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The Theory of Thermoelasticity with Double Porosity and Microtemperatures

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Abstract: The aim of the paper is to establish the basic governing equations for anisotropic thermoelastic medium with double porosity and microtemperatures and to construct the fundamental solution of a system of equations in cases of steady, pseudo-, quasi-static oscillations and equilibrium.

Key words: thermoelasticity, double porosity, steady oscillations

I. Introduction

Grot [1] extended the theory of thermodynamics of elastic bodies with microstructure with the assumption that the microelements have different temperatures. He modified Clausius-Duhem inequality to include microtemperatures and added first-order moment of energy equations to the basic balance laws for determining the microtemperatures of a continuum. Iesan and Quintanilla [2] constructed a linear theory for elastic materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures. They proved an existence theorem for initial boundary value problems via the semigroup theory and established the continuous dependence of solutions of the initial data and body loads. Iesan [3] established the field equations of a theory of microstretch thermoelastic bodies with microtemperatures. He proved a uniqueness theorem in the dynamic theory of anisotropic materials. Iesan [4] derived a linear theory of microstretch elastic solids with microtemperatures in which a microelement of a continuum is equipped with the mechanical degrees of freedom for rigid rotations and microdilatation in addition to the classical translation degrees of freedom. He also established a uniqueness result in the dynamical theory of anisotropic bodies.

The double porosity model represents a double porous structure: one with macro porosity which is connected to pores and the other with micro porosity which is connected to fissures. Wilson and Aifantis [5] developed the theory for deformable materials with double porosity. Iesan and Quintanilla [6] derived a non-linear theory of thermoelastic solids with double porous structure. They also linearized the above theory and formulated the basic initial-boundary-value problems. Kansal [7–9] developed linear generalized theories of thermoelastic diffusion and micropolar thermoelastic diffusion with double porosity and constructed fundamental solutions of a system of equations in case of steady oscillations. Svandaze and his co-workers [10–15] have also constructed fundamental solutions in the theory of thermoelasticity with double porosity as well as microtemperatures. Recently Kansal [16] has developed fundamental solutions of a system of equations of isotropic micromorphic thermoelastic diffusion materials with microtemperatures and microconcentrations in case of steady oscillations in terms of elementary functions.

In Sec. II, the constitutive relations, field equations for anisotropic thermoelastic bodies with double porosity and microtemperatures are derived. The system of linearlized equations of steady, peudo-, quasi-static oscillations and equilibrium in the theory of thermoelastic solids with double porosity and microtemperatures are presented in Sec. III. In Sec. IV and V, in terms of elementary functions, the fundamental solutions of basic governing equations in cases of steady, peudo-, quasi-static oscillations and equilibrium are constructed. Finally, some basic properties of fundamental matrix in case of steady oscillations are discussed in Sec. VI.

II. Basic Equations

Following [2, 6, 7], the balance of linear momentum, the balance of energy and the balance of the first moment of energy are given by

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i, \tag{1}$$

$$\rho\dot{U} = \sigma_{ji}\dot{e}_{ji} + \Omega_i\dot{\nu}_{1,i} + \chi_i\dot{\nu}_{2,i} - q_{i,i} - \xi\dot{\nu}_1 - \zeta\dot{\nu}_2, \tag{2}$$

$$\rho\dot{\varepsilon}_i = -Q_{ji,j} - q_i + Q_i, \tag{3}$$

where ξ and ζ satisfy the relations

$$\Omega_{i,i} + \xi + \rho\tilde{g} = \rho k_1 \ddot{\nu}_1, \tag{4}$$

$$\chi_{i,i} + \zeta + \rho\tilde{l} = \rho k_2 \ddot{\nu}_2. \tag{5}$$

Here U is the internal energy per unit mass, ρ is the density, q_i is the heat flux vector, Q_{ij} is the first heat flux moment tensor, Q_i is the micro heat flux average vector, u_i are the components of the displacement vector $\boldsymbol{u}$, F_i are the components of the external forces per unit mass, ε_i is the first moment of energy vector, $\sigma_{ij}(=\sigma_{ji})$ are the components of stress tensor, $e_{ij}(=e_{ji}) = \frac{1}{2}(u_{i,j}+u_{j,i})$ are components of strain tensor, ν_1 and ν_2 are the volume fraction fields corresponding to pores and fissures, respectively, k_1 and k_2 are coefficients of equilibrated inertia, $\tilde{g}$ and $\tilde{l}$ are, respectively, extrinsic equilibrated body forces per unit mass associated to macro pores and fissures, Ω_i, χ_i are, respectively, the components of equilibrated stress vectors corresponding to ν_1, ν_2.

The local form of the principle of entropy can be expressed in the form of an inequality called Clausius-Duhem inequality

$$\rho\dot{S} + \frac{q_{i,i}}{T} - \frac{q_i}{T^2}T_{,i} + \frac{Q_{ij,i}}{T}T_j - \frac{Q_{ij}}{T^2}T_{,i}T_j + \frac{Q_{ij}}{T}T_{j,i} \ge 0, \tag{6}$$

where S is entropy per unit mass, T is absolute temperature, T_i is the microtemperature vector.

In view of Eqs. (3) and (6), the balance of energy (2) reduces to

$$\rho[T\dot{S} - \dot{U} - T_i\dot{\varepsilon}_i] + \sigma_{ij}\dot{e}_{ij} + \Omega_i\dot{\nu}_{1,i} + \chi_i\dot{\nu}_{2,i} - \xi\dot{\nu}_1 - \zeta\dot{\nu}_2 - \frac{q_i}{T}T_{,i} - \frac{Q_{ij}}{T}T_{,i}T_j + Q_{ij}T_{j,i} - (q_i - Q_i)T_i \ge 0. \tag{7}$$

If we introduce Helmholtz free energy function Γ as

$$\Gamma = U + T_i\varepsilon_i - TS, \tag{8}$$

then relation (7) becomes

$$-\rho[\dot{\Gamma} + \dot{T}S - \dot{T}_i\varepsilon_i] + \sigma_{ij}\dot{e}_{ij} + \Omega_i\dot{\nu}_{1,i} + \chi_i\dot{\nu}_{2,i} - \xi\dot{\nu}_1 - \zeta\dot{\nu}_2 - \frac{q_i}{T}T_{,i} - \frac{Q_{ij}}{T}T_{,i}T_j + Q_{ij}T_{j,i} - (q_i - Q_i)T_i \ge 0. \tag{9}$$

The function Γ can be expressed in terms of independent variables $e_{ij}, \nu_1, \nu_{1,i}, \nu_2, \nu_{2,i}, T, T_{,i}, T_i$ and $T_{i,j}$. Therefore, we have

$$\begin{aligned}\dot{\Gamma} = {} & \frac{\partial\Gamma}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial\Gamma}{\partial\nu_1}\dot{\nu}_1 + \frac{\partial\Gamma}{\partial\nu_{1,i}}\dot{\nu}_{1,i} + \frac{\partial\Gamma}{\partial\nu_2}\dot{\nu}_2 + \\ & + \frac{\partial\Gamma}{\partial\nu_{2,i}}\dot{\nu}_{2,i} + \frac{\partial\Gamma}{\partial T}\dot{T} + \frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i} + \frac{\partial\Gamma}{\partial T_i}\dot{T}_i + \frac{\partial\Gamma}{\partial T_{i,j}}\dot{T}_{i,j}\ .\end{aligned} \tag{10}$$

Eq. (9) with the help of Eq. (10) becomes

$$\begin{aligned}&\left[\sigma_{ij} - \rho\frac{\partial\Gamma}{\partial e_{ij}}\right]\dot{e}_{ij} + \left[\Omega_i - \rho\frac{\partial\Gamma}{\partial\nu_{1,i}}\right]\dot{\nu}_{1,i} + \left[\chi_i - \rho\frac{\partial\Gamma}{\partial\nu_{2,i}}\right]\dot{\nu}_{2,i} - \left[\xi + \rho\frac{\partial\Gamma}{\partial\nu_1}\right]\dot{\nu}_1 + \\ &- \left[\zeta + \rho\frac{\partial\Gamma}{\partial\nu_2}\right]\dot{\nu}_2 - \rho\left[S + \frac{\partial\Gamma}{\partial T}\right]\dot{T} + \rho\left[\varepsilon_i - \frac{\partial\Gamma}{\partial T_i}\right]\dot{T}_i - \rho\frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i} - \rho\frac{\partial\Gamma}{\partial T_{i,j}}\dot{T}_{i,j} + \\ &- \frac{q_i}{T}T_{,i} - \frac{Q_{ij}}{T}T_{,i}T_j + Q_{ij}T_{j,i} - (q_i - Q_i)T_i \ge 0.\end{aligned}$$

The inequality should be confirmed for all rates $\dot{e}_{ij}, \dot{\nu}_1, \dot{\nu}_{1,i}, \dot{\nu}_2, \dot{\nu}_{2,i}, \dot{T}, \dot{T}_{,i}, \dot{T}_i$ and $\dot{T}_{i,j}$. Hence the coefficients of above variables must vanish, that is

$$\sigma_{ij} = \rho\frac{\partial\Gamma}{\partial e_{ij}},\ \ \Omega_i = \rho\frac{\partial\Gamma}{\partial \nu_{1,i}},\ \ \chi_i = \rho\frac{\partial\Gamma}{\partial \nu_{2,i}},\ \ \xi = -\rho\frac{\partial\Gamma}{\partial \nu_1}\ ,$$
$$\zeta = -\rho\frac{\partial\Gamma}{\partial \nu_2},\ \ S = -\frac{\partial\Gamma}{\partial T},\ \ \varepsilon_i = \frac{\partial\Gamma}{\partial T_i}\ , \tag{11}$$

$$\frac{\partial\Gamma}{\partial T_{,i}} = \frac{\partial\Gamma}{\partial T_{i,j}} = 0, \tag{12}$$

$$-q_iT_{,i} - Q_{ij}T_{,i}T_j + TQ_{ij}T_{j,i} - T(q_i - Q_i)T_i \ \geq\ 0. \tag{13}$$

Let us introduce the notations

$$\phi = \nu_1 - (\nu_1)_0,\quad \psi = \nu_2 - (\nu_2)_0,\quad \theta = T - T_0, \tag{14}$$

where T_0 is the reference temperature of the body chosen such that $|\frac{\theta}{T_0}| \ll 1$, $(\nu_1)_0$ and $(\nu_2)_0$ are the volume fraction fields in reference configuration.

In the linear theory of materials possessing a centre of symmetry, we can take Γ in the form

$$\begin{gathered}2\rho\Gamma = c_{ijpn}e_{ij}e_{pn} + d^*\phi^2 + f\psi^2 - \frac{a\theta^2}{T_0} + q_{ij}\phi_{,i}\phi_{,j} + f_{ij}\psi_{,i}\psi_{,j} - s_{ij}T_iT_j +\\ +2p_{ij}e_{ij}\phi + 2\gamma_{ij}e_{ij}\psi - 2a_{ij}e_{ij}\theta + 2\alpha_{ij}\phi_{,i}\psi_{,j} - r_{ij}\phi_{,i}T_j - d_{ij}\psi_{,i}T_j +\\ +2\alpha_1\phi\psi - 2\gamma_1\phi\theta - 2\gamma_2\psi\theta.\end{gathered} \tag{15}$$

From Eq. (11), it follows that

$$\begin{gathered}\sigma_{ij} = c_{ijpn}e_{pn} + p_{ij}\phi + \gamma_{ij}\psi - a_{ij}\theta,\\ \Omega_i = q_{ij}\phi_{,j} + \alpha_{ij}\psi_{,j} - r_{ij}T_j\ ,\\ \chi_i = \alpha_{ij}\phi_{,j} + f_{ij}\psi_{,j} - d_{ij}T_j\ ,\\ \xi = -p_{ij}e_{ij} - d^*\phi - \alpha_1\psi + \gamma_1\theta,\\ \zeta = -\gamma_{ij}e_{ij} - \alpha_1\phi - f\psi + \gamma_2\theta,\\ \rho S = a_{ij}e_{ij} + \gamma_1\phi + \gamma_2\psi + \frac{a\theta}{T_0},\\ \rho\varepsilon_i = -s_{ij}T_j - r_{ij}\phi_{,j} - d_{ij}\psi_{,j}\ .\end{gathered} \tag{16}$$

The linear expressions for q_i, Q_i and Q_{ij} are

$$\begin{gathered}q_i = -[k_{ij}\theta_{,j} + \kappa_{ij}T_j],\\ Q_i = (K_{ij} - k_{ij})\theta_{,j} + (-\kappa_{ij} + L_{ij})T_j\ ,\\ Q_{ij} = m_{ijpn}T_{n,p}\ .\end{gathered} \tag{17}$$

The linearized form of Eq. (6) is

$$\rho T_0\dot{S} = -q_{i,i}\ . \tag{18}$$

In view of Eqs. (16) and (17), Eqs. (1), (3)–(5) and (18) become

$$c_{ijpn}e_{pn,j} + p_{ij}\phi_{,j} + \gamma_{ij}\psi_{,j} - a_{ij}\theta_{,j} + \rho F_i = \rho\ddot{u}_i \ ,$$

$$-p_{ij}e_{ij} + q_{ij}\phi_{,ij} - d^*\phi + \alpha_{ij}\psi_{,ij} - \alpha_1\psi + \gamma_1\theta - r_{ij}T_{j,i} + \rho\tilde{g} = \rho k_1\ddot{\phi},$$

$$-\gamma_{ij}e_{ij} + \alpha_{ij}\phi_{,ij} - \alpha_1\phi + f_{ij}\psi_{,ij} - f\psi + \gamma_2\theta - d_{ij}T_{j,i} + \rho\tilde{l} = \rho k_2\ddot{\psi},$$

$$T_0[a_{ij}\dot{e}_{ij} + \gamma_1\dot{\phi} + \gamma_2\dot{\psi}] + a\dot{\theta} = k_{ij}\theta_{,ij} + \kappa_{ij}T_{j,i} \ ,$$

$$m_{jipn}T_{n,pj} - s_{ij}\dot{T}_j - r_{ij}\dot{\phi}_{,j} - d_{ij}\dot{\psi}_{,j} = K_{ij}\theta_{,j} + L_{ij}T_j \ . \tag{19}$$

In the case of an isotropic and homogeneous material, the constitutive equations become

$$\sigma_{ij} = \lambda e_{pp}\delta_{ij} + 2\mu e_{ij} - \beta\theta\delta_{ij} + p_1\phi\delta_{ij} + p_2\psi\delta_{ij} \ ,$$

$$\Omega_i = t_1\phi_{,i} + r_1\psi_{,i} - r_2T_i \ , \ \ \chi_i = r_1\phi_{,i} + t_2\psi_{,i} - r_3T_i \ ,$$

$$\xi = -p_1e_{pp} - d^*\phi - \alpha_1\psi + \gamma_1\theta, \ \ \zeta = -p_2e_{pp} - \alpha_1\phi - f\psi + \gamma_2\theta,$$

$$\rho S = \beta e_{pp} + \gamma_1\phi + \gamma_2\psi + \frac{a\theta}{T_0}, \ \ \rho\varepsilon_i = -\alpha T_i - r_2\phi_{,i} - r_3\psi_{,i} \ ,$$

$$q_i = -[k\theta_{,i} + \kappa_1T_i], \ \ Q_i = (\kappa_3 - k)\theta_{,i} + (-\kappa_1 + \kappa_2)T_i \ ,$$

$$Q_{ij} = \kappa_4T_{p,p}\delta_{ij} + \kappa_5T_{i,j} + \kappa_6T_{j,i} \ , \tag{20}$$

where

$$c_{ijpn} = \lambda\delta_{ij}\delta_{pn} + \mu\delta_{ip}\delta_{jn} + \mu\delta_{in}\delta_{jp}, \ \ a_{ij} = \beta\delta_{ij}, \ \ p_{ij} = p_1\delta_{ij}, \ \ \gamma_{ij} = p_2\delta_{ij} \ ,$$

$$q_{ij} = t_1\delta_{ij}, \ \ \alpha_{ij} = r_1\delta_{ij}, \ \ r_{ij} = r_2\delta_{ij}, \ \ f_{ij} = t_2\delta_{ij}, \ \ d_{ij} = r_3\delta_{ij}, \ \ s_{ij} = \alpha\delta_{ij} \ ,$$

$$m_{ijpn} = \kappa_4\delta_{ij}\delta_{pn} + \kappa_6\delta_{ip}\delta_{jn} + \kappa_5\delta_{in}\delta_{jp}, \ \ k_{ij} = k\delta_{ij}, \ \ \kappa_{ij} = \kappa_1\delta_{ij}, \ \ L_{ij} = \kappa_2\delta_{ij}, \ \ K_{ij} = \kappa_3\delta_{ij} \ .$$

Here $\lambda, \mu, \beta, p_1, p_2, t_1, t_2, r_1, r_2, r_3, d^*, \alpha_1, \gamma_1, \gamma_2, f, \alpha, a, k, \kappa_1, \ldots, \kappa_6$ are material constants.

Therefore, from Eq. (19) we obtain the basic governing equations for homogeneous isotropic thermoelastic material with double porosity and microtemperatures in the absence of body and equilibrated body forces as

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + p_1\,\mathrm{grad}\,\phi + p_2\,\mathrm{grad}\,\psi - \beta\,\mathrm{grad}\,\theta = \rho\ddot{\mathbf{u}},$$

$$-p_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta - d^*)\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2\,\mathrm{div}\,\mathbf{w} = \rho k_1\ddot{\phi},$$

$$-p_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta - f)\psi + \gamma_2\theta - r_3\,\mathrm{div}\,\mathbf{w} = \rho k_2\ddot{\psi},$$

$$T_0[\beta\,\mathrm{div}\,\dot{\mathbf{u}} + \gamma_1\dot{\phi} + \gamma_2\dot{\psi}] + a\dot{\theta} = k\Delta\theta + \kappa_1\mathrm{div}\,\mathbf{w},$$

$$\kappa_6\Delta\,\mathbf{w} + (\kappa_4 + \kappa_5)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{w} - \alpha\dot{\mathbf{w}} - r_2\,\mathrm{grad}\,\dot{\phi} - r_3\,\mathrm{grad}\dot{\psi} = \kappa_3\,\mathrm{grad}\,\theta + \kappa_2\,\mathbf{w}, \tag{21}$$

where $\mathbf{w} = (T_1, T_2, T_3)$ is the microrotation vector and Δ is the Laplacian operator.

III. Steady Oscillations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3, $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$, $\mathbf{D_x} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$. Let us assume the displacement vector, volume fraction fields, temperature change and microtemperature vector functions as:

$$\Big[\mathbf{u}(\mathbf{x},t), \phi(\mathbf{x},t), \psi(\mathbf{x},t), \theta(\mathbf{x},t), \mathbf{w}(\mathbf{x},t)\Big] = \mathrm{Re}\Big[(\mathbf{u}^*, \phi^*, \psi^*, \theta^*, \mathbf{w}^*)e^{-\iota\omega t}\Big], \tag{22}$$

where ω is oscillation frequency.

Therefore, from the system of Eq. (21) we obtain the system of linearlized equations of steady oscillations in the theory of thermoelastic solids with double porosity and microtemperatures as

$$\begin{gathered}
\Big[\mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div} + \rho\omega^2\Big]\mathbf{u} + p_1\,\mathrm{grad}\,\phi + p_2\,\mathrm{grad}\,\psi - \beta\,\mathrm{grad}\,\theta = \mathbf{0},\\
-p_1\,\mathrm{div}\,\mathbf{u} + \Big[t_1\Delta - d^* + \rho k_1\omega^2\Big]\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2\,\mathrm{div}\,\mathbf{w} = 0,\\
-p_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \alpha_1)\phi + \Big[t_2\Delta - f + \rho k_2\omega^2\Big]\psi + \gamma_2\theta - r_3\,\mathrm{div}\,\mathbf{w} = 0,\\
\iota\omega T_0[\beta\,\mathrm{div}\,\mathbf{u} + \gamma_1\phi + \gamma_2\psi] + \big[k\Delta + \iota\omega a\big]\theta + \kappa_1\,\mathrm{div}\,\mathbf{w} = 0,\\
\iota\omega[r_2\,\mathrm{grad}\,\phi + r_3\,\mathrm{grad}\,\psi] - \kappa_3\,\mathrm{grad}\,\theta + \Big[\kappa_6\Delta + (\kappa_4+\kappa_5)\,\mathrm{grad}\ \mathrm{div} - \kappa_2 + \iota\omega\alpha\Big]\mathbf{w} = \mathbf{0}\,.
\end{gathered} \tag{23}$$

If we replace ω by $-\iota\tau$, where τ is a complex number and $\mathrm{Re}(\tau) > 0$ in the system of Eq. (23), then the system of equations of pseudo-oscillations may be obtained as:

$$\begin{gathered}
\Big[\mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div} - \rho\tau^2\Big]\mathbf{u} + p_1\,\mathrm{grad}\,\phi + p_2\,\mathrm{grad}\,\psi - \beta\,\mathrm{grad}\,\theta = \mathbf{0},\\
-p_1\,\mathrm{div}\,\mathbf{u} + \Big[t_1\Delta - d^* - \rho k_1\tau^2\Big]\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2\,\mathrm{div}\,\mathbf{w} = 0,\\
-p_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \alpha_1)\phi + \Big[t_2\Delta - f - \rho k_2\tau^2\Big]\psi + \gamma_2\theta - r_3\,\mathrm{div}\,\mathbf{w} = 0,\\
\tau T_0[\beta\,\mathrm{div}\,\mathbf{u} + \gamma_1\phi + \gamma_2\psi] + \big[k\Delta + \tau a\big]\theta + \kappa_1\,\mathrm{div}\,\mathbf{w} = 0,\\
\tau[r_2\,\mathrm{grad}\,\phi + r_3\,\mathrm{grad}\,\psi] - \kappa_3\,\mathrm{grad}\,\theta + \Big[\kappa_6\Delta + (\kappa_4+\kappa_5)\,\mathrm{grad}\ \mathrm{div} - \kappa_2 + \tau\alpha\Big]\mathbf{w} = \mathbf{0}\,.
\end{gathered} \tag{24}$$

On taking $\rho = 0$ i.e. quasi-static case, we obtain the system of equations of quasi-static oscillations as:

$$\begin{gathered}
\Big[\mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\Big]\mathbf{u} + p_1\,\mathrm{grad}\,\phi + p_2\,\mathrm{grad}\,\psi - \beta\,\mathrm{grad}\,\theta = \mathbf{0},\\
-p_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta - d^*)\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2\,\mathrm{div}\,\mathbf{w} = 0,\\
-p_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta - f)\psi + \gamma_2\theta - r_3\,\mathrm{div}\,\mathbf{w} = 0,\\
\iota\omega T_0[\beta\,\mathrm{div}\,\mathbf{u} + \gamma_1\phi + \gamma_2\psi] + \big[k\Delta + \iota\omega a\big]\theta + \kappa_1\,\mathrm{div}\,\mathbf{w} = 0,\\
\iota\omega[r_2\,\mathrm{grad}\,\phi + r_3\,\mathrm{grad}\,\psi] - \kappa_3\,\mathrm{grad}\,\theta + \Big[\kappa_6\Delta + (\kappa_4+\kappa_5)\,\mathrm{grad}\ \mathrm{div} - \kappa_2 + \iota\omega\alpha\Big]\mathbf{w} = \mathbf{0}\,.
\end{gathered} \tag{25}$$

If we put $\omega = 0$ in the Eq. (23), the system of equations of the equilibrium theory of thermoelasticity with double porosity and microtemperatures as:

$$\begin{aligned}
&\Big[\mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\Big]\mathbf{u} + p_1\,\mathrm{grad}\,\phi + p_2\,\mathrm{grad}\,\psi - \beta\,\mathrm{grad}\,\theta = \mathbf{0},\\
&-p_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta - d^*)\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta - r_2\,\mathrm{div}\,\mathbf{w} = 0,\\
&-p_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta - f)\psi + \gamma_2\theta - r_3\,\mathrm{div}\,\mathbf{w} = 0,\\
&k\Delta\theta + \kappa_1\,\mathrm{div}\,\mathbf{w} = 0,\\
&-\kappa_3\,\mathrm{grad}\,\theta + \Big[\kappa_6\Delta + (\kappa_4+\kappa_5)\,\mathrm{grad}\ \mathrm{div} - \kappa_2\Big]\mathbf{w} = \mathbf{0}\,.
\end{aligned} \tag{26}$$

We introduce the second order matrix differential operators with constant coefficients

$$\mathbf{F}^{(i)}(\mathbf{D_x}) = \Big(F^{(i)}_{gh}(\mathbf{D_x})\Big)_{9\times 9},$$

where

$$\begin{aligned}
&F^{(1)}_{pq}(\mathbf{D_x}) = [\mu\Delta + \rho\omega^2]\delta_{pq} + (\lambda+\mu)\frac{\partial^2}{\partial x_p \partial x_q},\ \ F^{(1)}_{p4}(\mathbf{D_x}) = -F^{(1)}_{4p}(\mathbf{D_x}) = p_1\frac{\partial}{\partial x_p},\\
&F^{(1)}_{p5}(\mathbf{D_x}) = -F^{(1)}_{5p}(\mathbf{D_x}) = p_2\frac{\partial}{\partial x_p},\ \ F^{(1)}_{p6}(\mathbf{D_x}) = -\beta\frac{\partial}{\partial x_p},\ \ F^{(1)}_{44}(\mathbf{D_x}) = t_1\Delta - d^* + \rho k_1\omega^2,\\
&F^{(1)}_{45}(\mathbf{D_x}) = F^{(1)}_{54}(\mathbf{D_x}) = r_1\Delta - \alpha_1,\ \ F^{(1)}_{46}(\mathbf{D_x}) = \gamma_1,\ \ F^{(1)}_{4;q+6}(\mathbf{D_x}) = -r_2\frac{\partial}{\partial x_q},\\
&F^{(1)}_{55}(\mathbf{D_x}) = t_2\Delta - f + \rho k_2\omega^2,\ \ F^{(1)}_{56}(\mathbf{D_x}) = \gamma_2,\ \ F^{(1)}_{5;q+6}(\mathbf{D_x}) = -r_3\frac{\partial}{\partial x_q},\\
&F^{(1)}_{6q}(\mathbf{D_x}) = \iota\omega\beta T_0\frac{\partial}{\partial x_q},\ \ F^{(1)}_{64}(\mathbf{D_x}) = \iota\omega\gamma_1 T_0,\ \ F^{(1)}_{65}(\mathbf{D_x}) = \iota\omega\gamma_2 T_0,\\
&F^{(1)}_{66}(\mathbf{D_x}) = k\Delta + \iota\omega a,\ \ F^{(1)}_{6;q+6}(\mathbf{D_x}) = \kappa_1\frac{\partial}{\partial x_q},\ \ F^{(1)}_{p+6;q}(\mathbf{D_x}) = 0,\\
&F^{(1)}_{p+6;4}(\mathbf{D_x}) = \iota\omega r_2\frac{\partial}{\partial x_p},\ \ F^{(1)}_{p+6;5}(\mathbf{D_x}) = \iota\omega r_3\frac{\partial}{\partial x_p},\ \ F^{(1)}_{p+6;6}(\mathbf{D_x}) = -\kappa_3\frac{\partial}{\partial x_p},\\
&F^{(1)}_{p+6;q+6}(\mathbf{D_x}) = (\kappa_6\Delta - \kappa_2 + \iota\omega\alpha)\delta_{pq} + (\kappa_4+\kappa_5)\frac{\partial^2}{\partial x_p\partial x_q},\ \ F^{(1)}_{p;q+6}(\mathbf{D_x}) = 0,\ \ p,q = 1,2,3.
\end{aligned}$$

Here $i = 1,2,3,4$ corresponds to static, pseudo-, quasi-static oscillations and the equilibrium theory of thermoelasticity with double porosity and microtemperatures, respectively. The matrices $\mathbf{F}^{(i)}(\mathbf{D_x})$, $i = 2,3,4$ can be obtained from matrix $\mathbf{F}^{(1)}(\mathbf{D_x})$ by taking $\omega = -\iota\tau$, $\rho = 0$ and $\omega = 0$, respectively and

$$\tilde{\mathbf{F}}(\mathbf{D_x}) = \Big(\tilde{F}_{gh}(\mathbf{D_x})\Big)_{9\times 9},$$

where

$$\begin{aligned}
&\tilde{F}_{pq}(\mathbf{D_x}) = \mu\Delta\delta_{pq} + (\lambda+\mu)\frac{\partial^2}{\partial x_p\partial x_q},\ \ \tilde{F}_{pi}(\mathbf{D_x}) = \tilde{F}_{ip}(\mathbf{D_x}) = 0,\ \ \tilde{F}_{44}(\mathbf{D_x}) = t_1\Delta,\\
&\tilde{F}_{45}(\mathbf{D_x}) = \tilde{F}_{54}(\mathbf{D_x}) = r_1\Delta,\ \ \tilde{F}_{jn}(\mathbf{D_x}) = \tilde{F}_{nj}(\mathbf{D_x}) = 0,\\
&\tilde{F}_{55}(\mathbf{D_x}) = t_2\Delta,\ \ \tilde{F}_{66}(\mathbf{D_x}) = k\Delta,
\end{aligned}$$

$$\tilde{F}_{p+6;q+6}(\mathbf{D_x}) = (\kappa_6\Delta - \kappa_2 + \iota\omega\alpha)\delta_{pq} + (\kappa_4 + \kappa_5)\frac{\partial^2}{\partial x_p \partial x_q}, \;\; \tilde{F}_{6;q+6}(\mathbf{D_x}) = \tilde{F}_{q+6,6}(\mathbf{D_x}) = 0,$$

$$p, q = 1,2,3; \;\; i = 4,\dots,9; \;\; j = 4,5; \;\; n = 6,7,8,9.$$

The system of Eqs. (23)–(26) can be represented as

$$\mathbf{F}^{(i)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \;\; i = 1,2,3,4,$$

where $\mathbf{U} = (\mathbf{u}, \phi, \psi, \theta, \mathbf{w})$ is a nine-component vector function on E^3. The matrix $\tilde{\mathbf{F}}(\mathbf{D_x})$ is called the principal part of operator $\mathbf{F}^{(i)}(\mathbf{D_x})$.

Definition 1. The operator $\mathbf{F}^{(i)}(\mathbf{D_x})$, $i = 1,2,3,4$ is said to be elliptic if $|\tilde{\mathbf{F}}(\boldsymbol{v})| \neq 0$, where $\boldsymbol{v} = (v_1, v_2, v_3)$. Since $|\tilde{\mathbf{F}}(\boldsymbol{v})| = \mu^2\tilde{\lambda}\sigma k\kappa_6^2\kappa_7|\boldsymbol{v}|^{18}, \tilde{\lambda} = \lambda + 2\mu, \sigma = t_1t_2 - r_1^2, \kappa_7 = \kappa_4 + \kappa_5 + \kappa_6$, therefore operator $\mathbf{F}^{(i)}(\mathbf{D_x})$ is an elliptic differential operator iff

$$\mu\tilde{\lambda}\sigma k\kappa_6\kappa_7 \neq 0. \tag{27}$$

Definition 2. The fundamental solutions of the system of Eqs. (23)–(26) (fundamental matrices of operators $\mathbf{F}^{(i)}$) are the matrices $\mathbf{G}^{(i)}(\mathbf{x}) = \left(G_{gh}^{(i)}(\mathbf{x})\right)_{9\times 9}$ satisfying conditions

$$\mathbf{F}^{(i)}(\mathbf{D_x})\mathbf{G}^{(i)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}), \;\; i = 1,2,3,4, \tag{28}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{I} = (\delta_{gh})_{9\times 9}$ is the unit matrix and $\mathbf{x} \in \mathrm{E}^3$.

IV. Construction of $\mathbf{G}(\mathbf{x})$ in Terms of Elementary Functions

Let us consider the system of non-homogeneous equations

$$[\mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div} + \rho\omega^2]\mathbf{u} - p_1\,\mathrm{grad}\,\phi - p_2\,\mathrm{grad}\,\psi + \iota\omega\beta T_0\,\mathrm{grad}\,\theta = \mathbf{H},$$

$$p_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta + d_1)\phi + (r_1\Delta - \alpha_1)\psi + \iota\omega\gamma_1 T_0\theta + \iota\omega r_2\,\mathrm{div}\,\mathbf{w} = L,$$

$$p_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta + d_2)\psi + \iota\omega\gamma_2 T_0\theta + \iota\omega r_3\,\mathrm{div}\,\mathbf{w} = M,$$

$$-\beta\,\mathrm{div}\,\mathbf{u} + \gamma_1\phi + \gamma_2\psi + (k\Delta + \iota\omega a)\theta - \kappa_3\,\mathrm{div}\,\mathbf{w} = Z,$$

$$-r_2\,\mathrm{grad}\,\phi - r_3\,\mathrm{grad}\,\psi + \kappa_1\,\mathrm{grad}\,\theta + [\kappa_6\Delta + (\kappa_4+\kappa_5)\,\mathrm{grad}\,\mathrm{div} + \kappa_8]\mathbf{w} = \mathbf{X}, \tag{29}$$

where $d_1 = -d^* + \rho k_1\omega^2, \;\; d_2 = -f + \rho k_2\omega^2, \;\; \kappa_8 = -\kappa_2 + \iota\omega\alpha$ and $\mathbf{H}, \mathbf{X}$ are three-component vector functions on E^3; L, M and Z are scalar functions on E^3.

The system of Eq. (29) may also be written in the form

$$\mathbf{F}^{(1)^{tr}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \tag{30}$$

where $\mathbf{F}^{(1)^{tr}}$ is the transpose of matrix $\mathbf{F}^{(1)}$, $\mathbf{Q} = (\mathbf{H}, L, M, Z, \mathbf{X})$ and $\mathbf{x} \in \mathrm{E}^3$.

Applying operator div to the Eqs. $(29)_1$ and $(29)_5$, we obtain

$$[\tilde{\lambda}\Delta + \rho\omega^2]\,\mathrm{div}\,\mathbf{u} - p_1\Delta\phi - p_2\Delta\psi + \iota\omega\beta T_0\Delta\theta = \mathrm{div}\,\mathbf{H}, \tag{31}$$

$$-r_2\Delta\phi - r_3\Delta\psi + \kappa_1\Delta\theta + [\kappa_7\Delta + \kappa_8]\mathrm{div}\,\mathbf{w} = \mathrm{div}\,\mathbf{X}. \tag{32}$$

The Eqs. $(29)_2$–$(29)_4$, (31) and (32) may be expressed in the form

$$\mathbf{N}^{(1)}(\Delta)\mathbf{S} = \tilde{\mathbf{Q}}, \tag{33}$$

where $\mathbf{S} = (\operatorname{div}\mathbf{u}, \phi, \psi, \theta, \operatorname{div}\mathbf{w})$, $\tilde{\mathbf{Q}} = (w_1, \ldots, w_5) = (\operatorname{div}\mathbf{H}, L, M, Z, \operatorname{div}\mathbf{X})$ and

$$\mathbf{N}^{(1)}(\Delta) = \left(N_{gh}^{(1)}(\Delta)\right)_{5\times 5} = \\ = \begin{pmatrix} \tilde{\lambda}\Delta + \rho\omega^2 & -p_1\Delta & -p_2\Delta & \iota\omega\beta T_0\Delta & 0 \\ p_1 & t_1\Delta + d_1 & r_1\Delta - \alpha_1 & \iota\omega\gamma_1 T_0 & \iota\omega r_2 \\ p_2 & r_1\Delta - \alpha_1 & t_2\Delta + d_2 & \iota\omega\gamma_2 T_0 & \iota\omega r_3 \\ -\beta & \gamma_1 & \gamma_2 & k\Delta + \iota\omega a & -\kappa_3 \\ 0 & -r_2\Delta & -r_3\Delta & \kappa_1\Delta & \kappa_7\Delta + \kappa_8 \end{pmatrix}_{5\times 5}. \tag{34}$$

The Eqs. $(29)_2$-$(29)_4$, (31) and (32) may also be written as

$$\Gamma^{(1)}(\Delta)\mathbf{S} = \mathbf{\Psi}, \tag{35}$$

where

$$\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_5), \quad \Psi_p = \frac{1}{M^*}\sum_{i=1}^{5} N_{ip}^{(1)^*} w_i, \\ \Gamma^{(1)}(\Delta) = \frac{1}{M^*}\,|\,\mathbf{N}^{(1)}(\Delta)|, \; M^* = \tilde{\lambda}k\kappa_7\sigma \quad p = 1, \ldots, 5, \tag{36}$$

and $N_{ip}^{(1)^*}$ is the cofactor of the element $N_{ip}^{(1)}$ of the matrix $\mathbf{N}^{(1)}$.

From Eqs. (34) and (36), we see that

$$\Gamma^{(1)}(\Delta) = \prod_{i=1}^{5}(\Delta + \lambda_i^2),$$

where $\lambda_i^2,\; i = 1, \ldots, 5$ are the roots of the equation $\Gamma^{(1)}(-v) = 0$ (with respect to v).

Applying operator $\Gamma^{(1)}(\Delta)$ to the Eqs. $(29)_1$ and $(29)_5$, respectively, we obtain

$$\Gamma^{(1)}(\Delta)(\Delta + \lambda_6^2)\mathbf{u} = \mathbf{\Psi}', \\ \Gamma^{(1)}(\Delta)(\Delta + \lambda_7^2)\mathbf{w} = \mathbf{\Psi}'', \tag{37}$$

where $\lambda_6^2 = \frac{\rho\omega^2}{\mu}$, $\lambda_7^2 = \frac{\kappa_8}{\kappa_6}$ and

$$\mathbf{\Psi}' = \frac{1}{\mu}\left\{\Gamma^{(1)}(\Delta)\mathbf{H} - \operatorname{grad}\left[(\lambda+\mu)\Psi_1 - p_1\,\Psi_2 - p_2\,\Psi_3 + \iota\omega\beta T_0\,\Psi_4\right]\right\}, \\ \mathbf{\Psi}'' = \frac{1}{\kappa_6}\left\{\Gamma^{(1)}(\Delta)\mathbf{X} - \operatorname{grad}\left[(\kappa_4+\kappa_5)\Psi_5 - r_2\,\Psi_2 - r_3\,\Psi_3 + \kappa_1\,\Psi_4\right]\right\}. \tag{38}$$

From Eqs. (36) and (38), we obtain

$$\mathbf{\Theta}^{(1)}(\Delta)\mathbf{U}(\mathbf{x}) = \hat{\mathbf{\Psi}}(\mathbf{x}), \tag{39}$$

where $\hat{\mathbf{\Psi}} = (\mathbf{\Psi}', \Psi_2, \Psi_3, \Psi_4, \mathbf{\Psi}'')$ and

$$\mathbf{\Theta}^{(1)}(\Delta) = \left(\Theta_{gh}^{(1)}(\Delta)\right)_{9\times 9},$$

$$\Theta^{(1)}_{pp}(\Delta) = \Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2) = \prod_{i=1}^{6}(\Delta+\lambda_i^2),$$
$$\Theta^{(1)}_{p+3;p+3}(\Delta) = \Gamma^{(1)}(\Delta) = \prod_{i=1}^{5}(\Delta+\lambda_i^2),$$
$$\Theta^{(1)}_{p+6;p+6}(\Delta) = \Gamma^{(1)}(\Delta)(\Delta+\lambda_7^2) = \prod_{i=1, i\neq 6}^{7}(\Delta+\lambda_i^2),$$
$$\Theta^{(1)}_{gh}(\Delta) = 0;\;\; p=1,2,3;\;\; g,h=1,\ldots,9;\;\; g\neq h.$$

The Eqs. (36) and (38) can be rewritten in the form

$$\mathbf{\Psi}' = \left[\frac{1}{\mu}\Gamma^{(1)}(\Delta)\mathbf{J} + w^{(1)}_{11}(\Delta)\,\mathrm{grad\,div}\right]\mathbf{H} + \sum_{i=2}^{5} w^{(1)}_{i1}(\Delta)\,\mathrm{grad}\,w_i,$$
$$\mathbf{\Psi}'' = \sum_{i=1}^{4} w^{(1)}_{i5}(\Delta)\,\mathrm{grad}\,w_i + \left[\frac{1}{\kappa_6}\Gamma^{(1)}(\Delta)\mathbf{J} + w^{(1)}_{55}(\Delta)\,\mathrm{grad\,div}\right]\mathbf{X},$$
$$\Psi_l = w^{(1)}_{1l}(\Delta)\,\mathrm{div}\,\mathbf{H} + \sum_{i=2}^{4} w^{(1)}_{il}(\Delta) w_i + w^{(1)}_{5l}(\Delta)\,\mathrm{div}\,\mathbf{X};\;\; l=2,3,4, \tag{40}$$

where, we have used the following notations:

$$w^{(1)}_{p1}(\Delta) = -\frac{1}{M^*\mu}\left[(\lambda+\mu)\tilde{N}^{(1)}_{p1}(\Delta) - p_1\tilde{N}^{(1)}_{p2}(\Delta) - p_2\tilde{N}^{(1)}_{p2}(\Delta) + \iota\omega\beta T_0\tilde{N}^{(1)}_{p4}(\Delta)\right],$$
$$w^{(1)}_{p5}(\Delta) = -\frac{1}{M^*\kappa_6}\left[(\kappa_4+\kappa_5)\tilde{N}^{(1)}_{p5}(\Delta) - r_2\tilde{N}^{(1)}_{p2}(\Delta) - r_3\tilde{N}^{(1)}_{p3}(\Delta) + \kappa_1\tilde{N}^{(1)}_{p4}(\Delta)\right],$$
$$w^{(1)}_{pq}(\Delta) = \frac{\tilde{N}^{(1)}_{pq}(\Delta)}{M^*};\;\; p=1,\ldots,5;\;\; q=2,3,4.$$

From Eqs. (40), we have

$$\hat{\mathbf{\Psi}}(\mathbf{x}) = {\mathbf{R}^{(1)}}^{tr}(\mathbf{D}_\mathbf{x})\mathbf{Q}(\mathbf{x}), \tag{41}$$

where

$$\mathbf{R}^{(1)}(\mathbf{D}_\mathbf{x}) = \left(R^{(1)}_{gh}(\mathbf{D}_\mathbf{x})\right)_{9\times 9},$$
$$R^{(1)}_{ij}(\mathbf{D}_\mathbf{x}) = \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{ij} + w^{(1)}_{11}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$R^{(1)}_{i;p+2}(\mathbf{D}_\mathbf{x}) = w^{(1)}_{1p}(\Delta)\frac{\partial}{\partial x_i},\;\; R^{(1)}_{p+2;i}(\mathbf{D}_\mathbf{x}) = w^{(1)}_{p1}(\Delta)\frac{\partial}{\partial x_i},$$
$$R^{(1)}_{i;j+6}(\mathbf{D}_\mathbf{x}) = w^{(1)}_{15}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},\;\; R^{(1)}_{i+6;j}(\mathbf{D}_\mathbf{x}) = w^{(1)}_{51}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$R^{(1)}_{i+6;p+2}(\mathbf{D}_\mathbf{x}) = w^{(1)}_{5p}(\Delta)\frac{\partial}{\partial x_i},\;\; R^{(1)}_{p+2;i+6}(\mathbf{D}_\mathbf{x}) = w^{(1)}_{p5}(\Delta)\frac{\partial}{\partial x_i},$$
$$R^{(1)}_{i+6;j+6}(\mathbf{D}_\mathbf{x}) = \frac{1}{\kappa_6}\Gamma^{(1)}(\Delta)\delta_{ij} + w^{(1)}_{55}(\Delta)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$R^{(1)}_{p+2;n+2}(\mathbf{D}_\mathbf{x}) = w^{(1)}_{pn}(\Delta);\;\; i,j=1,2,3;\;\; p,n=2,3,4. \tag{42}$$

From Eqs. (30), (39) and (41), we obtain

$$\mathbf{\Theta}^{(1)}\mathbf{U} = \mathbf{R}^{(1)^{tr}}\mathbf{F}^{(1)^{tr}}\mathbf{U}.$$

The above relation implies

$$\mathbf{R}^{(1)^{tr}}\mathbf{F}^{(1)^{tr}} = \mathbf{\Theta}^{(1)}.$$

Therefore, we obtain

$$\mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{R}^{(1)}(\mathbf{D_x}) = \mathbf{\Theta}^{(1)}(\Delta). \tag{43}$$

We assume that

$$\lambda_p^2 \neq \lambda_q^2 \neq 0;\ \ p,q = 1,\dots,7;\ \ p \neq q.$$

Let

$$\mathbf{Y}^{(1)}(\mathbf{x}) = \left(Y_{ij}^{(1)}(\mathbf{x})\right)_{9\times 9},\ Y_{pp}^{(1)}(\mathbf{x}) = \sum_{g=1}^{6} r_{1g}^{(1)}\varsigma_g(\mathbf{x}),$$

$$Y_{p+3;p+3}^{(1)}(\mathbf{x}) = \sum_{g=1}^{5} r_{2g}^{(1)}\varsigma_g(\mathbf{x}),\ \ Y_{p+6;p+6}^{(1)}(\mathbf{x}) = \sum_{g=1, g\neq 6}^{7} r_{3g}^{(1)}\varsigma_g(\mathbf{x}),$$

$$Y_{ij}^{(1)}(\mathbf{x}) = 0;\ \ p = 1,2,3;\ \ i,j = 1,\dots,9;\ \ i \neq j,$$

where

$$\varsigma_n(\mathbf{x}) = -\frac{e^{\iota\lambda_n|\mathbf{x}|}}{4\pi|\mathbf{x}|},\ \ r_{1g}^{(1)} = \prod_{i=1, i\neq g}^{6}(\lambda_i^2-\lambda_g^2)^{-1},\ \ r_{2h}^{(1)} = \prod_{i=1, i\neq h}^{5}(\lambda_i^2-\lambda_h^2)^{-1},$$

$$r_{3l}^{(1)} = \prod_{i=1, i\neq 6, l}^{7}(\lambda_i^2-\lambda_l^2)^{-1};\ \ n = 1,\dots,7;\ \ g = 1,\dots,6;\ \ h = 1,\dots,5;\ \ l = 1,\dots,5,7. \tag{44}$$

Lemma 1. The matrix $\mathbf{Y}^{(1)}$ defined above is the fundamental matrix of operator $\mathbf{\Theta}^{(1)}(\Delta)$, i.e.

$$\mathbf{\Theta}^{(1)}(\Delta)\mathbf{Y}^{(1)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}). \tag{45}$$

Proof: To prove the lemma, it is sufficient to prove that

$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2)Y_{11}^{(1)}(\mathbf{x}) = \delta(\mathbf{x}),\ \ \Gamma^{(1)}(\Delta)Y_{44}^{(1)}(\mathbf{x}) = \delta(\mathbf{x}),\ \ \Gamma^{(1)}(\Delta)(\Delta+\lambda_7^2)Y_{77}^{(1)}(\mathbf{x}) = \delta(\mathbf{x}). \tag{46}$$

Consider

$$\sum_{i=1}^{6} r_{1i}^{(1)} = \frac{\sum_{j=1}^{6}(-1)^j z_j}{z_7},$$

where

$$z_1 = \prod_{i=3}^{6}(\lambda_2^2-\lambda_i^2)\prod_{j=4}^{6}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{6}(\lambda_4^2-\lambda_l^2)(\lambda_5^2-\lambda_6^2),$$

$$z_2 = \prod_{i=3}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{6}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{6}(\lambda_4^2-\lambda_l^2)(\lambda_5^2-\lambda_6^2),$$

$$z_3 = \prod_{i=2,i\neq 3}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{6}(\lambda_2^2-\lambda_j^2)\prod_{l=5}^{6}(\lambda_4^2-\lambda_l^2)(\lambda_5^2-\lambda_6^2),$$
$$z_4 = \prod_{i=2,i\neq 4}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq 4}^{6}(\lambda_2^2-\lambda_j^2)\prod_{l=5}^{6}(\lambda_3^2-\lambda_l^2)(\lambda_5^2-\lambda_6^2),$$
$$z_5 = \prod_{i=2,i\neq 5}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq 5}^{6}(\lambda_2^2-\lambda_j^2)\prod_{l=4,l\neq 5}^{6}(\lambda_3^2-\lambda_l^2)(\lambda_4^2-\lambda_6^2),$$
$$z_6 = \prod_{i=2,i\neq 6}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{5}(\lambda_2^2-\lambda_j^2)\prod_{l=4}^{5}(\lambda_3^2-\lambda_l^2)(\lambda_4^2-\lambda_5^2),$$
$$z_7 = \prod_{i=2}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{6}(\lambda_2^2-\lambda_j^2)\prod_{l=4}^{6}(\lambda_3^2-\lambda_l^2)\prod_{p=5}^{6}(\lambda_4^2-\lambda_p^2)(\lambda_5^2-\lambda_6^2).$$

On simplifying the right hand side of the above relation, we obtain

$$\sum_{i=1}^{6} r_{1i}^{(1)} = 0. \tag{47}$$

Similarly, we find that

$$\sum_{i=2}^{6} r_{1i}^{(1)}(\lambda_1^2-\lambda_i^2) = 0, \;\; \sum_{i=3}^{6} r_{1i}^{(1)}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_i^2)\right] = 0,$$
$$\sum_{i=4}^{6} r_{1i}^{(1)}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_i^2)\right] = 0, \;\; \sum_{i=5}^{6} r_{1i}^{(1)}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_i^2)\right] = 0,$$
$$\prod_{j=1}^{5} r_{16}^{(1)}(\lambda_j^2-\lambda_6^2) = 1. \tag{48}$$

Also,

$$(\Delta+\lambda_p^2)\varsigma_g(\mathbf{x}) = \delta(\mathbf{x}) + (\lambda_p^2-\lambda_g^2)\varsigma_g(\mathbf{x}); \;\; p,g = 1,\dots,7. \tag{49}$$

Now consider

$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2)Y_{11}^{(1)}(\mathbf{x}) = \prod_{i=1}^{6}(\Delta+\lambda_i^2)\sum_{g=1}^{6} r_{1g}^{(1)}\varsigma_g(\mathbf{x}) =$$
$$= \prod_{i=2}^{6}(\Delta+\lambda_i^2)\sum_{g=1}^{6} r_{1g}^{(1)}\left[\delta(\mathbf{x}) + (\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] =$$
$$= \prod_{i=2}^{6}(\Delta+\lambda_i^2)\left[\delta(\mathbf{x})\sum_{g=1}^{6} r_{1g}^{(1)} + \sum_{g=2}^{6} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right].$$

Using Eqs. (47)–(49) in the above relation, we obtain

$$\Gamma^{(1)}(\Delta)(\Delta+\lambda_6^2)Y_{11}^{(1)}(\mathbf{x}) = \prod_{i=2}^{6}(\Delta+\lambda_i^2)\left[\sum_{g=2}^{6} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right] =$$
$$= \prod_{i=3}^{6}(\Delta+\lambda_i^2)\left[\sum_{g=2}^{6} r_{1g}^{(1)}(\lambda_1^2-\lambda_g^2)\left[\delta(\mathbf{x}) + (\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\right] =$$

$$= \prod_{i=3}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big] =$$

$$= \prod_{i=4}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=3}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big] =$$

$$= \prod_{i=4}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big] =$$

$$= \prod_{i=5}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=4}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big] =$$

$$= \prod_{i=5}^{6}(\Delta+\lambda_i^2)\Big[\sum_{g=5}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big] =$$

$$= (\Delta+\lambda_6^2)\Big[\sum_{g=5}^{6} r_{1g}^{(1)}\Big[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\Big]\Big[\delta(\mathbf{x})+(\lambda_5^2-\lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big] =$$

$$= (\Delta+\lambda_6^2)\varsigma_6(\mathbf{x}) = \delta(\mathbf{x}).$$

The Eqs. $(46)_2$ and $(46)_3$ can be proved in the similar way.

We introduce the matrix

$$\mathbf{G}^{(1)}(\mathbf{x}) = \mathbf{R}^{(1)}(\mathbf{D_x})\mathbf{Y}^{(1)}(\mathbf{x}). \tag{50}$$

From Eqs. (43), (45) and (50), we obtain

$$\mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{G}^{(1)}(\mathbf{x}) = \mathbf{F}^{(1)}(\mathbf{D_x})\mathbf{R}^{(1)}(\mathbf{D_x})\mathbf{Y}^{(1)}(\mathbf{x}) = \mathbf{\Theta}^{(1)}(\Delta)\mathbf{Y}^{(1)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(1)}(\mathbf{x})$ is a solution to Eq. $(28)_1$.

Theorem 1. If the condition (27) is satisfied, then the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ defined by the Eq. (50) is the fundamental solution of the system of Eq. (23) and the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ is represented in the following form:

$$\mathbf{G}^{(1)}(\mathbf{x}) = \Big(G_{pq}^{(1)}(\mathbf{x})\Big)_{9\times 9},$$

$$\mathbf{G}_{gh}^{(1)}(\mathbf{x}) = R_{gh}^{(1)}(\mathbf{D_x})Y_{11}^{(1)}(\mathbf{x}),\;\; \mathbf{G}_{g;h+3}^{(1)}(\mathbf{x}) = R_{g;h+3}^{(1)}(\mathbf{D_x})Y_{44}^{(1)}(\mathbf{x}),$$

$$\mathbf{G}_{g;h+6}^{(1)}(\mathbf{x}) = R_{g;h+6}^{(1)}(\mathbf{D_x})Y_{77}^{(1)}(\mathbf{x});\;\; g=1,\dots,9;\;\; h=1,2,3.$$

V. Construction of Matrices $\mathbf{G}^{(i)}(\mathbf{x})$ for $\mathbf{i}=\mathbf{2},\mathbf{3},\mathbf{4}$

V. 1. Pseudo-Oscillations

We introduce the matrix

$$\mathbf{G}^{(2)}(\mathbf{x}) = \mathbf{R}^{(2)}(\mathbf{D_x})\mathbf{Y}^{(2)}(\mathbf{x}), \tag{51}$$

where the matrices $\mathbf{R}^{(2)}(\mathbf{D_x})$ and $\mathbf{Y}^{(2)}(\mathbf{x})$ can be obtained from matrices $\mathbf{R}^{(1)}(\mathbf{D_x})$ and $\mathbf{Y}^{(1)}(\mathbf{x})$, respectively, by taking $\omega = -\iota\tau$ and repeating the above procedure after Eq. (28).

Theorem 2. If the condition (27) is satisfied, then the matrix $\mathbf{G}^{(2)}(\mathbf{x})$ defined by the Eq. (51) is the fundamental solution of the system of Eq. (24).

V. 2. Quasi-Static Oscillations

In this case the matrix $\mathbf{N}^{(3)}(\Delta)$, operator $\Gamma^{(3)}(\Delta)$ and matrix operators $\mathbf{\Theta}^{(3)}(\Delta)$, $\mathbf{R}^{(3)}(\mathbf{D_x})$, $\mathbf{Y}^{(3)}(\mathbf{x})$ and $\mathbf{G}^{(3)}(\mathbf{x})$ are obtained as:

$$(\mathbf{i})\quad \hat{\mathbf{N}}^{(3)}(\Delta)=\left(\hat{N}^{(3)}_{gh}(\Delta)\right)_{5\times5},\quad \mathbf{N}^{(3)}(\Delta)=\left(N^{(3)}_{gh}(\Delta)\right)_{5\times5},$$

$$\hat{N}^{(3)}_{11}(\Delta)=\tilde{\lambda},\hat{N}^{(3)}_{12}(\Delta)=-p_1,\ \ \hat{N}^{(3)}_{13}(\Delta)=-p_2,\ \ \hat{N}^{(3)}_{14}(\Delta)=\iota\omega\beta T_0,\ \ \hat{N}^{(3)}_{15}(\Delta)=0,$$

$$N^{(3)}_{1q}(\Delta)=\Delta\hat{N}^{(3)}_{1q}(\Delta),\ \ N^{(3)}_{l1}(\Delta)=\hat{N}^{(3)}_{l1}(\Delta)=N^{(1)}_{l1}(\Delta),$$

$$N^{(3)}_{22}(\Delta)=\hat{N}^{(3)}_{22}(\Delta)=t_1\Delta-d^*,\ \ N^{(3)}_{23}(\Delta)=\hat{N}^{(3)}_{23}(\Delta)=N^{(3)}_{32}(\Delta)=\hat{N}^{(3)}_{32}(\Delta)=r_1\Delta-\alpha_1,$$

$$N^{(3)}_{33}(\Delta)=\hat{N}^{(3)}_{33}(\Delta)=t_2\Delta-f,\ \ N^{(3)}_{lp}(\Delta)=\hat{N}^{(3)}_{lp}(\Delta)=N^{(1)}_{lp}(\Delta),$$

$$N^{(3)}_{pq}(\Delta)=\hat{N}^{(3)}_{pq}(\Delta)=N^{(1)}_{pq}(\Delta);\ \ i=1,2,3;\ \ l=2,3;\ \ p=4,5;\ \ q=1,\dots,5.$$

$$(\mathbf{ii})\quad \Gamma^{(3)}(\Delta)=\Delta\prod_{i=1}^{4}(\Delta+\mu_i^2),$$

where $\mu_i^2,\ i=1,\dots,4$ are the roots of the equation $|\hat{\mathbf{N}}^{(3)}(-v)|=0$ (with respect to v).

$$(\mathbf{iii})\quad \mathbf{\Theta}^{(3)}(\Delta)=\left(\Theta^{(3)}_{gh}(\Delta)\right)_{9\times9},$$

$$\Theta^{(3)}_{pp}(\Delta)=\Gamma^{(3)}(\Delta)\Delta=\Delta^2\prod_{i=1}^{4}(\Delta+\mu_i^2),$$

$$\Theta^{(3)}_{p+3;p+3}(\Delta)=\Gamma^{(3)}(\Delta)=\Delta\prod_{i=1}^{4}(\Delta+\mu_i^2),\ \ \Theta^{(6)}_{p+6;p+6}(\Delta)=\Gamma^{(3)}(\Delta)(\Delta+\mu_5^2)=\Delta\prod_{i=1}^{5}(\Delta+\mu_i^2),$$

$$\Theta^{(3)}_{gh}(\Delta)=0,\ \ \mu_5^2=\frac{\kappa_8}{\kappa_6};\ \ p=1,2,3;\ \ g,h=1,\dots,9;\ \ g\neq h.$$

$$(\mathbf{iv})\quad w^{(3)}_{p1}(\Delta)=-\frac{1}{M^*\mu}\Big[(\lambda+\mu)\,\tilde{N}^{(3)}_{p1}(\Delta)-p_1\tilde{N}^{(3)}_{p2}(\Delta)-p_2\tilde{N}^{(3)}_{p3}(\Delta)+\iota\omega\beta T_0\tilde{N}^{(3)}_{p4}(\Delta)\Big],$$

$$w^{(3)}_{p5}(\Delta)=-\frac{1}{M^*\kappa_6}\Big[(\kappa_4+\kappa_5)\tilde{N}^{(3)}_{p5}(\Delta)-r_2\tilde{N}^{(3)}_{p2}(\Delta)-r_3\tilde{N}^{(3)}_{p3}(\Delta)+\kappa_1\tilde{N}^{(3)}_{p4}(\Delta)\Big],$$

$$w^{(3)}_{pq}(\Delta)=\frac{\tilde{N}^{(3)}_{pq}(\Delta)}{M^*};\ \ p=1,\dots,5;\ \ q=2,3,4,$$

where $\tilde{N}^{(3)}_{ij};\ \ i,j=1,\dots,5$ is the cofactor of the element $N^{(3)}_{ij}$ of the matrix $\mathbf{N}^{(3)}$.

$$(\mathbf{v})\quad \mathbf{R}^{(3)}(\mathbf{D_x})=\left(R^{(3)}_{gh}(\mathbf{D_x})\right)_{9\times9},$$

$$R^{(3)}_{ij}(\mathbf{D_x})=\frac{1}{\mu}\Gamma^{(3)}(\Delta)\delta_{ij}+w^{(3)}_{11}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R^{(3)}_{i;p+2}(\mathbf{D_x})=w^{(3)}_{1p}(\Delta)\frac{\partial}{\partial x_i},\ \ R^{(3)}_{p+2;i}(\mathbf{D_x})=w^{(3)}_{p1}(\Delta)\frac{\partial}{\partial x_i},$$

$$R^{(3)}_{i+6;j}(\mathbf{D_x})=w^{(3)}_{51}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},\ \ R^{(3)}_{i;j+6}(\mathbf{D_x})=w^{(3)}_{15}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R^{(3)}_{i+6;j+6}(\mathbf{D_x})=\frac{1}{\kappa_6}\Gamma^{(3)}(\Delta)\delta_{ij}+w^{(3)}_{55}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R^{(3)}_{i+6;p+2}(\mathbf{D_x}) = w^{(3)}_{5p}(\Delta)\frac{\partial}{\partial x_i},\;\; R^{(3)}_{p+2;i+6}(\mathbf{D_x}) = w^{(3)}_{p5}(\Delta)\frac{\partial}{\partial x_i},$$

$$R^{(3)}_{p+2;q+2}(\mathbf{D_x}) = w^{(3)}_{pq}(\Delta);\;\; i,j = 1,2,3;\;\; p,q = 2,3,4.$$

$$(\mathbf{vi})\;\;\mathbf{Y}^{(3)}(\mathbf{x}) = \left(Y^{(3)}_{ij}(\mathbf{x})\right)_{9\times 9},\;\; Y^{(3)}_{pp}(\mathbf{x}) = r^{(3)}_{11}\varsigma^*_1(\mathbf{x}) + r^{(3)}_{12}\varsigma^*_2(\mathbf{x}) + \sum_{g=1}^{4} r^{(3)}_{1;g+2}\tilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{p+3;p+3}(\mathbf{x}) = r^{(3)}_{21}\varsigma^*_1(\mathbf{x}) + \sum_{g=1}^{4} r^{(3)}_{2;g+1}\tilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{p+6;p+6}(\mathbf{x}) = r^{(3)}_{31}\varsigma^*_1(\mathbf{x}) + \sum_{g=1}^{5} r^{(3)}_{3;g+1}\tilde{\varsigma}_g(\mathbf{x}),$$

$$Y^{(3)}_{ij}(\mathbf{x}) = 0;\;\; p = 1,2,3;\;\; i,j = 1,\ldots,9;\;\; i \neq j,$$

where

$$\varsigma^*_1 = -\frac{1}{4\pi|\mathbf{x}|},\;\; \varsigma^*_2 = -\frac{|\mathbf{x}|}{8\pi},\;\; \tilde{\varsigma}_g(\mathbf{x}) = -\frac{e^{\iota\mu_g|\mathbf{x}|}}{4\pi|\mathbf{x}|};\;\; g = 1,\ldots,5,$$

$$r^{(3)}_{11} = -\frac{\mu_1^2\mu_2^2\mu_3^2 + \mu_1^2\mu_2^2\mu_4^2 + \mu_1^2\mu_3^2\mu_4^2 + \mu_2^2\mu_3^2\mu_4^2}{\mu_1^2\mu_2^2\mu_3^2\mu_4^2},$$

$$r^{(3)}_{12} = r^{(3)}_{21} = \prod_{i=1}^{4}\mu_i^{-2},\;\; r^{(3)}_{1;l+2} = \mu_l^{-4}\prod_{i=1,i\neq l}^{4}(\mu_i^2 - \mu_l^2)^{-1},$$

$$r^{(3)}_{2;l+1} = -\mu_l^{-2}\prod_{i=1,i\neq l}^{4}(\mu_i^2 - \mu_l^2)^{-1},\;\; r^{(3)}_{31} = \prod_{i=1}^{5}\mu_i^{-2},$$

$$r^{(3)}_{3;n+1} = -\mu_l^{-2}\prod_{i=1,i\neq n}^{5}(\mu_i^2 - \mu_n^2)^{-1};\;\; l = 1,\ldots,4;\;\; n = 1,\ldots,5.$$

On introducing the matrix

$$\mathbf{G}^{(3)}(\mathbf{x}) = \mathbf{R}^{(3)}(\mathbf{D_x})\mathbf{Y}^{(3)}(\mathbf{x}), \tag{52}$$

we obtain

$$\mathbf{F}^{(3)}(\mathbf{D_x})\mathbf{G}^{(3)}(\mathbf{x}) = \mathbf{F}^{(3)}(\mathbf{D_x})\mathbf{R}^{(3)}(\mathbf{D_x})\mathbf{Y}^{(3)}(\mathbf{x}) = \mathbf{\Theta}^{(3)}(\Delta)\mathbf{Y}^{(3)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(3)}(\mathbf{x})$ is a fundamental solution to Eq. $(28)_3$.

Theorem 3. If the condition (27) is satisfied, then the matrix $\mathbf{G}^{(3)}(\mathbf{x})$ defined by the Eq. (52) is the fundamental solution of the system of Eq. (25).

V. 3. Equilibrium Theory

In this case, the matrix $\mathbf{N}^{(4)}(\Delta)$, operator $\Gamma^{(4)}(\Delta)$ and matrix operators $\mathbf{\Theta}^{(4)}(\Delta)$, $\mathbf{R}^{(4)}(\mathbf{D_x})$, $\mathbf{Y}^{(4)}(\mathbf{x})$ and $\mathbf{G}^{(4)}(\mathbf{x})$ are obtained as:

$$(\mathbf{i})\;\;\hat{\mathbf{N}}^{(4)}(\Delta) = \left(\hat{N}^{(4)}_{gh}(\Delta)\right)_{6\times 6},\;\; \mathbf{N}^{(4)}(\Delta) = \left(N^{(4)}_{gh}(\Delta)\right)_{6\times 6},$$

$$\hat{N}^{(4)}_{pi}(\Delta) = \hat{N}^{(3)}_{pi}(\Delta),\;\; N^{(4)}_{pi}(\Delta) = N^{(3)}_{pi}(\Delta),\;\; N^{(4)}_{il}(\Delta) = \hat{N}^{(4)}_{il}(\Delta) = 0,$$

$$\hat{N}^{(4)}_{44}(\Delta) = k,\;\; N^{(4)}_{45}(\Delta) = \hat{N}^{(4)}_{45}(\Delta) = -\kappa_3,$$

$$\hat{N}^{(4)}_{54}(\Delta) = \kappa_1,\;\; N^{(4)}_{55}(\Delta) = \hat{N}^{(4)}_{55}(\Delta) = \kappa_7\Delta - \kappa_2,\;\; N^{(4)}_{l4}(\Delta) = \Delta\hat{N}^{(4)}_{l4}(\Delta),$$

$$p = 1,\ldots,5;\;\; i = 1,2,3;\;\; l = 4,5.$$

$$(\mathbf{ii})\quad \Gamma^{(4)}(\Delta)=\Delta^2\prod_{i=1}^{3}(\Delta+\omega_i^2),$$

where $\omega_i^2,\ i=1,2,3$ are the roots of the equation $|\hat{\mathbf{N}}^{(4)}(-\kappa)|=0$ (with respect to κ).

$$(\mathbf{iii})\quad \mathbf{\Theta}^{(4)}(\Delta)=\Big(\Theta_{gh}^{(4)}(\Delta)\Big)_{9\times 9},$$

$$\Theta_{pp}^{(4)}(\Delta)=\Gamma^{(4)}(\Delta)\Delta=\Delta^3\prod_{i=1}^{3}(\Delta+\omega_i^2),$$

$$\Theta_{p+3;p+3}^{(4)}(\Delta)=\Gamma^{(4)}(\Delta)=\Delta^2\prod_{i=1}^{3}(\Delta+\omega_i^2),$$

$$\Theta_{p+6;p+6}^{(4)}(\Delta)=\Gamma^{(4)}(\Delta)(\Delta+\omega_4^2)=\Delta^2\prod_{i=1}^{4}(\Delta+\omega_i^2),\ \ \Theta_{gh}^{(4)}(\Delta)=0,$$

$$\omega_4^2=-\frac{\kappa_2}{\kappa_6};\ \ p=1,2,3;\ \ g,h=1,\dots,9;\ \ g\neq h.$$

$$(\mathbf{iv})\quad w_{p1}^{(4)}(\Delta)=-\frac{1}{M^*\mu}\Big[(\lambda+\mu)\tilde{N}_{p1}^{(4)}(\Delta)-p_1\tilde{N}_{p2}^{(4)}(\Delta)-p_2\tilde{N}_{p3}^{(4)}(\Delta)\Big],$$

$$w_{p5}^{(4)}(\Delta)=-\frac{1}{M^*\kappa_6}\Big[(\kappa_4+\kappa_5)\tilde{N}_{p5}^{(4)}(\Delta)-r_2\tilde{N}_{p2}^{(4)}(\Delta)-r_3\tilde{N}_{p3}^{(4)}(\Delta)+\kappa_1\tilde{N}_{p4}^{(4)}(\Delta)\Big],$$

$$w_{pq}^{(4)}(\Delta)=\frac{\tilde{N}_{pq}^{(4)}(\Delta)}{M^*};\ \ p=1,\dots,5;\ \ q=2,3,4,$$

where $\tilde{N}_{ij}^{(4)};\ \ i,j=1,\dots,5$ is the cofactor of the element $N_{ij}^{(4)}$ of the matrix $\mathbf{N}^{(4)}$.

$$(\mathbf{v})\quad \mathbf{R}^{(4)}(\mathbf{D_x})=\Big(R_{gh}^{(4)}(\mathbf{D_x})\Big)_{9\times 9},$$

$$R_{ij}^{(4)}(\mathbf{D_x})=\frac{1}{\mu}\Gamma^{(4)}(\Delta)\delta_{ij}+w_{11}^{(4)}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i;p+2}^{(4)}(\mathbf{D_x})=w_{1p}^{(4)}(\Delta)\frac{\partial}{\partial x_i},\ \ R_{p+2;i}^{(4)}(\mathbf{D_x})=w_{p1}^{(4)}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{i+6;j}^{(4)}(\mathbf{D_x})=w_{51}^{(4)}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},\ \ R_{i;j+6}^{(4)}(\mathbf{D_x})=w_{15}^{(4)}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i+6;j+6}^{(4)}(\mathbf{D_x})=\frac{1}{\kappa_6}\Gamma^{(4)}(\Delta)\delta_{ij}+w_{55}^{(4)}(\Delta)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$R_{i+6;p+2}^{(4)}(\mathbf{D_x})=w_{5p}^{(4)}(\Delta)\frac{\partial}{\partial x_i},\ \ R_{p+2;i+6}^{(4)}(\mathbf{D_x})=w_{p5}^{(4)}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{p+2;q+2}^{(4)}(\mathbf{D_x})=w_{pq}^{(4)}(\Delta);\ \ i,j=1,2,3;\ \ p,q=2,3,4.$$

$$(\mathbf{vi})\quad \mathbf{Y}^{(4)}(\mathbf{x})=\Big(Y_{ij}^{(4)}(\mathbf{x})\Big)_{9\times 9},\ \ Y_{pp}^{(4)}(\mathbf{x})=r_{11}^{(4)}\varsigma_1^*(\mathbf{x})+r_{12}^{(4)}\varsigma_2^*(\mathbf{x})+\sum_{g=1}^{3}r_{1;g+2}^{(4)}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y_{p+3;p+3}^{(4)}(\mathbf{x})=r_{21}^{(4)}\varsigma_1^*(\mathbf{x})+r_{22}^{(4)}\varsigma_2^*(\mathbf{x})+\sum_{g=1}^{3}r_{2;g+2}^{(4)}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y_{p+6;p+6}^{(4)}(\mathbf{x})=r_{31}^{(4)}\varsigma_1^*(\mathbf{x})+r_{32}^{(4)}\varsigma_2^*(\mathbf{x})+\sum_{g=1}^{4}r_{3;g+2}^{(4)}\hat{\varsigma}_g(\mathbf{x}),$$

$$Y_{ij}^{(4)}(\mathbf{x})=0;\ \ p=1,2,3;\ \ i,j=1,\dots,9;\ \ i\neq j,$$

where

$$\hat{\varsigma}_g(\mathbf{x}) = -\frac{e^{\iota\omega_g|\mathbf{x}|}}{4\pi|\mathbf{x}|};\ \ g = 1, \ldots, 4,$$

$$r^{(4)}_{11} = -\frac{\omega_1^4\omega_2^2(\omega_2^2+\omega_3^2)+\omega_2^4\omega_3^2(\omega_3^2+\omega_1^2)+\omega_3^4\omega_1^2(\omega_1^2+\omega_2^2)}{\omega_1^6\omega_2^6\omega_3^6},$$

$$r^{(4)}_{12} = r^{(4)}_{22} = \frac{1}{\omega_1^2\omega_2^2\omega_3^2},\ \ r^{(4)}_{1;l+2} = -\omega_l^{-6}\prod_{i=1,i\neq l}^{3}(\omega_i^2-\omega_l^2)^{-1},$$

$$r^{(4)}_{21} = -\frac{\omega_1^2\omega_2^2+\omega_2^2\omega_3^2+\omega_3^2\omega_1^2}{\omega_1^4\omega_2^4\omega_3^4},\ \ r^{(4)}_{2;l+2} = \omega_l^{-4}\prod_{i=1,i\neq l}^{3}(\omega_i^2-\omega_l^2)^{-1},$$

$$r^{(4)}_{31} = -\frac{\omega_1^2\omega_2^2\omega_3^2+\omega_1^2\omega_2^2\omega_4^2+\omega_1^2\omega_3^2\omega_4^2+\omega_2^2\omega_3^2\omega_4^2}{\omega_1^2\omega_2^2\omega_3^2\omega_4^2},\ \ r^{(4)}_{32} = \prod_{i=1}^{4}\omega_i^{-2},$$

$$r^{(4)}_{3;n+2} = \omega_n^{-4}\prod_{i=1,i\neq n}^{4}(\omega_i^2-\omega_n^2)^{-1};\ \ l=1,2,3;\ \ n=1,\ldots,4.$$

If we introduce the matrix

$$\mathbf{G}^{(4)}(\mathbf{x}) = \mathbf{R}^{(4)}(\mathbf{D_x})\mathbf{Y}^{(4)}(\mathbf{x}). \tag{53}$$

then we obtain

$$\mathbf{F}^{(4)}(\mathbf{D_x})\mathbf{G}^{(4)}(\mathbf{x}) = \mathbf{F}^{(4)}(\mathbf{D_x})\mathbf{R}^{(4)}(\mathbf{D_x})\mathbf{Y}^{(4)}(\mathbf{x}) = \mathbf{\Theta}^{(4)}(\Delta)\mathbf{Y}^{(4)}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}^{(4)}(\mathbf{x})$ is a solution to Eq. $(28)_4$.

Theorem 4. If the condition (27) is satisfied, then the matrix $\mathbf{G}^{(4)}(\mathbf{x})$ defined by the Eq. (53) is the fundamental solution of the system of Eq. (26).

VI. Basic Properties of $\mathbf{G}^{(1)}(\mathbf{x})$

Theorem 5. Each column of the matrix $\mathbf{G}^{(1)}(\mathbf{x})$ is a solution of the system of Eq. (23) at every point $\mathbf{x} \in \mathrm{E}^3$ except the origin.

Theorem 6. If the condition (27) is satisfied, then the fundamental solution of the system $\tilde{\mathbf{F}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$ is the matrix

$$\mathbf{W}(\mathbf{x}) = \Big(W_{gh}(\mathbf{x})\Big)_{9\times 9},$$

$$W_{ij}(\mathbf{x}) = \left[\frac{1}{\tilde{\lambda}}\frac{\partial^2}{\partial x_i\partial x_j} - \frac{1}{\mu}\tilde{R}_{ij}\right]\varsigma_2^*(\mathbf{x}),$$

$$W_{i+6,j+6}(\mathbf{x}) = \left[\frac{1}{\kappa_7}\frac{\partial^2}{\partial x_i\partial x_j} - \frac{1}{\kappa_6}\tilde{R}_{ij}\right]\varsigma_2^*(\mathbf{x}),$$

$$W_{44}(\mathbf{x}) = \frac{t_2}{\sigma}\,\varsigma_1^*(\mathbf{x}),\ \ W_{55}(\mathbf{x}) = \frac{t_1}{\sigma}\,\varsigma_1^*(\mathbf{x}),$$

$$W_{45}(\mathbf{x}) = W_{54}(\mathbf{x}) = -\frac{r_1}{\sigma}\,\varsigma_1^*(\mathbf{x}),\ \ W_{66}(\mathbf{x}) = \frac{\varsigma_1^*}{k},$$

$$W_{i;q+3}(\mathbf{x}) = W_{q+3;j}(\mathbf{x}) = W_{pn}(\mathbf{x}) = W_{np}(\mathbf{x}) = W_{6,i+6}(\mathbf{x}) = W_{i+6,6}(\mathbf{x}) = 0,$$

$$\tilde{R}_{ij} = \frac{\partial^2}{\partial x_i\partial x_j} - \Delta\delta_{ij};\ \ i,j=1,2,3;\ \ q=1,\ldots,6;\ \ p=6,7,8,9;\ \ n=4,5. \tag{54}$$

Lemma 2. If condition (27) is satisfied, then

$$\Delta w^{(1)}_{p1}(\Delta) = \frac{1}{M^*}(\Delta + \lambda_6^2)\tilde{N}^{(1)}_{p1}(\Delta) - \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{p1},$$
$$\Delta w^{(1)}_{p5}(\Delta) = \frac{1}{M^*}(\Delta + \lambda_7^2)\tilde{N}^{(1)}_{p5}(\Delta) - \frac{1}{\kappa_6}\Gamma^{(1)}(\Delta)\delta_{p5};\ \ p = 1, \ldots, 5. \tag{55}$$

Proof: Consider

$$w^{(1)}_{p1}(\Delta) = -\frac{1}{M^*\mu}\Big\{\tilde{\lambda}\,\tilde{N}^{(1)}_{p1}(\Delta) - p_1\tilde{N}^{(1)}_{12}(\Delta) - p_2\tilde{N}^{(1)}_{13}(\Delta) + \iota\omega\beta T_0\tilde{N}^{(1)}_{p4}(\Delta)\Big\}.$$

Now

$$\Gamma^{(1)}(\Delta)\delta_{p1} = \frac{1}{M^*}\det\mathbf{N}^{(1)}(\Delta)\delta_{p1} = \frac{1}{M^*}\Big\{(\tilde{\lambda}\,\Delta + \rho\omega^2)\tilde{N}^{(1)}_{p1}(\Delta) +$$
$$-p_1\Delta\tilde{N}^{(1)}_{12}(\Delta) - p_2\Delta\tilde{N}^{(1)}_{13}(\Delta) + \iota\omega\beta T_0\Delta\tilde{N}^{(1)}_{p4}(\Delta)\Big\}.$$

Therefore,

$$\Delta w^{(1)}_{p1}(\Delta) = -\frac{1}{M^*\mu}\Big\{\tilde{\lambda}\Delta\tilde{N}^{(1)}_{p1}(\Delta) - p_1\Delta\tilde{N}^{(1)}_{12}(\Delta) - p_2\Delta\tilde{N}^{(1)}_{13}(\Delta) + \iota\omega\beta T_0\Delta\tilde{N}^{(1)}_{p4}(\Delta)\Big\} =$$
$$= -\frac{1}{M^*\mu}\Big[M^*\,\Gamma^{(1)}(\Delta)\delta_{p1} - (\mu\Delta + \rho\omega^2)\tilde{N}^{(1)}_{p1}(\Delta)\Big] =$$
$$= \frac{1}{M^*}(\Delta + \lambda_6^2)\tilde{N}^{(1)}_{p1}(\Delta) - \frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{p1}\,.$$

Similarly, we can prove Eq. $(55)_2$.

Theorem 7. If condition (27) is satisfied and $\mathbf{x} \in \mathrm{E}^3 - \{\mathbf{0}\}$, then

$$G^{(1)}_{gh}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{p=1}^{5} c_{p11}\varsigma_p(\mathbf{x}) + \tilde{R}_{gh}\,c_{611}\,\varsigma_6(\mathbf{x}),$$
$$G^{(1)}_{g;l+2}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{p=1}^{5} c_{p1l}\varsigma_p(\mathbf{x}),\ \ G^{(1)}_{l+2;g}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{p=1}^{5} c_{pl1}\varsigma_p(\mathbf{x}),$$
$$G^{(1)}_{l+2;n+2}(\mathbf{x}) = \sum_{p=1}^{5} c_{pln}\varsigma_p(\mathbf{x}),\ \ G^{(1)}_{g;h+6}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{p=1}^{5} c_{p15}\varsigma_p(\mathbf{x}),$$
$$G^{(1)}_{g+6;h}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{p=1}^{5} c_{p51}\varsigma_p(\mathbf{x}),\ \ G^{(1)}_{l+2;g+6}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{p=1}^{5} c_{pl5}\varsigma_p(\mathbf{x}),$$
$$G^{(1)}_{g+6;l+2}(\mathbf{x}) = \frac{\partial}{\partial x_g}\sum_{p=1}^{5} c_{p5l}\varsigma_p(\mathbf{x}),$$
$$G^{(1)}_{g+6;h+6}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{p=1}^{5} c_{p55}\varsigma_p(\mathbf{x}) + \tilde{R}_{gh}\,c_{755}\,\varsigma_7(\mathbf{x});\ \ g, h = 1, 2, 3;\ \ l, n = 2, 3, 4,$$

where

$$c_{pgh} = -\frac{r^{(1)}_{2p}}{M^*\lambda_p^2}\tilde{N}^{(1)}_{gh}(-\lambda_p^2),\ \ c_{pgl} = \frac{r^{(1)}_{2p}}{M^*}\tilde{N}^{(1)}_{gl}(-\lambda_p^2),$$
$$c_{611} = \frac{1}{\rho\omega^2} = \frac{1}{\mu\lambda_6^2},\ \ c_{755} = \frac{1}{\kappa_8} = \frac{1}{\kappa_6\lambda_7^2};\ \ g, p = 1, \ldots, 5;\ \ h = 1, 5;\ \ l = 2, 3, 4. \tag{56}$$

Proof: From Eq. (49),

$$\Delta\varsigma_j(\mathbf{x}) = -\lambda_j^2\varsigma_j(\mathbf{x});\;\; j = 1,\ldots,7. \tag{57}$$

Thus, we have

$$-\frac{1}{\lambda_j^2}\left(\frac{\partial^2}{\partial x_g \partial x_h} - \tilde{R}_{gh}\right)\varsigma_j(\mathbf{x}) = \delta_{gh}\,\varsigma_j(\mathbf{x}),\;\; \mathbf{x} \neq \mathbf{0}. \tag{58}$$

Consider

$$G_{gh}^{(1)}(\mathbf{x}) = R_{gh}^{(1)}(\mathbf{D_x})Y_{11}^{(1)}(\mathbf{x}) =$$

$$= \left[\frac{1}{\mu}\Gamma^{(1)}(\Delta)\delta_{gh} + w_{11}^{(1)}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}\right]\sum_{j=1}^{6} r_{1j}^{(1)}\varsigma_j(\mathbf{x}) =$$

$$= \sum_{j=1}^{6} r_{1j}^{(1)}\left\{\left[-\frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2) + w_{11}^{(1)}(-\lambda_j^2)\right]\frac{\partial^2}{\partial x_g \partial x_h} + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2)\tilde{R}_{gh}\right\}\varsigma_j(\mathbf{x}). \tag{59}$$

From Eq. $(55)_1$, we have

$$w_{11}^{(1)}(-\lambda_j^2) = -\frac{1}{M^*\lambda_j^2}(-\lambda_j^2 + \lambda_6^2)\tilde{N}_{11}^{(1)}(-\lambda_j^2) + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2). \tag{60}$$

Using Eq. (60) in Eq. (59), we get

$$G_{gh}^{(1)}(\mathbf{x}) = \sum_{j=1}^{6} r_{1j}^{(1)}\left\{\left[-\frac{1}{M^*\lambda_j^2}(-\lambda_j^2 + \lambda_6^2)\tilde{N}_{11}^{(1)}(-\lambda_j^2)\right]\frac{\partial^2}{\partial x_g \partial x_h} + \frac{1}{\mu\lambda_j^2}\Gamma^{(1)}(-\lambda_j^2)\tilde{R}_{gh}\right\}\varsigma_j(\mathbf{x}). \tag{61}$$

Now,

$$\Gamma^{(1)}(-\lambda_j^2)r_{1j}^{(1)} = 0;\;\; j = 1,\ldots,5,$$

$$\Gamma^{(1)}(-\lambda_j^2)r_{1j}^{(1)} = 1;\;\; j = 6,$$

and

$$(-\lambda_j^2 + \lambda_6^2)r_{1j}^{(1)} = r_{2j}^{(1)};\;\; j = 1,\ldots,5,$$

$$(-\lambda_j^2 + \lambda_6^2)r_{1j}^{(1)} = 0;\;\; j = 6. \tag{62}$$

By virtue of Eq. (62), Eq. (61) becomes

$$G_{gh}^{(1)}(\mathbf{x}) = \frac{\partial^2}{\partial x_g \partial x_h}\sum_{j=1}^{5}\left[-\frac{1}{M^*\lambda_j^2}r_{2j}^{(1)}\tilde{N}_{11}^{(1)}(-\lambda_j^2)\right]\varsigma_j(\mathbf{x}) + \tilde{R}_{gh}\frac{1}{\mu\lambda_6^2}\varsigma_6(\mathbf{x}) =$$

$$= \frac{\partial^2}{\partial x_g \partial x_h}\sum_{j=1}^{5} c_{j11}\varsigma_j(\mathbf{x}) + \tilde{R}_{gh}\,c_{611}\,\varsigma_6(\mathbf{x}).$$

The remaining formulae of the above theorem can be proved in the similar way.

Lemma 3. If the condition (27) is satisfied, then

$$\sum_{p=1}^{5} r_{2p}^{(1)} = \sum_{p=1}^{5} r_{2p}^{(1)}\lambda_p^2 = \sum_{p=1}^{5} r_{2p}^{(1)}\lambda_p^4 = \sum_{p=1}^{5} r_{2p}^{(1)}\lambda_p^6 = 0, \;\; \sum_{p=1}^{5} r_{2p}^{(1)}\lambda_p^8 = 1,$$
$$\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{\lambda_p^2} = \prod_{i=1}^{5}\lambda_i^{-2} = \frac{M^*}{\rho\omega^2 \tilde{N}_{11}^{(1)}(0)} = \frac{M^*}{\kappa_8 \tilde{N}_{55}^{(1)}(0)}, \tag{63}$$

and

$$\sum_{p=1}^{5} c_{p11} = -(\rho\omega^2)^{-1}, \;\; \sum_{p=1}^{5} c_{p11}\lambda_p^2 = -\tilde{\lambda}^{-1},$$
$$\sum_{p=1}^{5} c_{p55} = -\kappa_8^{-1}, \;\; \sum_{p=1}^{5} c_{p55}\lambda_p^2 = -\kappa_7^{-1},$$
$$\sum_{p=1}^{5} c_{p22} = \frac{t_2}{\sigma}, \;\; \sum_{p=1}^{5} c_{p33} = \frac{t_1}{\sigma}, \;\; \sum_{p=1}^{5} c_{p44} = k^{-1}. \tag{64}$$

Proof: Consider

$$\tilde{N}_{11}^{(1)}(-\lambda_p^2) = k\kappa_7\sigma\lambda_p^8 + M_1^*\lambda_p^6 + M_2^*\lambda_p^4 + M_3^*\lambda_p^2 + \tilde{N}_{11}^{(1)}(0), \tag{65}$$

where $M_p^*,\ p = 1, 2, 3$ are coefficients, independent of λ_p and skipped due to lengthy calculations.

It is easier to prove the relations (63) using Eq. (44). From Eqs. (63) and (65), we get

$$\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{\lambda_p^2}\tilde{N}_{11}^{(1)}(-\lambda_p^2) = \sum_{p=1}^{5} r_{2p}^{(1)}[k\kappa_7\sigma\lambda_p^6 + M_1^*\lambda_p^4 + M_2^*\lambda_p^2 + M_3^* + \tilde{N}_{11}^{(1)}(0)\lambda_p^{-2}] =$$
$$= \tilde{N}_{11}^{(1)}(0)\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{\lambda_p^2} = \frac{M^*}{\rho\omega^2},$$

and

$$\sum_{p=1}^{5} r_{2p}^{(1)}\tilde{N}_{11}^{(1)}(-\lambda_p^2) = \sum_{p=1}^{5} r_{2p}^{(1)}\left[k\kappa_7\sigma\lambda_p^8 + M_1^*\lambda_p^6 + M_2^*\lambda_p^4 + M_3^*\lambda_p^2 + \tilde{N}_{11}^{(1)}(0)\right] = k\kappa_7\sigma.$$

Therefore, from Eq. (56), we have

$$\sum_{p=1}^{5} c_{p11} = -\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{M^*\lambda_p^2}\tilde{N}_{11}^{(1)}(-\lambda_p^2) = -(\rho\omega^2)^{-1},$$
$$\sum_{p=1}^{5} c_{p11}\lambda_p^2 = -\sum_{p=1}^{5} \frac{r_{2p}^{(1)}}{M^*}\tilde{N}_{11}^{(1)}(-\lambda_p^2) = -\frac{k\kappa_7\sigma}{M^*} = -\tilde{\lambda}^{-1}.$$

Similarly, we can prove Eqs. $(64)_2$ and $(64)_3$.

Theorem 8. The relations

$$G_{pq}^{(1)}(\mathbf{x}) - W_{pq}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|); \;\; p, q = 1, \ldots, 9, \tag{66}$$

hold in the neighbourhood of the origin.

Proof: For $p, q = 1, 2, 3$, consider

$$G^{(1)}_{pq}(\mathbf{x}) - W_{pq}(\mathbf{x}) = \frac{\partial^2}{\partial x_p \partial x_q}\bar{Y}_{11}(\mathbf{x}) + \tilde{R}_{pq}\,\bar{Y}_{22}(\mathbf{x}), \tag{67}$$

where

$$\bar{Y}_{11}(\mathbf{x}) = \sum_{j=1}^{5} c_{j11}\varsigma_j(\mathbf{x}) - \frac{\varsigma_2^*(\mathbf{x})}{\tilde{\lambda}},$$
$$\bar{Y}_{22}(\mathbf{x}) = c_{611}\,\varsigma_6(\mathbf{x}) + \frac{\varsigma_2^*(\mathbf{x})}{\mu}. \tag{68}$$

From Eq. (68), we have

$$\bar{Y}_{11}(\mathbf{x}) = \sum_{j=1}^{5}\frac{-c_{j11}}{4\pi}\sum_{l=0}^{\infty}\frac{\iota^l\lambda_j^l}{l!}|\mathbf{x}|^{l-1} + \frac{|\mathbf{x}|}{8\pi\tilde{\lambda}} =$$
$$= -\frac{1}{8\pi}\left[2\sum_{j=1}^{5}c_{j11}\sum_{l=0}^{\infty}\frac{\iota^l\lambda_j^l}{l!}|\mathbf{x}|^{l-1} - \frac{|\mathbf{x}|}{\tilde{\lambda}}\right] =$$
$$= -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{j=1}^{5}c_{j11} - |\mathbf{x}|\left(\sum_{j=1}^{5}c_{j11}\lambda_j^2 + \frac{1}{\tilde{\lambda}}\right)\right] - \frac{\iota}{4\pi}\sum_{j=1}^{5}c_{j11}\lambda_j + \bar{Y}_{33}(\mathbf{x}). \tag{69}$$

Similarly,

$$\bar{Y}_{22}(\mathbf{x}) = -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}c_{611} - |\mathbf{x}|\left(c_{611}\lambda_6^2 - \frac{1}{\mu}\right)\right] - \frac{\iota}{4\pi}c_{611}\lambda_6 + \bar{Y}_{44}(\mathbf{x}), \tag{70}$$

where

$$\bar{Y}_{33}(\mathbf{x}) = -\frac{1}{4\pi}\sum_{j=1}^{5}c_{j11}\sum_{l=3}^{\infty}\frac{\iota^l\lambda_j^l}{l!}|\mathbf{x}|^{l-1},$$
$$\bar{Y}_{44}(\mathbf{x}) = -\frac{1}{4\pi}c_{611}\sum_{l=3}^{\infty}\frac{\iota^l\lambda_6^l}{l!}|\mathbf{x}|^{l-1}. \tag{71}$$

Clearly

$$\bar{Y}_{hh}(\mathbf{x}) = O(|\mathbf{x}|^2),\;\; \frac{\partial}{\partial x_e}\bar{Y}_{hh}(\mathbf{x}) = O(|\mathbf{x}|),$$
$$\frac{\partial^2}{\partial x_e \partial x_i}\bar{Y}_{hh}(\mathbf{x}) = \text{ constant} + O(|\mathbf{x}|);\;\; e, i = 1, 2, 3;\;\; h = 3, 4. \tag{72}$$

Consider

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{x_i}{|\mathbf{x}|^3},\;\; \frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = \left[\frac{3x_i^2}{|\mathbf{x}|^5} - \frac{1}{|\mathbf{x}|^3}\right].$$

Hence,

$$\Delta\frac{1}{|\mathbf{x}|} = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = 0.$$

Therefore,

$$\left(\frac{\partial^2}{\partial x_p \partial x_q} - \tilde{R}_{pq}\right)\frac{1}{|\mathbf{x}|} = \delta_{gh}\Delta\frac{1}{|\mathbf{x}|} = \mathbf{0}. \tag{73}$$

Eq. (67) with the aid of Eqs. (64), (69)–(73) becomes

$$G^{(1)}_{pq}(\mathbf{x}) - W_{pq}(\mathbf{x}) = \frac{\partial^2}{\partial x_p \partial x_q}\bar{Y}_{33}(\mathbf{x}) + \tilde{R}_{pq}\,\bar{Y}_{44}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|).$$

Similarly other formulae of Eq. (66) can be proved.

Therefore, matrix $\mathbf{W}(\mathbf{x})$ is the singular part of the fundamental matrix $\mathbf{G}^{(1)}(\mathbf{x})$ in the neighbourhood of the origin.

VII. Conclusions

The current paper gives the following outcomes:

1. Without utilizing Darcy's law, the linear theory of thermoelasticity with double porosity and microtemperatures is derived. This theory can be useful for finding fundamental solutions, studying wave phenomenon, etc.
2. After reducing the governing equations in isotropic medium, the fundamental matrix of system of equations in cases of steady, pseudo-, quasi-static oscillations and equilibrium are obtained and properties of fundamental matrix are discussed.

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