij}, K_{ij} = \kappa_3\delta_{ij}.$$

Here $\lambda, \mu, \beta, p_1, p_2, t_1, t_2, r_1, r_2, r_3, d^*, \alpha_1, \gamma_1, \gamma_2, f, \alpha, a, k, \kappa_1, ..., \kappa_6$ are material constants.

Therefore from equation (19), we obtain the basic governing equations for homogeneous isotropic thermoelastic material with double porosity and microtemperatures in the absence of body and equilibrated body forces as

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + p_1\,\mathrm{grad}\,\phi + p_2\,\mathrm{grad}\,\psi - \beta\,\mathrm{grad}\,\theta = \rho\ddot{\mathbf{u}},$$

$$-p_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta - d^*)\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2\,\mathrm{div}\,\mathbf{w} = \rho k_1\ddot{\phi},$$

$$-p_2\;\mathrm{div}\;\mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta - f)\psi + \gamma_2\theta - r_3\,\mathrm{div}\,\mathbf{w} = \rho k_2\ddot{\psi},$$

$$T_0[\beta\,\mathrm{div}\,\dot{\mathbf{u}} + \gamma_1\dot{\phi} + \gamma_2\dot{\psi}] + a\dot{\theta} = k\Delta\theta + \kappa_1\mathrm{div}\,\mathbf{w},$$

$$\kappa_6\Delta\,\mathbf{w} + (\kappa_4+\kappa_5)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{w} - \alpha\dot{\mathbf{w}} - r_2\,\mathrm{grad}\,\dot{\phi} - r_3\,\mathrm{grad}\dot{\psi} = \kappa_3\,\mathrm{grad}\;\theta + \kappa_2\;\mathbf{w}, \quad (21)$$

where $\mathbf{w} = (T_1, T_2, T_3)$ is microrotation vector and Δ is Laplacian operator.

2. STEADY OSCILLATIONS

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3,

$|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$, $\mathbf{D_x} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$.

Let us assume the displacement vector, volume fraction fields, temperature change and microtemperature vector functions as:

$$\Big[\mathbf{u}(\mathbf{x},t), \phi(\mathbf{x},t), \psi(\mathbf{x},t), \theta(\mathbf{x},t), \mathbf{w}(\mathbf{x},t)\Big] = \mathrm{Re}\Big[(\mathbf{u}^*, \phi^*, \psi^*, \theta^*, \mathbf{w}^*)e^{-\iota\omega t}\Big], \quad (22)$$

where ω is oscillation frequency.

Therefore from system of equations (21), we obtain the system of linearlized equations of steady oscillations in the theory of thermoelastic solids with double porosity and microtemperatures as

$$\Big[\mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div} + \rho\omega^2\Big]\mathbf{u} + p_1\,\mathrm{grad}\,\phi + p_2\,\mathrm{grad}\,\psi - \beta\,\mathrm{grad}\,\theta = \mathbf{0},$$

$$-p_1\,\mathrm{div}\,\mathbf{u} + \Big[t_1\Delta - d^* + \rho k_1\omega^2\Big]\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2\,\mathrm{div}\,\mathbf{w} = 0,$$

$$-p_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \alpha_1)\phi + \Big[t_2\Delta - f + \rho k_2\omega^2\Big]\psi + \gamma_2\theta - r_3\,\mathrm{div}\,\mathbf{w} = 0,$$

$$\iota\omega T_0[\beta\,\mathrm{div}\,\mathbf{u} + \gamma_1\phi + \gamma_2\psi] + [k\Delta + \iota\omega a]\theta + \kappa_1\,\mathrm{div}\,\mathbf{w} = 0,$$

$$\iota\omega[r_2 \operatorname{grad}\phi + r_3 \operatorname{grad}\psi] - \kappa_3 \operatorname{grad}\theta + \left[\kappa_6\Delta + (\kappa_4 + \kappa_5) \operatorname{grad}\operatorname{div} - \kappa_2 + \iota\omega\alpha\right]\mathbf{w} = \mathbf{0}\,. \tag{23}$$

If we replace ω by $-\iota\tau$, where τ is a complex number and $Re(\tau) > 0$ in the system of equations (23), then the system of equations of pseudo-oscillations may be obtained as:

$$\left[\mu\Delta + (\lambda+\mu)\operatorname{grad}\operatorname{div} - \rho\tau^2\right]\mathbf{u} + p_1 \operatorname{grad}\phi + p_2 \operatorname{grad}\psi - \beta \operatorname{grad}\theta = \mathbf{0},$$

$$-p_1 \operatorname{div}\mathbf{u} + \left[t_1\Delta - d^* - \rho k_1\tau^2\right]\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2 \operatorname{div}\mathbf{w} = 0,$$

$$-p_2 \operatorname{div}\mathbf{u} + (r_1\Delta - \alpha_1)\phi + \left[t_2\Delta - f - \rho k_2\tau^2\right]\psi + \gamma_2\theta - r_3 \operatorname{div}\mathbf{w} = 0,$$

$$\tau T_0[\beta \operatorname{div}\mathbf{u} + \gamma_1\phi + \gamma_2\psi] + [k\Delta + \tau a]\theta + \kappa_1 \operatorname{div}\mathbf{w} = 0,$$

$$\tau[r_2 \operatorname{grad}\phi + r_3 \operatorname{grad}\psi] - \kappa_3 \operatorname{grad}\theta + \left[\kappa_6\Delta + (\kappa_4+\kappa_5)\operatorname{grad}\operatorname{div} - \kappa_2 + \tau\alpha\right]\mathbf{w} = \mathbf{0}\,. \tag{24}$$

On taking $\rho = 0$ i.e. quasi-static case, we obtain the system of equations of quasi-static oscillations as:

$$\left[\mu\Delta + (\lambda+\mu)\operatorname{grad}\operatorname{div}\right]\mathbf{u} + p_1 \operatorname{grad}\phi + p_2 \operatorname{grad}\psi - \beta \operatorname{grad}\theta = \mathbf{0},$$

$$-p_1 \operatorname{div}\mathbf{u} + (t_1\Delta - d^*)\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2 \operatorname{div}\mathbf{w} = 0,$$

$$-p_2 \operatorname{div}\mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta - f)\psi + \gamma_2\theta - r_3 \operatorname{div}\mathbf{w} = 0,$$

$$\iota\omega T_0[\beta \operatorname{div}\mathbf{u} + \gamma_1\phi + \gamma_2\psi] + [k\Delta + \iota\omega a]\theta + \kappa_1 \operatorname{div}\mathbf{w} = 0,$$

$$\iota\omega[r_2 \operatorname{grad}\phi + r_3 \operatorname{grad}\psi] - \kappa_3 \operatorname{grad}\theta + \left[\kappa_6\Delta + (\kappa_4 + \kappa_5) \operatorname{grad}\operatorname{div} - \kappa_2 + \iota\omega\alpha\right]\mathbf{w} = \mathbf{0}\,. \tag{25}$$

If we put $\omega = 0$ in the equations (23), the system of equations of equilibrium theory of thermoelasticity with double porosity and microtemperatures as:

$$\left[\mu\Delta + (\lambda+\mu)\operatorname{grad}\operatorname{div}\right]\mathbf{u} + p_1 \operatorname{grad}\phi + p_2 \operatorname{grad}\psi - \beta \operatorname{grad}\theta = \mathbf{0},$$

$$-p_1 \operatorname{div}\mathbf{u} + (t_1\Delta - d^*)\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2 \operatorname{div}\mathbf{w} = 0,$$

$$-p_2 \operatorname{div}\mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta - f)\psi + \gamma_2\theta - r_3 \operatorname{div}\mathbf{w} = 0,$$

$$k\Delta\theta + \kappa_1 \operatorname{div}\mathbf{w} = 0,$$

$$-\kappa_3 \,\mathrm{grad}\,\theta + \Big[\kappa_6\Delta + (\kappa_4+\kappa_5)\,\mathrm{grad\ div} - \kappa_2\Big]\mathbf{w} = \mathbf{0}\ . \tag{26}$$

We introduce the second order matrix differential operators with constant coefficients

$$\mathbf{F}^{(i)}(\mathbf{D_x}) = \Big(F^{(i)}_{gh}(\mathbf{D_x})\Big)_{9\times 9},$$

where

$$F^{(1)}_{pq}(\mathbf{D_x}) = [\mu\Delta + \rho\omega^2]\delta_{pq} + (\lambda+\mu)\frac{\partial^2}{\partial x_p \partial x_q}, F^{(1)}_{p4}(\mathbf{D_x}) = -F^{(1)}_{4p}(\mathbf{D_x}) = p_1\frac{\partial}{\partial x_p},$$

$$F^{(1)}_{p5}(\mathbf{D_x}) = -F^{(1)}_{5p}(\mathbf{D_x}) = p_2\frac{\partial}{\partial x_p}, F^{(1)}_{p6}(\mathbf{D_x}) = -\beta\frac{\partial}{\partial x_p}, F^{(1)}_{44}(\mathbf{D_x}) = t_1\Delta - d^* + \rho k_1\omega^2,$$

$$F^{(1)}_{45}(\mathbf{D_x}) = F^{(1)}_{54}(\mathbf{D_x}) = r_1\Delta - \alpha_1, F^{(1)}_{46}(\mathbf{D_x}) = \gamma_1, F^{(1)}_{4;q+6}(\mathbf{D_x}) = -r_2\frac{\partial}{\partial x_q},$$

$$F^{(1)}_{55}(\mathbf{D_x}) = t_2\Delta - f + \rho k_2\omega^2, F^{(1)}_{56}(\mathbf{D_x}) = \gamma_2, F^{(1)}_{5;q+6}(\mathbf{D_x}) = -r_3\frac{\partial}{\partial x_q},$$

$$F^{(1)}_{6q}(\mathbf{D_x}) = \iota\omega\beta T_0\frac{\partial}{\partial x_q}, F^{(1)}_{64}(\mathbf{D_x}) = \iota\omega\gamma_1 T_0, F^{(1)}_{65}(\mathbf{D_x}) = \iota\omega\gamma_2 T_0,$$

$$F^{(1)}_{66}(\mathbf{D_x}) = k\Delta + \iota\omega a, F^{(1)}_{6;q+6}(\mathbf{D_x}) = \kappa_1\frac{\partial}{\partial x_q}, F^{(1)}_{p+6;q}(\mathbf{D_x}) = 0,$$

$$F^{(1)}_{p+6;4}(\mathbf{D_x}) = \iota\omega r_2\frac{\partial}{\partial x_p}, F^{(1)}_{p+6;5}(\mathbf{D_x}) = \iota\omega r_3\frac{\partial}{\partial x_p}, F^{(1)}_{p+6;6}(\mathbf{D_x}) = -\kappa_3\frac{\partial}{\partial x_p},$$

$$F^{(1)}_{p+6;q+6}(\mathbf{D_x}) = (\kappa_6\Delta - \kappa_2 + \iota\omega\alpha)\delta_{pq} + (\kappa_4+\kappa_5)\frac{\partial^2}{\partial x_p\partial x_q}, F^{(1)}_{p;q+6}(\mathbf{D_x}) = 0\ p,q = 1,2,3,$$

Here $i = 1,2,3,4$ corresponds to static, pseudo-, quasi-static oscillations and equilibrium theory of thermoelasticity with double porosity and microtemperatures respectively. The matrices $\mathbf{F}^{(i)}(\mathbf{D_x})$, $i = 2,3,4$ can be obtained from matrix $\mathbf{F}^{(1)}(\mathbf{D_x})$ by taking $\omega = -\iota\tau$, $\rho = 0$ and $\omega = 0$ respectively.

and

$$\tilde{\mathbf{F}}(\mathbf{D_x}) = \Big(\tilde{F}_{gh}(\mathbf{D_x})\Big)_{9\times 9},$$

where

$$\tilde{F}_{pq}(\mathbf{D_x}) = \mu\Delta\delta_{pq} + (\lambda+\mu)\frac{\partial^2}{\partial x_p\partial x_q}, \tilde{F}_{pi}(\mathbf{D_x}) = \tilde{F}_{ip}(\mathbf{D_x}) = 0, \tilde{F}_{44}(\mathbf{D_x}) = t_1\Delta,$$

$$\tilde{F}_{45}(\mathbf{D_x}) = \tilde{F}_{54}(\mathbf{D_x}) = r_1\Delta, \tilde{F}_{jn}(\mathbf{D_x}) = \tilde{F}_{nj}(\mathbf{D_x}) = 0,$$

$$\tilde{F}_{55}(\mathbf{D_x}) = t_2\Delta, \tilde{F}_{66}(\mathbf{D_x}) = k\Delta,$$

$$\tilde{F}_{p+6;q+6}(\mathbf{D_x}) = (\kappa_6\Delta - \kappa_2 + \iota\omega\alpha)\delta_{pq} + (\kappa_4 + \kappa_5)\frac{\partial^2}{\partial x_p \partial x_q}, \tilde{F}_{6;q+6}(\mathbf{D_x}) = \tilde{F}_{q+6,6}(\mathbf{D_x}) = 0,$$

$$p, q = 1, 2, 3 \;\; i = 4, ..., 9 \;\; j = 4, 5 \;\; n = 6, 7, 8, 9.$$

The system of equations (23)-(26) can be represented as

$$\mathbf{F}^{(i)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \;\; i = 1, 2, 3, 4$$

where $\mathbf{U} = (\mathbf{u}, \phi, \psi, \theta, \mathbf{w})$ is a nine-component vector function on E^3. The matrix $\tilde{\mathbf{F}}(\mathbf{D_x})$ is called the principal part of operator $\mathbf{F}^{(i)}(\mathbf{D_x})$.

DEFINITION 1: The operator $\mathbf{F}^{(i)}(\mathbf{D_x})$, $i = 1, 2, 3, 4$ is said to be elliptic if $|\tilde{\mathbf{F}}(\boldsymbol{v})| \neq 0$, where $\boldsymbol{v} = (v_1, v_2, v_3)$.

Since $|\tilde{\mathbf{F}}(\boldsymbol{v})| = \mu^2\tilde{\lambda}\sigma k\kappa_6^2\kappa_7|\boldsymbol{v}|^{18}, \tilde{\lambda} = \lambda + 2\mu, \sigma = t_1t_2 - r_1^2, \kappa_7 = \kappa_4 + \kappa_5 + \kappa_6,$

therefore operator $\mathbf{F}^{(i)}(\mathbf{D_x})$ is an elliptic differential operator iff

$$\mu\tilde{\lambda}\sigma k\kappa_6\kappa_7 \neq 0. \tag{27}$$

DEFINITION 2: The fundamental solutions of the system of equations (23)-(26) (fundamental matrices of operators $\mathbf{F}^{(i)}$) are the matrices $\mathbf{G}^{(i)}(\mathbf{x}) = \left(G_{gh}^{(i)}(\mathbf{x})\right)_{9\times 9}$ satisfying conditions

$$\mathbf{F}^{(i)}(\mathbf{D_x})\mathbf{G}^{(i)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}), \;\; i = 1, 2, 3, 4 \tag{28}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{I} = (\delta_{gh})_{9\times 9}$ is the unit matrix and $\mathbf{x} \in \mathrm{E}^3$.

3. CONSTRUCTION OF G(x) IN TERMS OF ELEMENTARY FUNCTIONS

Let us consider the system of non-homogeneous equations

$$[\mu\Delta + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div} + \rho\omega^2]\mathbf{u} - p_1\,\mathrm{grad}\,\phi - p_2\,\mathrm{grad}\,\psi + \iota\omega\beta T_0\,\mathrm{grad}\,\theta = \mathbf{H},$$

$$p_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta + d_1)\phi + (r_1\Delta - \alpha_1)\psi + \iota\omega\gamma_1 T_0\theta + \iota\omega r_2\,\mathrm{div}\,\mathbf{w} = L,$$

$$p_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta + d_2)\psi + \iota\omega\gamma_2 T_0\theta + \iota\omega r_3\,\mathrm{div}\,\mathbf{w} = M,$$

$$-\beta\,\mathrm{div}\,\mathbf{u} + \gamma_1\phi + \gamma_2\psi + (k\Delta + \iota\omega a)\theta - \kappa_3\,\mathrm{div}\,\mathbf{w} = Z,$$

$$-r_2\,\mathrm{grad}\,\phi - r_3\,\mathrm{grad}\,\psi + \kappa_1\,\mathrm{grad}\,\theta + [\kappa_6\Delta + (\kappa_4 + \kappa_5)\,\mathrm{grad}\,\mathrm{div} + \kappa_8]\mathbf{w} = \mathbf{X}, \tag{29}$$

where $d_1 = -d^* + \rho k_1\omega^2, d_2 = -f + \rho k_2\omega^2, \kappa_8 = -\kappa_2 + \imath\omega\alpha$ and $\mathbf{H}, \mathbf{X}$ are three-component vector functions on E^3; L, M and Z are scalar functions on E^3.

The system of equations (29) may also be written in the form

$$\mathbf{F}^{(1)^{tr}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \tag{30}$$

where $\mathbf{F}^{(1)^{tr}}$ is the transpose of matrix $\mathbf{F}^{(1)}$, $\mathbf{Q} = (\mathbf{H}, L, M, Z, \mathbf{X})$ and $\mathbf{x} \in \mathrm{E}^3$.

Applying operator div to the equations $(29)_1$ and $(29)_5$, we obtain

$$[\tilde{\lambda}\Delta + \rho\omega^2]\,\mathrm{div}\,\mathbf{u} - p_1\Delta\phi - p_2\Delta\psi + \imath\omega\beta T_0\Delta\theta = \mathrm{div}\,\mathbf{H}, \tag{31}$$

$$-r_2\Delta\phi - r_3\Delta\psi + \kappa_1\Delta\theta + [\kappa_7\Delta + \kappa_8]\mathrm{div}\,\mathbf{w} = \mathrm{div}\,\mathbf{X}. \tag{32}$$

The equations $(29)_2$-$(29)_4$, (31) and(32) may be expressed in the form

$$\mathbf{N}^{(1)}(\Delta)\mathbf{S} = \tilde{\mathbf{Q}}, \tag{33}$$

where $\mathbf{S} = (\mathrm{div}\,\mathbf{u}, \phi, \psi, \theta, \mathrm{div}\,\mathbf{w})$, $\tilde{\mathbf{Q}} = (w_1, ..., w_5) = (\mathrm{div}\,\mathbf{H}, L, M, Z, \mathrm{div}\,\mathbf{X})$ and

$$\mathbf{N}^{(1)}(\Delta) = \left(N_{gh}^{(1)}(\Delta)\right)_{5\times5} =$$

$$\begin{pmatrix} \tilde{\lambda}\Delta + \rho\omega^2 & -p_1\Delta & -p_2\Delta & \imath\omega\beta T_0\Delta & 0 \\ p_1 & t_1\Delta + d_1 & r_1\Delta - \alpha_1 & \imath\omega\gamma_1 T_0 & \imath\omega r_2 \\ p_2 & r_1\Delta - \alpha_1 & t_2\Delta + d_2 & \imath\omega\gamma_2 T_0 & \imath\omega r_3 \\ -\beta & \gamma_1 & \gamma_2 & k\Delta + \imath\omega a & -\kappa_3 \\ 0 & -r_2\Delta & -r_3\Delta & \kappa_1\Delta & \kappa_7\Delta + \kappa_8 \end{pmatrix}_{5\times5} \tag{34}$$

The equations $(29)_2$-$(29)_4$, (31) and (32) may also be written as

$$\Gamma^{(1)}(\Delta)\mathbf{S} = \mathbf{\Psi}, \tag{35}$$

where

$$\mathbf{\Psi} = (\Psi_1, ..., \Psi_5), \Psi_p = \frac{1}{M^*}\sum_{i=1}^{5} N_{ip}^{(1)^*} w_i,$$

$$\Gamma^{(1)}(\Delta) = \frac{1}{M^*}\,|\,\mathbf{N}^{(1)}(\Delta)|,\ M^* = \tilde{\lambda}k\kappa_7\sigma \quad p = 1, ..., 5 \tag{36}$$

and $N_{ip}^{(1)^*}$ is the cofactor of the element $N_{ip}^{(1)}$ of the matrix $\mathbf{N}^{(1)}$.

From equations (34) and (36), we see that

$$\Gamma^{(1)}(\Delta) = \prod_{i=1}^{5}(\Delta + \lambda_i^2),$$

where $\lambda_i^2,\ i=1,....,5$ are the roots of the equation $\Gamma^{(1)}(-v)=0$(with respect to v).

Applying operator $\Gamma^{(1)}(\Delta)$ to the equations $(29)_1$ and $(29)_5$, respectively, we obtain

$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2)\mathbf{u}=\mathbf{\Psi}',$$
$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_7^2)\mathbf{w}=\mathbf{\Psi}'', \tag{37}$$

where $\lambda_6^2=\frac{\rho\omega^2}{\mu}$, $\lambda_7^2=\frac{\kappa_8}{\kappa_6}$ and

$$\mathbf{\Psi}'=\frac{1}{\mu}\Big\{\Gamma^{(1)}(\Delta)\mathbf{H}-\operatorname{grad}\Big[(\lambda+\mu)\Psi_1-p_1\,\Psi_2-p_2\,\Psi_3+\imath\omega\beta T_0\,\Psi_4\Big]\Big\},$$
$$\mathbf{\Psi}''=\frac{1}{\kappa_6}\Big\{\Gamma^{(1)}(\Delta)\mathbf{X}-\operatorname{grad}\Big[(\kappa_4+\kappa_5)\Psi_5-r_2\,\Psi_2-r_3\,\Psi_3+\kappa_1\,\Psi_4\Big]\Big\}. \tag{38}$$

From equations (36) and (38), we obtain

$$\mathbf{\Theta}^{(1)}(\Delta)\mathbf{U}(\mathbf{x})=\hat{\mathbf{\Psi}}(\mathbf{x}), \tag{39}$$

where $\hat{\mathbf{\Psi}}=(\mathbf{\Psi}',\Psi_2,\Psi_3,\Psi_4,\mathbf{\Psi}'')$ and

$$\mathbf{\Theta}^{(1)}(\Delta)=\Big(\Theta_{gh}^{(1)}(\Delta)\Big)_{9\times 9},$$

$$\Theta_{pp}^{(1)}(\Delta)=\Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2)=\prod_{i=1}^{6}(\Delta+\lambda_i^2),$$

$$\Theta_{p+3;p+3}^{(1)}(\Delta)=\Gamma^{(1)}(\Delta)=\prod_{i=1}^{5}(\Delta+\lambda_i^2),$$

$$\Theta_{p+6;p+6}^{(1)}(\Delta)=\Gamma^{(1)}(\Delta)(\Delta+\lambda_7^2)=\prod_{i=1,i\neq 6}^{7}(\Delta+\lambda_i^2),$$

$$\Theta_{gh}^{(1)}(\Delta)=0\ \ p=1,2,3\ \ g,h=1,.....,9\ \ g\neq h$$

The equations (36) and (38) can be rewritten in the form

$$\mathbf{\Psi}'=\Big[\frac{1}{\mu}\Gamma^{(1)}(\Delta)\mathbf{J}+w_{11}^{(1)}(\Delta)\operatorname{grad}\operatorname{div}\Big]\mathbf{H}+\sum_{i=2}^{5}w_{i1}^{(1)}(\Delta)\operatorname{grad}w_i,$$

$$\mathbf{\Psi}''=\sum_{i=1}^{4}w_{i5}^{(1)}(\Delta)\operatorname{grad}w_i+\Big[\frac{1}{\kappa_6}\Gamma^{(1)}(\Delta)\mathbf{J}+w_{55}^{(1)}(\Delta)\operatorname{grad}\operatorname{div}\Big]\mathbf{X},$$

$$\Psi_l = w^{(1)}_{1l}(\Delta)\,\mathrm{div}\,\mathbf{H} + \sum_{i=2}^{4} w^{(1)}_{il}(\Delta)w_i + w^{(1)}_{5l}(\Delta)\,\mathrm{div}\,\mathbf{X}\;\; l = 2,3,4 \tag{40}$$

where, we have used the following notations:

$$w^{(1)}_{p1}(\Delta) = -\frac{1}{M^*\mu}\Big[(\lambda+\mu)\tilde{N}^{(1)}_{p1}(\Delta) - p_1\tilde{N}^{(1)}_{p2}(\Delta) - p_2\tilde{N}^{(1)}_{p2}(\Delta) + \iota\omega\beta T_0\tilde{N}^{(1)}_{p4}(\Delta)\Big],$$

$$w^{(1)}_{p5}(\Delta) = -\frac{1}{M^*\kappa_6}\Big[(\kappa_4+\kappa_5)\tilde{N}^{(1)}_{p5}(\Delta) - r_2\tilde{N}^{(1)}_{p2}(\Delta) - r_3\tilde{N}^{(1)}_{p3}(\Delta) + \kappa_1\tilde{N}^{(1)}_{p4}(\Delta)\Big],$$

$$w^{(1)}_{pq}(\Delta) = \frac{\tilde{N}^{(1)}_{pq}(\Delta)}{M^*}\;\; p = 1,....,5\;\; q = 2,3,4$$

From equations (40), we have

$$\hat{\mathbf{\Psi}}(\mathbf{x}) = \mathbf{R}^{(1)^{tr}}(\mathbf{D_x})\mathbf{Q}(\mathbf{x}), \tag{41}$$

where

$$\mathbf{R}^{(1)}(\mathbf{D_x}) = \Big(R^{(1)}_{gh}(\mathbf{D_x})\Big)_{9\times 9},$$

$$R^{(1)}_{ij}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{ij} + w^{(1)}_{11}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R^{(1)}_{i;p+2}(\mathbf{D_x}) = w^{(1)}_{1p}(\Delta)\frac{\partial}{\partial x_i},\, R^{(1)}_{p+2;i}(\mathbf{D_x}) = w^{(1)}_{p1}(\Delta)\frac{\partial}{\partial x_i},$$

$$R^{(1)}_{i;j+6}(\mathbf{D_x}) = w^{(1)}_{15}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},\, R^{(1)}_{i+6;j}(\mathbf{D_x}) = w^{(1)}_{51}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R^{(1)}_{i+6;p+2}(\mathbf{D_x}) = w^{(1)}_{5p}(\Delta)\frac{\partial}{\partial x_i},\, R^{(1)}_{p+2;i+6}(\mathbf{D_x}) = w^{(1)}_{p5}(\Delta)\frac{\partial}{\partial x_i},$$

$$R^{(1)}_{i+6;j+6}(\mathbf{D_x}) = \frac{1}{\kappa_6}\Gamma^{(1)}(\Delta)\delta_{ij} + w^{(1)}_{55}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R^{(1)}_{p+2;n+2}(\mathbf{D_x}) = w^{(1)}_{pn}(\Delta)\;\; i,j = 1,2,3\;\; p,n = 2,3,4 \tag{42}$$

From equations (30), (39) and (41), we obtain

$$\mathbf{\Theta}^{(1)}\mathbf{U} = \mathbf{R}^{(1)^{tr}}\mathbf{F}^{(1)^{tr}}\mathbf{U}.$$

The above relation implies

$$\mathbf{R}^{(1)^{tr}}\mathbf{F}^{(1)^{tr}} = \mathbf{\Theta}^{(1)}.$$

Therefore, we obtain

$$\mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{R}^{(1)}(\mathbf{D_x}) = \mathbf{\Theta}^{(1)}(\Delta). \tag{43}$$

We assume that

$$\lambda_p^2 \neq \lambda_q^2 \neq 0 \ p, q = 1, ..., 7 \ p \neq q.$$

Let

$$\mathbf{Y}^{(1)}(\mathbf{x}) = \Big(Y_{ij}^{(1)}(\mathbf{x})\Big)_{9\times 9}, \ Y_{pp}^{(1)}(\mathbf{x}) = \sum_{g=1}^{6} r_{1g}^{(1)}\varsigma_g(\mathbf{x}),$$

$$Y_{p+3;p+3}^{(1)}(\mathbf{x}) = \sum_{g=1}^{5} r_{2g}^{(1)}\varsigma_g(\mathbf{x}), Y_{p+6;p+6}^{(1)}(\mathbf{x}) = \sum_{g=1,g\neq 6}^{7} r_{3g}^{(1)}\varsigma_g(\mathbf{x}),$$

$$Y_{ij}^{(1)}(\mathbf{x}) = 0 \ p = 1, 2, 3 \ i, j = 1, ..., 9 \ i \neq j$$

where

$$\varsigma_n(\mathbf{x}) = -\frac{e^{\iota\lambda_n|\mathbf{x}|}}{4\pi|\mathbf{x}|}, r_{1g}^{(1)} = \prod_{i=1,i\neq g}^{6}(\lambda_i^2 - \lambda_g^2)^{-1}, r_{2h}^{(1)} = \prod_{i=1,i\neq h}^{5}(\lambda_i^2 - \lambda_h^2)^{-1},$$

$$r_{3l}^{(1)} = \prod_{i=1,i\neq 6,l}^{7}(\lambda_i^2 - \lambda_l^2)^{-1} \ n = 1, ..., 7 \ g = 1, ..., 6 \ h = 1, ..., 5 \ l = 1, ..., 5, 7 \tag{44}$$

LEMMA 1: The matrix $\mathbf{Y}^{(1)}$ defined above is the fundamental matrix of operator $\mathbf{\Theta}^{(1)}(\Delta)$, i.e.

$$\mathbf{\Theta}^{(1)}(\Delta)\mathbf{Y}^{(1)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}) \tag{45}$$

PROOF: To prove the lemma, it is sufficient to prove that

$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2)Y_{11}^{(1)}(\mathbf{x}) = \delta(\mathbf{x}), \Gamma^{(1)}(\Delta)Y_{44}^{(1)}(\mathbf{x}) = \delta(\mathbf{x}), \Gamma^{(1)}(\Delta)(\Delta+\lambda_7^2)Y_{77}^{(1)}(\mathbf{x}) = \delta(\mathbf{x}). \tag{46}$$

Consider

$$\sum_{i=1}^{6} r_{1i}^{(1)} = \frac{\sum_{j=1}^{6}(-1)^j z_j}{z_7},$$

where

$$z_1 = \prod_{i=3}^{6}(\lambda_2^2 - \lambda_i^2)\prod_{j=4}^{6}(\lambda_3^2 - \lambda_j^2)\prod_{l=5}^{6}(\lambda_4^2 - \lambda_l^2)(\lambda_5^2 - \lambda_6^2),$$

$$z_2 = \prod_{i=3}^{6}(\lambda_1^2 - \lambda_i^2)\prod_{j=4}^{6}(\lambda_3^2 - \lambda_j^2)\prod_{l=5}^{6}(\lambda_4^2 - \lambda_l^2)(\lambda_5^2 - \lambda_6^2),$$

$$z_3 = \prod_{i=2,i\neq 3}^{6}(\lambda_1^2 - \lambda_i^2)\prod_{j=4}^{6}(\lambda_2^2 - \lambda_j^2)\prod_{l=5}^{6}(\lambda_4^2 - \lambda_l^2)(\lambda_5^2 - \lambda_6^2),$$

$$z_4 = \prod_{i=2,i\neq 4}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq 4}^{6}(\lambda_2^2-\lambda_j^2)\prod_{l=5}^{6}(\lambda_3^2-\lambda_l^2)(\lambda_5^2-\lambda_6^2),$$

$$z_5 = \prod_{i=2,i\neq 5}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq 5}^{6}(\lambda_2^2-\lambda_j^2)\prod_{l=4,l\neq 5}^{6}(\lambda_3^2-\lambda_l^2)(\lambda_4^2-\lambda_6^2),$$

$$z_6 = \prod_{i=2,i\neq 6}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{5}(\lambda_2^2-\lambda_j^2)\prod_{l=4}^{5}(\lambda_3^2-\lambda_l^2)(\lambda_4^2-\lambda_5^2),$$

$$z_7 = \prod_{i=2}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{6}(\lambda_2^2-\lambda_j^2)\prod_{l=4}^{6}(\lambda_3^2-\lambda_l^2)\prod_{p=5}^{6}(\lambda_4^2-\lambda_p^2)(\lambda_5^2-\lambda_6^2).$$

On simplifying the right hand side of above relation, we obtain

$$\sum_{i=1}^{6} r_{1i}^{(1)} = 0. \tag{47}$$

Similarly, we find that

$$\sum_{i=2}^{6} r_{1i}^{(1)}(\lambda_1^2-\lambda_i^2)=0, \sum_{i=3}^{6} r_{1i}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_i^2)\Big]=0,$$

$$\sum_{i=4}^{6} r_{1i}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_i^2)\Big]=0, \sum_{i=5}^{6} r_{1i}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_i^2)\Big]=0,$$

$$\prod_{j=1}^{5} r_{16}^{(1)}(\lambda_j^2-\lambda_6^2)=1. \tag{48}$$

Also,

$$(\Delta+\lambda_p^2)\varsigma_g(\mathbf{x}) = \delta(\mathbf{x}) + (\lambda_p^2-\lambda_g^2)\varsigma_g(\mathbf{x}) \; p,g=1,...,7. \tag{49}$$

Now consider

$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2)Y_{11}^{(1)}(\mathbf{x}) = \prod_{i=1}^{6}(\Delta+\lambda_i^2)\sum_{g=1}^{6} r_{1g}^{(1)}\varsigma_g(\mathbf{x})$$

$$= \prod_{i=2}^{6}(\Delta+\lambda_i^2)\sum_{g=1}^{6} r_{1g}^{(1)}\Big[\delta(\mathbf{x}) + (\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]$$

$$= \prod_{i=2}^{6}(\Delta+\lambda_i^2)\Big[\delta(\mathbf{x})\sum_{g=1}^{6} r_{1g}^{(1)} + \sum_{g=2}^{6} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]$$

Using equations (47)-(49) in the above relation, we obtain

$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2)Y_{11}^{(1)}(\mathbf{x}) = \prod_{i=2}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=2}^{6} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]$$

$$= \prod_{i=3}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=2}^{6} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\Big[\delta(\mathbf{x})+(\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= \prod_{i=3}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$

$$= \prod_{i=4}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= \prod_{i=4}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$

$$= \prod_{i=5}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= \prod_{i=5}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=5}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$

$$= (\Delta+\lambda_6^2)\Big[\sum_{g=5}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_5^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$

$$= (\Delta+\lambda_6^2)\varsigma_6(\mathbf{x}) = \delta(\mathbf{x}).$$

The equations $(46)_2$ and $(46)_3$ can be proved in the similar way.

We introduce the matrix

$$\mathbf{G}^{(1)}(\mathbf{x}) = \mathbf{R}^{(1)}(\mathbf{D}_\mathbf{x})\mathbf{Y}^{(1)}(\mathbf{x}). \tag{50}$$

From equations (43), (45) and (50), we obtain

$$\mathbf{F}^{(1)}(\mathbf{D}_\mathbf{x})\mathbf{G}^{(1)}(\mathbf{x}) = \mathbf{F}^{(1)}(\mathbf{D}_\mathbf{x})\mathbf{R}^{(1)}(\mathbf{D}_\mathbf{x})\mathbf{Y}^{(1)}(\mathbf{x}) = \mathbf{\Theta}^{(1)}(\Delta)\mathbf{Y}^{(1)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x})$$

Hence, $\mathbf{G}^{(1)}(\mathbf{x})$ is a solution to equation $(28)_1$.

THEOREM 1: If the condition (27) is satisfied, then the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ defined by the equation (50) is the fundamental solution of the system of equations (23) and the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ is represented in the following form:

$$\mathbf{G}^{(1)}(\mathbf{x}) = \left(G^{(1)}_{pq}(\mathbf{x})\right)_{9\times 9},$$

$$\mathbf{G}^{(1)}_{gh}(\mathbf{x}) = R^{(1)}_{gh}(\mathbf{D_x})Y^{(1)}_{11}(\mathbf{x}), \mathbf{G}^{(1)}_{g;h+3}(\mathbf{x}) = R^{(1)}_{g;h+3}(\mathbf{D_x})Y^{(1)}_{44}(\mathbf{x}),$$
$$\mathbf{G}^{(1)}_{g;h+6}(\mathbf{x}) = R^{(1)}_{g;h+6}(\mathbf{D_x})Y^{(1)}_{77}(\mathbf{x}) \ g = 1, ..., 9 \ h = 1, 2, 3$$

4. CONSTRUCTION OF MATRICES $\mathbf{G}^{(i)}(\mathbf{x})$ $i = 2, 3, 4$

PSEUDO-OSCILLATIONS

We introduce the matrix

$$\mathbf{G}^{(2)}(\mathbf{x}) = \mathbf{R}^{(2)}(\mathbf{D_x})\mathbf{Y}^{(2)}(\mathbf{x}), \tag{51}$$

where, the matrices $\mathbf{R}^{(2)}(\mathbf{D_x})$ and $\mathbf{Y}^{(2)}(\mathbf{x})$ can be obtained from matrices $\mathbf{R}^{(1)}(\mathbf{D_x})$ and $\mathbf{Y}^{(1)}(\mathbf{x})$ respectively by taking $\omega = -\iota\tau$ and repeating the above procedure after equation (28).

THEOREM 2: If the condition (27) is satisfied, then the matrix $\mathbf{G}^{(2)}(\mathbf{x})$ defined by the equation (51) is the fundamental solution of the system of equations (24).

QUASI-STATIC OSCILLATIONS

In this case, the matrix $\mathbf{N}^{(3)}(\Delta)$, operator $\Gamma^{(3)}(\Delta)$ and matrix operators $\boldsymbol{\Theta}^{(3)}(\Delta)$, $\mathbf{R}^{(3)}(\mathbf{D_x})$, $\mathbf{Y}^{(3)}(\mathbf{x})$ and $\mathbf{G}^{(3)}(\mathbf{x})$ are obtained as:

(i) $$\hat{\mathbf{N}}^{(3)}(\Delta) = \left(\hat{N}^{(3)}_{gh}(\Delta)\right)_{5\times 5}, \ \mathbf{N}^{(3)}(\Delta) = \left(N^{(3)}_{gh}(\Delta)\right)_{5\times 5},$$

$$\hat{N}^{(3)}_{11}(\Delta) = \tilde{\lambda}, \hat{N}^{(3)}_{12}(\Delta) = -p_1, \hat{N}^{(3)}_{13}(\Delta) = -p_2, \hat{N}^{(3)}_{14}(\Delta) = \iota\omega\beta T_0, \hat{N}^{(3)}_{15}(\Delta) = 0,$$
$$N^{(3)}_{1q}(\Delta) = \Delta\hat{N}^{(3)}_{1q}(\Delta), N^{(3)}_{l1}(\Delta) = \hat{N}^{(3)}_{l1}(\Delta) = N^{(1)}_{l1}(\Delta),$$
$$N^{(3)}_{22}(\Delta) = \hat{N}^{(3)}_{22}(\Delta) = t_1\Delta - d^*, N^{(3)}_{23}(\Delta) = \hat{N}^{(3)}_{23}(\Delta) = N^{(3)}_{32}(\Delta) = \hat{N}^{(3)}_{32}(\Delta) = r_1\Delta - \alpha_1,$$
$$N^{(3)}_{33}(\Delta) = \hat{N}^{(3)}_{33}(\Delta) = t_2\Delta - f, N^{(3)}_{lp}(\Delta) = \hat{N}^{(3)}_{lp}(\Delta) = N^{(1)}_{lp}(\Delta),$$
$$N^{(3)}_{pq}(\Delta) = \hat{N}^{(3)}_{pq}(\Delta) = N^{(1)}_{pq}(\Delta) \ i = 1, 2, 3 \ l = 2, 3 \ p = 4, 5 \ q = 1, ..., 5.$$

(ii)
$$\Gamma^{(3)}(\Delta) = \Delta \prod_{i=1}^{4} (\Delta + \mu_i^2),$$

where $\mu_i^2,\ i = 1, ..., 4$ are the roots of the equation $|\hat{\mathbf{N}}^{(3)}(-v)| = 0$(with respect to v).

(iii)
$$\mathbf{\Theta}^{(3)}(\Delta) = \left(\Theta_{gh}^{(3)}(\Delta)\right)_{9\times 9},$$

$$\Theta_{pp}^{(3)}(\Delta) = \Gamma^{(3)}(\Delta)\Delta = \Delta^2 \prod_{i=1}^{4} (\Delta + \mu_i^2),$$

$$\Theta_{p+3;p+3}^{(3)}(\Delta) = \Gamma^{(3)}(\Delta) = \Delta \prod_{i=1}^{4} (\Delta + \mu_i^2), \Theta_{p+6;p+6}^{(6)}(\Delta) = \Gamma^{(3)}(\Delta)(\Delta + \mu_5^2) = \Delta \prod_{i=1}^{5} (\Delta + \mu_i^2),$$

$$\Theta_{gh}^{(3)}(\Delta) = 0, \mu_5^2 = \frac{\kappa_8}{\kappa_6} \ p = 1, 2, 3 \ g, h = 1, ..., 9 \ g \neq h$$

$$(iv)\ \ w_{p1}^{(3)}(\Delta) = -\frac{1}{M^*\mu}\left[(\lambda + \mu)\, \tilde{N}_{p1}^{(3)}(\Delta) - p_1 \tilde{N}_{p2}^{(3)}(\Delta) - p_2 \tilde{N}_{p3}^{(3)}(\Delta) + \iota\omega\beta T_0 \tilde{N}_{p4}^{(3)}(\Delta)\right],$$

$$w_{p5}^{(3)}(\Delta) = -\frac{1}{M^*\kappa_6}\left[(\kappa_4 + \kappa_5)\tilde{N}_{p5}^{(3)}(\Delta) - r_2 \tilde{N}_{p2}^{(3)}(\Delta) - r_3 \tilde{N}_{p3}^{(3)}(\Delta) + \kappa_1 \tilde{N}_{p4}^{(3)}(\Delta)\right],$$

$$w_{pq}^{(3)}(\Delta) = \frac{\tilde{N}_{pq}^{(3)}(\Delta)}{M^*} \ p = 1, ..., 5 \ q = 2, 3, 4$$

where $\tilde{N}_{ij}^{(3)}\ i, j = 1, ..., 5$ is the cofactor of the element $N_{ij}^{(3)}$ of the matrix $\mathbf{N}^{(3)}$.

(v)
$$\mathbf{R}^{(3)}(\mathbf{D_x}) = \left(R_{gh}^{(3)}(\mathbf{D_x})\right)_{9\times 9},$$

$$R_{ij}^{(3)}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma^{(3)}(\Delta)\delta_{ij} + w_{11}^{(3)}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$R_{i;p+2}^{(3)}(\mathbf{D_x}) = w_{1p}^{(3)}(\Delta)\frac{\partial}{\partial x_i}, R_{p+2;i}^{(3)}(\mathbf{D_x}) = w_{p1}^{(3)}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{i+6;j}^{(3)}(\mathbf{D_x}) = w_{51}^{(3)}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j}, R_{i;j+6}^{(3)}(\mathbf{D_x}) = w_{15}^{(3)}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$R_{i+6;j+6}^{(3)}(\mathbf{D_x}) = \frac{1}{\kappa_6}\Gamma^{(3)}(\Delta)\delta_{ij} + w_{55}^{(3)}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$R_{i+6;p+2}^{(3)}(\mathbf{D_x}) = w_{5p}^{(3)}(\Delta)\frac{\partial}{\partial x_i}, R_{p+2;i+6}^{(3)}(\mathbf{D_x}) = w_{p5}^{(3)}(\Delta)\frac{\partial}{\partial x_i},$$

$$R^{(3)}_{p+2;q+2}(\mathbf{D_x}) = w^{(3)}_{pq}(\Delta)\ i,j=1,2,3\ p,q=2,3,4$$

$$(vi)\qquad \mathbf{Y}^{(3)}(\mathbf{x}) = \left(Y^{(3)}_{ij}(\mathbf{x})\right)_{9\times 9},\ Y^{(3)}_{pp}(\mathbf{x}) = r^{(3)}_{11}\varsigma^*_1(\mathbf{x}) + r^{(3)}_{12}\varsigma^*_2(\mathbf{x}) + \sum_{g=1}^{4} r^{(3)}_{1;g+2}\tilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{p+3;p+3}(\mathbf{x}) = r^{(3)}_{21}\varsigma^*_1(\mathbf{x}) + \sum_{g=1}^{4} r^{(3)}_{2;g+1}\tilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{p+6;p+6}(\mathbf{x}) = r^{(3)}_{31}\varsigma^*_1(\mathbf{x}) + \sum_{g=1}^{5} r^{(3)}_{3;g+1}\tilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{ij}(\mathbf{x}) = 0\ p=1,2,3\ i,j=1,.....,9\ i\neq j$$

where

$$\varsigma^*_1 = -\frac{1}{4\pi|\mathbf{x}|}, \varsigma^*_2 = -\frac{|\mathbf{x}|}{8\pi}, \tilde{\varsigma}_g(\mathbf{x}) = -\frac{e^{\iota\mu_g|\mathbf{x}|}}{4\pi|\mathbf{x}|}\quad g=1,.....,5$$

$$r^{(3)}_{11} = -\frac{\mu_1^2\mu_2^2\mu_3^2 + \mu_1^2\mu_2^2\mu_4^2 + \mu_1^2\mu_3^2\mu_4^2 + \mu_2^2\mu_3^2\mu_4^2}{\mu_1^2\mu_2^2\mu_3^2\mu_4^2},$$

$$r^{(3)}_{12} = r^{(3)}_{21} = \prod_{i=1}^{4}\mu_i^{-2}, r^{(3)}_{1;l+2} = \mu_l^{-4}\prod_{i=1,i\neq l}^{4}(\mu_i^2-\mu_l^2)^{-1},$$

$$r^{(3)}_{2;l+1} = -\mu_l^{-2}\prod_{i=1,i\neq l}^{4}(\mu_i^2-\mu_l^2)^{-1}, r^{(3)}_{31} = \prod_{i=1}^{5}\mu_i^{-2},$$

$$r^{(3)}_{3;n+1} = -\mu_l^{-2}\prod_{i=1,i\neq n}^{5}(\mu_i^2-\mu_n^2)^{-1}\ l=1,.....,4\ n=1,....,5.$$

On introducing the matrix

$$\mathbf{G}^{(3)}(\mathbf{x}) = \mathbf{R}^{(3)}(\mathbf{D_x})\mathbf{Y}^{(3)}(\mathbf{x}), \tag{52}$$

we obtain

$$\mathbf{F}^{(3)}(\mathbf{D_x})\mathbf{G}^{(3)}(\mathbf{x}) = \mathbf{F}^{(3)}(\mathbf{D_x})\mathbf{R}^{(3)}(\mathbf{D_x})\mathbf{Y}^{(3)}(\mathbf{x}) = \mathbf{\Theta}^{(3)}(\Delta)\mathbf{Y}^{(3)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x})$$

Hence, $\mathbf{G}^{(3)}(\mathbf{x})$ is a fundamental solution to equation $(28)_3$.

THEOREM 3: If the condition (27) is satisfied, then the matrix $\mathbf{G}^{(3)}(\mathbf{x})$ defined by the equation (52) is the fundamental solution of the system of equations (25).

EQUILIBRIUM THEORY

In this case, the matrix $\mathbf{N}^{(4)}(\Delta)$, operator $\Gamma^{(4)}(\Delta)$ and matrix operators $\boldsymbol{\Theta}^{(4)}(\Delta)$, $\mathbf{R}^{(4)}(\mathbf{D_x})$, $\mathbf{Y}^{(4)}(\mathbf{x})$ and $\mathbf{G}^{(4)}(\mathbf{x})$ are obtained as:

(i)
$$\hat{\mathbf{N}}^{(4)}(\Delta) = \left(\hat{N}^{(4)}_{gh}(\Delta)\right)_{6\times 6},\ \mathbf{N}^{(4)}(\Delta) = \left(N^{(4)}_{gh}(\Delta)\right)_{6\times 6},$$

$$\hat{N}^{(4)}_{pi}(\Delta) = \hat{N}^{(3)}_{pi}(\Delta), N^{(4)}_{pi}(\Delta) = N^{(3)}_{pi}(\Delta), N^{(4)}_{il}(\Delta) = \hat{N}^{(4)}_{il}(\Delta) = 0,$$
$$\hat{N}^{(4)}_{44}(\Delta) = k, N^{(4)}_{45}(\Delta) = \hat{N}^{(4)}_{45}(\Delta) = -\kappa_3,$$
$$\hat{N}^{(4)}_{54}(\Delta) = \kappa_1, N^{(4)}_{55}(\Delta) = \hat{N}^{(4)}_{55}(\Delta) = \kappa_7\Delta - \kappa_2, N^{(4)}_{l4}(\Delta) = \Delta\hat{N}^{(4)}_{l4}(\Delta),$$
$$p = 1, ..., 5\ i = 1, 2, 3\ l = 4, 5.$$

(ii)
$$\Gamma^{(4)}(\Delta) = \Delta^2 \prod_{i=1}^{3}(\Delta+\omega_i^2),$$

where $\omega_i^2,\ i = 1, 2, 3$ are the roots of the equation $|\hat{\mathbf{N}}^{(4)}(-\kappa)| = 0$(with respect to κ).

(iii)
$$\boldsymbol{\Theta}^{(4)}(\Delta) = \left(\Theta^{(4)}_{gh}(\Delta)\right)_{9\times 9},$$

$$\Theta^{(4)}_{pp}(\Delta) = \Gamma^{(4)}(\Delta)\Delta = \Delta^3 \prod_{i=1}^{3}(\Delta + \omega_i^2),$$

$$\Theta^{(4)}_{p+3;p+3}(\Delta) = \Gamma^{(4)}(\Delta) = \Delta^2 \prod_{i=1}^{3}(\Delta + \omega_i^2),$$

$$\Theta^{(4)}_{p+6;p+6}(\Delta) = \Gamma^{(4)}(\Delta)(\Delta + \omega_4^2) = \Delta^2 \prod_{i=1}^{4}(\Delta + \omega_i^2), \Theta^{(4)}_{gh}(\Delta) = 0,$$

$$\omega_4^2 = -\frac{\kappa_2}{\kappa_6}\ p = 1, 2, 3\ g, h = 1, ..., 9\ g \neq h$$

$$(iv)\ \ w^{(4)}_{p1}(\Delta) = -\frac{1}{M^*\mu}\Big[(\lambda + \mu)\tilde{N}^{(4)}_{p1}(\Delta) - p_1\tilde{N}^{(4)}_{p2}(\Delta) - p_2\tilde{N}^{(4)}_{p3}(\Delta)\Big],$$

$$w^{(4)}_{p5}(\Delta) = -\frac{1}{M^*\kappa_6}\Big[(\kappa_4 + \kappa_5)\tilde{N}^{(4)}_{p5}(\Delta) - r_2\tilde{N}^{(4)}_{p2}(\Delta) - r_3\tilde{N}^{(4)}_{p3}(\Delta) + \kappa_1\tilde{N}^{(4)}_{p4}(\Delta)\Big],$$

$$w^{(4)}_{pq}(\Delta) = \frac{\tilde{N}^{(4)}_{pq}(\Delta)}{M^*}\ p = 1, ..., 5\ q = 2, 3, 4$$

where $\tilde{N}^{(4)}_{ij}\ i, j = 1, ..., 5$ is the cofactor of the element $N^{(4)}_{ij}$ of the matrix $\mathbf{N}^{(4)}$.

(v)
$$\mathbf{R}^{(4)}(\mathbf{D_x}) = \left(R^{(4)}_{gh}(\mathbf{D_x})\right)_{9\times 9},$$

$$R^{(4)}_{ij}(\mathbf{D_x}) = \frac{1}{\mu}\Gamma^{(4)}(\Delta)\delta_{ij} + w^{(4)}_{11}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$R^{(4)}_{i;p+2}(\mathbf{D_x}) = w^{(4)}_{1p}(\Delta)\frac{\partial}{\partial x_i}, R^{(4)}_{p+2;i}(\mathbf{D_x}) = w^{(4)}_{p1}(\Delta)\frac{\partial}{\partial x_i},$$

$$R^{(4)}_{i+6;j}(\mathbf{D_x}) = w^{(4)}_{51}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j}, R^{(4)}_{i;j+6}(\mathbf{D_x}) = w^{(4)}_{15}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$R^{(4)}_{i+6;j+6}(\mathbf{D_x}) = \frac{1}{\kappa_6}\Gamma^{(4)}(\Delta)\delta_{ij} + w^{(4)}_{55}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$R^{(4)}_{i+6;p+2}(\mathbf{D_x}) = w^{(4)}_{5p}(\Delta)\frac{\partial}{\partial x_i}, R^{(4)}_{p+2;i+6}(\mathbf{D_x}) = w^{(4)}_{p5}(\Delta)\frac{\partial}{\partial x_i},$$

$$R^{(4)}_{p+2;q+2}(\mathbf{D_x}) = w^{(4)}_{pq}(\Delta) \; i,j = 1,2,3 \; p,q = 2,3,4$$

(vi) $$\mathbf{Y}^{(4)}(\mathbf{x}) = \left(Y^{(4)}_{ij}(\mathbf{x})\right)_{9\times 9}, \; Y^{(4)}_{pp}(\mathbf{x}) = r^{(4)}_{11}\varsigma^*_1(\mathbf{x}) + r^{(4)}_{12}\varsigma^*_2(\mathbf{x}) + \sum_{g=1}^{3} r^{(4)}_{1;g+2}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y^{(4)}_{p+3;p+3}(\mathbf{x}) = r^{(4)}_{21}\varsigma^*_1(\mathbf{x}) + r^{(4)}_{22}\varsigma^*_2(\mathbf{x}) + \sum_{g=1}^{3} r^{(4)}_{2;g+2}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y^{(4)}_{p+6;p+6}(\mathbf{x}) = r^{(4)}_{31}\varsigma^*_1(\mathbf{x}) + r^{(4)}_{32}\varsigma^*_2(\mathbf{x}) + \sum_{g=1}^{4} r^{(4)}_{3;g+2}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y^{(4)}_{ij}(\mathbf{x}) = 0 \; p = 1,2,3 \; i,j = 1,.....,9 \; i \neq j$$

where

$$\hat{\varsigma}_g(\mathbf{x}) = -\frac{e^{\iota\omega_g|\mathbf{x}|}}{4\pi|\mathbf{x}|} \quad g = 1,.....,4$$

$$r^{(4)}_{11} = -\frac{\omega_1^4\omega_2^2(\omega_2^2+\omega_3^2) + \omega_2^4\omega_3^2(\omega_3^2+\omega_1^2) + \omega_3^4\omega_1^2(\omega_1^2+\omega_2^2)}{\omega_1^6\omega_2^6\omega_3^6},$$

$$r^{(4)}_{12} = r^{(4)}_{22} = \frac{1}{\omega_1^2\omega_2^2\omega_3^2}, r^{(4)}_{1;l+2} = -\omega_l^{-6}\prod_{i=1, i\neq l}^{3}(\omega_i^2 - \omega_l^2)^{-1},$$

$$r^{(4)}_{21} = -\frac{\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\omega_1^2}{\omega_1^4\omega_2^4\omega_3^4}, r^{(4)}_{2;l+2} = \omega_l^{-4}\prod_{i=1, i\neq l}^{3}(\omega_i^2 - \omega_l^2)^{-1},$$

$$r^{(4)}_{31} = -\frac{\omega_1^2\omega_2^2\omega_3^2 + \omega_1^2\omega_2^2\omega_4^2 + \omega_1^2\omega_3^2\omega_4^2 + \omega_2^2\omega_3^2\omega_4^2}{\omega_1^2\omega_2^2\omega_3^2\omega_4^2}, r^{(4)}_{32} = \prod_{i=1}^{4}\omega_i^{-2},$$

$$r^{(4)}_{3;n+2} = \omega_n^{-4} \prod_{i=1, i\neq n}^{4} (\omega_i^2 - \omega_n^2)^{-1} \; l = 1,2,3 \; n = 1, ..., 4.$$

If we introduce the matrix

$$\mathbf{G}^{(4)}(\mathbf{x}) = \mathbf{R}^{(4)}(\mathbf{D_x})\mathbf{Y}^{(4)}(\mathbf{x}). \tag{53}$$

then, we obtain

$$\mathbf{F}^{(4)}(\mathbf{D_x})\mathbf{G}^{(4)}(\mathbf{x}) = \mathbf{F}^{(4)}(\mathbf{D_x})\mathbf{R}^{(4)}(\mathbf{D_x})\mathbf{Y}^{(4)}(\mathbf{x}) = \mathbf{\Theta}^{(4)}(\Delta)\mathbf{Y}^{(4)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x})$$

Hence, $\mathbf{G}^{(4)}(\mathbf{x})$ is a solution to equation $(28)_4$.

THEOREM 4: If the condition (27) is satisfied, then the matrix $\mathbf{G}^{(4)}(\mathbf{x})$ defined by the equation (53) is the fundamental solution of the system of equations (26).

5. BASIC PROPERTIES OF $\mathbf{G}^{(1)}(\mathbf{x})$

THEOREM 5: Each column of the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ is a solution of the system of equations (23) at every point $\mathbf{x} \in \mathrm{E}^3$ except the origin.

THEOREM 6: If the condition (27) is satisfied, then the fundamental solution of the system $\tilde{\mathbf{F}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$ is the matrix

$$\mathbf{W}(\mathbf{x}) = \Big(W_{gh}(\mathbf{x})\Big)_{9\times 9},$$

$$W_{ij}(\mathbf{x}) = \left[\frac{1}{\tilde{\lambda}} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{\mu}\tilde{R}_{ij}\right]\varsigma_2^*(\mathbf{x}),$$

$$W_{i+6,j+6}(\mathbf{x}) = \left[\frac{1}{\kappa_7} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{\kappa_6}\tilde{R}_{ij}\right]\varsigma_2^*(\mathbf{x}),$$

$$W_{44}(\mathbf{x}) = \frac{t_2}{\sigma}\,\varsigma_1^*(\mathbf{x}), W_{55}(\mathbf{x}) = \frac{t_1}{\sigma}\,\varsigma_1^*(\mathbf{x}),$$

$$W_{45}(\mathbf{x}) = W_{54}(\mathbf{x}) = -\frac{r_1}{\sigma}\,\varsigma_1^*(\mathbf{x}), W_{66}(\mathbf{x}) = \frac{\varsigma_1^*}{k},$$

$$W_{i;q+3}(\mathbf{x}) = W_{q+3;j}(\mathbf{x}) = W_{pn}(\mathbf{x}) = W_{np}(\mathbf{x}) = W_{6,i+6}(\mathbf{x}) = W_{i+6,6}(\mathbf{x}) = 0,$$

$$\tilde{R}_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} - \Delta\delta_{ij} \; i,j = 1,2,3 \; q = 1, ..., 6 \; p = 6,7,8,9 \; n = 4,5 \tag{54}$$

LEMMA 2: If condition (27) is satisfied, then

$$\Delta w_{p1}^{(1)}(\Delta) = \frac{1}{M^*}(\Delta + \lambda_6^2)\tilde{N}_{p1}^{(1)}(\Delta) - \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{p1},$$

$$\Delta w_{p5}^{(1)}(\Delta) = \frac{1}{M^*}(\Delta + \lambda_7^2)\tilde{N}_{p5}^{(1)}(\Delta) - \frac{1}{\kappa_6}\Gamma^{(1)}(\Delta)\delta_{p5},\ p = 1, ..., 5 \tag{55}$$

PROOF: Consider

$$w_{p1}^{(1)}(\Delta) = -\frac{1}{M^*\mu}\Big\{\tilde{\lambda}\,\tilde{N}_{p1}^{(1)}(\Delta) - p_1\tilde{N}_{12}^{(1)}(\Delta) - p_2\tilde{N}_{13}^{(1)}(\Delta) + \iota\omega\beta T_0\tilde{N}_{p4}^{(1)}(\Delta)\Big\}.$$

Now

$$\Gamma^{(1)}(\Delta)\delta_{p1} = \frac{1}{M^*}\ \det\mathbf{N}^{(1)}(\Delta)\delta_{p1} = \frac{1}{M^*}\Big\{(\tilde{\lambda}\,\Delta + \rho\omega^2)\tilde{N}_{p1}^{(1)}(\Delta)$$

$$-p_1\Delta\tilde{N}_{12}^{(1)}(\Delta) - p_2\Delta\tilde{N}_{13}^{(1)}(\Delta) + \iota\omega\beta T_0\Delta\tilde{N}_{p4}^{(1)}(\Delta)\Big\}.$$

Therefore,

$$\Delta w_{p1}^{(1)}(\Delta) = -\frac{1}{M^*\mu}\Big\{\tilde{\lambda}\Delta\tilde{N}_{p1}^{(1)}(\Delta) - p_1\Delta\tilde{N}_{12}^{(1)}(\Delta) - p_2\Delta\tilde{N}_{13}^{(1)}(\Delta) + \iota\omega\beta T_0\Delta\tilde{N}_{p4}^{(1)}(\Delta)\Big\}$$

$$= -\frac{1}{M^*\mu}\Big[M^*\,\Gamma^{(1)}(\Delta)\delta_{p1} - (\mu\Delta + \rho\omega^2)\tilde{N}_{p1}^{(1)}(\Delta)\Big]$$

$$= \frac{1}{M^*}(\Delta + \lambda_6^2)\tilde{N}_{p1}^{(1)}(\Delta) - \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{p1}$$

Similarly, we can prove equation $(55)_2$.

THEOREM 7: If condition (27) is satisfied and $\mathbf{x} \in \mathrm{E}^3 - \{\mathbf{0}\}$, then

$$G_{gh}^{(1)}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{p=1}^{5} c_{p11}\varsigma_p(\mathbf{x}) + \tilde{R}_{gh}\,c_{611}\,\varsigma_6(\mathbf{x}),$$

$$G_{g;l+2}^{(1)}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{p=1}^{5} c_{p1l}\varsigma_p(\mathbf{x}), G_{l+2;g}^{(1)}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{p=1}^{5} c_{pl1}\varsigma_p(\mathbf{x}),$$

$$G_{l+2;n+2}^{(1)}(\mathbf{x}) = \sum_{p=1}^{5} c_{pln}\varsigma_p(\mathbf{x}), G_{g;h+6}^{(1)}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{p=1}^{5} c_{p15}\varsigma_p(\mathbf{x}),$$

$$G^{(1)}_{g+6;h}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{p=1}^{5} c_{p51}\varsigma_p(\mathbf{x}), G^{(1)}_{l+2;g+6}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{p=1}^{5} c_{pl5}\varsigma_p(\mathbf{x}),$$

$$G^{(1)}_{g+6;l+2}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{p=1}^{5} c_{p5l}\varsigma_p(\mathbf{x}),$$

$$G^{(1)}_{g+6;h+6}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{p=1}^{5} c_{p55}\varsigma_p(\mathbf{x}) + \tilde{R}_{gh}\, c_{755}\, \varsigma_7(\mathbf{x}),\ \ g, h = 1, 2, 3\ l, n = 2, 3, 4$$

where

$$c_{pgh} = -\frac{r^{(1)}_{2p}}{M^*\lambda_p^2}\tilde{N}^{(1)}_{gh}(-\lambda_p^2), c_{pgl} = \frac{r^{(1)}_{2p}}{M^*}\tilde{N}^{(1)}_{gl}(-\lambda_p^2),$$

$$c_{611} = \frac{1}{\rho\omega^2} = \frac{1}{\mu\lambda_6^2}, c_{755} = \frac{1}{\kappa_8} = \frac{1}{\kappa_6\lambda_7^2},\ \ g, p = 1.,,,5\ h = 1, 5\ l = 2, 3, 4 \tag{56}$$

PROOF: From equation (49),

$$\Delta\varsigma_j(\mathbf{x}) = -\lambda_j^2\varsigma_j(\mathbf{x})\ j = 1, ..., 7 \tag{57}$$

Thus, we have

$$-\frac{1}{\lambda_j^2}\Big(\frac{\partial^2}{\partial x_g \partial x_h} - \tilde{R}_{gh}\Big)\varsigma_j(\mathbf{x}) = \delta_{gh}\, \varsigma_j(\mathbf{x}),\ \ \mathbf{x} \neq \mathbf{0} \tag{58}$$

Consider

$$G^{(1)}_{gh}(\mathbf{x}) = R^{(1)}_{gh}(\mathbf{D_x})Y^{(1)}_{11}(\mathbf{x})$$

$$= \Big[\frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{gh} + w^{(1)}_{11}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}\Big]\sum_{j=1}^{6} r^{(1)}_{1j}\,\varsigma_j(\mathbf{x})$$

$$= \sum_{j=1}^{6} r^{(1)}_{1j}\Big\{\Big[-\frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2) + w^{(1)}_{11}(-\lambda_j^2)\Big]\frac{\partial^2}{\partial x_g \partial x_h} + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2)\tilde{R}_{gh}\Big\}\varsigma_j(\mathbf{x}) \tag{59}$$

From equation $(55)_1$, we have

$$w^{(1)}_{11}(-\lambda_j^2) = -\frac{1}{M^*\lambda_j^2}(-\lambda_j^2 + \lambda_6^2)\tilde{N}^{(1)}_{11}(-\lambda_j^2) + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2). \tag{60}$$

Using equation (60) in equation (59), we get

$$G^{(1)}_{gh}(\mathbf{x}) = \sum_{j=1}^{6} r^{(1)}_{1j}\Big\{\Big[-\frac{1}{M^*\lambda_j^2}(-\lambda_j^2 + \lambda_6^2)\tilde{N}^{(1)}_{11}(-\lambda_j^2)\Big]\frac{\partial^2}{\partial x_g \partial x_h} + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2)\tilde{R}_{gh}\Big\}\varsigma_j(\mathbf{x})$$

$$(61)$$

Now,

$$\Gamma^{(1)}(-\lambda_j^2) r_{1j}^{(1)} = 0 \quad j = 1, ..., 5$$

$$\Gamma^{(1)}(-\lambda_j^2) r_{1j}^{(1)} = 1 \quad j = 6$$

and

$$(-\lambda_j^2 + \lambda_6^2) r_{1j}^{(1)} = r_{2j}^{(1)} \quad j = 1, ..., 5$$

$$(-\lambda_j^2 + \lambda_6^2) r_{1j}^{(1)} = 0 \quad j = 6 \tag{62}$$

By virtue of equation (62), equation (61) becomes

$$G_{gh}^{(1)}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h} \sum_{j=1}^{5} \Big[-\frac{1}{M^* \lambda_j^2} r_{2j}^{(1)} \tilde{N}_{11}^{(1)}(-\lambda_j^2) \Big] \varsigma_j(\mathbf{x}) + \tilde{R}_{gh} \frac{1}{\mu \lambda_6^2} \varsigma_6(\mathbf{x})$$

$$= \frac{\partial^2}{\partial x_g \partial x_h} \sum_{j=1}^{5} c_{j11} \varsigma_j(\mathbf{x}) + \tilde{R}_{gh}\, c_{611}\, \varsigma_6(\mathbf{x}).$$

The remaining formulae of above theorem can be proved in the similar way.

LEMMA 3: If the condition (27) is satisfied, then

$$\sum_{p=1}^{5} r_{2p}^{(1)} = \sum_{p=1}^{5} r_{2p}^{(1)} \lambda_p^2 = \sum_{p=1}^{5} r_{2p}^{(1)} \lambda_p^4 = \sum_{p=1}^{5} r_{2p}^{(1)} \lambda_p^6 = 0, \sum_{p=1}^{5} r_{2p}^{(1)} \lambda_p^8 = 1,$$

$$\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{\lambda_p^2} = \prod_{i=1}^{5} \lambda_i^{-2} = \frac{M^*}{\rho \omega^2 \tilde{N}_{11}^{(1)}(0)} = \frac{M^*}{\kappa_8 \tilde{N}_{55}^{(1)}(0)}, \tag{63}$$

and

$$\sum_{p=1}^{5} c_{p11} = -(\rho \omega^2)^{-1}, \sum_{p=1}^{5} c_{p11} \lambda_p^2 = -\tilde{\lambda}^{-1}$$

$$\sum_{p=1}^{5} c_{p55} = -\kappa_8^{-1}, \sum_{p=1}^{5} c_{p55} \lambda_p^2 = -\kappa_7^{-1},$$

$$\sum_{p=1}^{5} c_{p22} = \frac{t_2}{\sigma}, \sum_{p=1}^{5} c_{p33} = \frac{t_1}{\sigma}, \sum_{p=1}^{5} c_{p44} = k^{-1}. \tag{64}$$

PROOF: Consider

$$\tilde{N}_{11}^{(1)}(-\lambda_p^2) = k \kappa_7 \sigma \lambda_p^8 + M_1^* \lambda_p^6 + M_2^* \lambda_p^4 + M_3^* \lambda_p^2 + \tilde{N}_{11}^{(1)}(0), \tag{65}$$

where M_p^*, $p = 1, 2, 3$ are coefficients, independent of λ_p and skipped due to lengthy calculations.

It is easier to prove the relations (63) using equation (44).

From equations (63) and (65), we get

$$\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{\lambda_p^2} \tilde{N}_{11}^{(1)}(-\lambda_p^2) = \sum_{p=1}^{5} r_{2p}^{(1)} [k\kappa_7 \sigma \lambda_p^6 + M_1^* \lambda_p^4 + M_2^* \lambda_p^2 + M_3^* + \tilde{N}_{11}^{(1)}(0) \lambda_p^{-2}]$$

$$= \tilde{N}_{11}^{(1)}(0) \sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{\lambda_p^2} = \frac{M^*}{\rho\omega^2}$$

and

$$\sum_{p=1}^{5} r_{2p}^{(1)} \tilde{N}_{11}^{(1)}(-\lambda_p^2) = \sum_{p=1}^{5} r_{2p}^{(1)} [k\kappa_7 \sigma \lambda_p^8 + M_1^* \lambda_p^6 + M_2^* \lambda_p^4 + M_3^* \lambda_p^2 + \tilde{N}_{11}^{(1)}(0)] = k\kappa_7 \sigma$$

Therefore, from equation (56), we have

$$\sum_{p=1}^{5} c_{p11} = -\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{M^* \lambda_p^2} \tilde{N}_{11}^{(1)}(-\lambda_p^2) = -(\rho\omega^2)^{-1},$$

$$\sum_{p=1}^{5} c_{p11} \lambda_p^2 = -\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{M^*} \tilde{N}_{11}^{(1)}(-\lambda_p^2) = -\frac{k\kappa_7 \sigma}{M^*} = -\tilde{\lambda}^{-1}.$$

Similarly, we can prove equations $(64)_2$ and $(64)_3$.

THEOREM 8: The relations

$$G_{pq}^{(1)}(\mathbf{x}) - W_{pq}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|)\ p, q = 1, ..., 9 \tag{66}$$

hold in the neighborhood of the origin.

PROOF: For $p, q = 1, 2, 3$, consider

$$G_{pq}^{(1)}(\mathbf{x}) - W_{pq}(\mathbf{x}) = \frac{\partial^2}{\partial x_p \partial x_q} \bar{Y}_{11}(\mathbf{x}) + \tilde{R}_{pq} \bar{Y}_{22}(\mathbf{x}), \tag{67}$$

where

$$\bar{Y}_{11}(\mathbf{x}) = \sum_{j=1}^{5} c_{j11} \varsigma_j(\mathbf{x}) - \frac{\varsigma_2^*(\mathbf{x})}{\tilde{\lambda}},$$

$$\bar{Y}_{22}(\mathbf{x}) = c_{611}\,\varsigma_6(\mathbf{x}) + \frac{\varsigma_2^*(\mathbf{x})}{\mu}. \tag{68}$$

From equation (68), we have

$$\bar{Y}_{11}(\mathbf{x}) = \sum_{j=1}^{5}\frac{-c_{j11}}{4\pi}\sum_{l=0}^{\infty}\frac{\iota^l\lambda_j^l}{l!}|\mathbf{x}|^{l-1} + \frac{|\mathbf{x}|}{8\pi\tilde{\lambda}}$$

$$= -\frac{1}{8\pi}\Big[2\sum_{j=1}^{5}c_{j11}\sum_{l=0}^{\infty}\frac{\iota^l\lambda_j^l}{l!}|\mathbf{x}|^{l-1} - \frac{|\mathbf{x}|}{\tilde{\lambda}}\Big]$$

$$= -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{j=1}^{5}c_{j11} - |\mathbf{x}|\left(\sum_{j=1}^{5}c_{j11}\lambda_j^2 + \frac{1}{\tilde{\lambda}}\right)\right] - \frac{\iota}{4\pi}\sum_{j=1}^{5}c_{j11}\lambda_j + \ \bar{Y}_{33}(\mathbf{x}), \tag{69}$$

Similarly,

$$\bar{Y}_{22}(\mathbf{x}) = -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}c_{611} - |\mathbf{x}|\left(c_{611}\lambda_6^2 - \frac{1}{\mu}\right)\right] - \frac{\iota}{4\pi}c_{611}\lambda_6 + \ \bar{Y}_{44}(\mathbf{x}), \tag{70}$$

where

$$\bar{Y}_{33}(\mathbf{x}) = -\frac{1}{4\pi}\sum_{j=1}^{5}c_{j11}\sum_{l=3}^{\infty}\frac{\iota^l\lambda_j^l}{l!}|\mathbf{x}|^{l-1},$$

$$\bar{Y}_{44}(\mathbf{x}) = -\frac{1}{4\pi}c_{611}\sum_{l=3}^{\infty}\frac{\iota^l\lambda_6^l}{l!}|\mathbf{x}|^{l-1}. \tag{71}$$

Clearly

$$\bar{Y}_{hh}(\mathbf{x}) = O(|\mathbf{x}|^2),\ \frac{\partial}{\partial x_e}\bar{Y}_{hh}(\mathbf{x}) = O(|\mathbf{x}|),$$

$$\frac{\partial^2}{\partial x_e\partial x_i}\bar{Y}_{hh}(\mathbf{x}) = \ \text{constant} + O(|\mathbf{x}|)\ e,i = 1,2,3\ h = 3,4 \tag{72}$$

Consider

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{x_i}{|\mathbf{x}|^3},\ \frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = \left[\frac{3x_i^2}{|\mathbf{x}|^5} - \frac{1}{|\mathbf{x}|^3}\right]$$

Hence,

$$\Delta\frac{1}{|\mathbf{x}|} = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = 0.$$

Therefore,

$$\left(\frac{\partial^2}{\partial x_p\partial x_q} - \tilde{R}_{pq}\right)\frac{1}{|\mathbf{x}|} = \delta_{gh}\Delta\frac{1}{|\mathbf{x}|} = \mathbf{0}. \tag{73}$$

Equation (67) with the aid of equations (64),(69)-(73) becomes

$$G_{pq}^{(1)}(\mathbf{x}) - W_{pq}(\mathbf{x}) = \frac{\partial^2}{\partial x_p \partial x_q}\bar{Y}_{33}(\mathbf{x}) + \tilde{R}_{pq}\,\bar{Y}_{44}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|).$$

Similarly other formulae of equation (66) can be proved.

Therefore, matrix $\mathbf{W}(\mathbf{x})$ is the singular part of the fundamental matrix $\mathbf{G}^{(1)}(\mathbf{x})$ in the neighborhood of the origin.

CONCLUSIONS

The current paper gives the following outcomes:

1. Without utilizing Darcy's law, the linear theory of thermoelasticity with double porosity and microtemperatures is derived. This theory can be useful for finding fundamental solutions, studying wave phenomenon etc.

2. After reducing the governing equations in isotropic medium, the fundamental matrix of system of equations in cases of steady, pseudo-, quasi-static oscillations and equilibrium are obtained and properties of fundamental matrix are discussed.

REFERENCES

[1] R.A. Grot, "Thermodynamics of a continuum with microstructure," Int. J. Engg. Sci. **7**, 801-814 (1969).

[2] D. Iesan, and R. Quintanilla, "On a theory of thermoelasticity with microtemperatures," J. Ther. Stress. **23**, 199-215 (2000).

[3] D. Iesan, "On a theory of micromorphic elastic solids with microtemperatures," J. Ther. Stress. **24**, 737-752 (2001).

[4] D. Iesan, "Thermoelasticity of bodies with microstructure and microtemperatures," Int. J. Solids Struct. **44**, 8648-8662 (2007).

[5] R. K. Wilson, and E.C. Aifantis, "On the theory of consolidation with double porosity-I," Int. J. Engg. Sci. **20**, 1009-1035 (1982).

[6] D. Iesan, and R. Quintanilla, "On a theory of thermoelastic materials with a double porosity structure," J. Ther. Stress. **37**, 1017-1036 (2014).

[7] T. Kansal, "Generalized theory of thermoelastic diffusion with double porosity," Arch. Mech. **70**, 241-268 (2018).

[8] T. Kansal, "Fundamental solution of the system of equations of pseudo oscillations in the theory of thermoelastic diffusion materials with double porosity," MMMS **15**, 317-336 (2019).

[9] T. Kansal, "The theory of generalized micropolar thermoelastic diffusion

with double porosity," Theo. and Appl. Mech. **49**, 85-109 (2022).
[10] M. Svanadze, "Fundamental solutions of the equations of the theory of thermoelasticity with microtemperatures," J. Ther. Stress. **27**, 151-170 (2004).
[11] M. Svanadze, "Fundamental solution in the theory of micromorphic elastic solids with microtemperatures," J. Ther. Stress. **27**, 345-366 (2004).
[12] M. Svanadze, "Fundamental solution in the theory of consolidation with double porosity," J. Mech. Beh. Mat. **16**, 123-130 (2005).
[13] M. Svanadze, and S. De. Cicco, "Fundamental solutions in the full coupled linear theory of elasticity for solid with double porosity," Arch. Mech. **65**, 367-390 (2013).
[14] M. Svanadze, "Fundamental Solution in the linear theory of consolidation for elastic solids with double porosity," J. Math. Sci. **195**, 258-268 (2013).
[15] E. Scarpetta, M. Svanadze, and V. Zampoli, "Fundamental solutions in the theory of thermoelasticity for solids with double porosity," J. Ther. Stress. **37**, 727-748 (2014).
[16] T. Kansal, "Fundamental solutions in the theory of micromorphic thermoelastic diffusion materials with microtemperatures and microconcentrations," CMST **28**, 11-25 (2022).