$1000 SNOOK PRIZES FOR 2021: The Information Dimensions of a Two-Dimensional Baker Map

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Abstract: The fractal information dimension can be computed in three ways: (1) mapping points, (2) mapping regions (two-dimensional areas here), and (3) applying the Kaplan-Yorke conjecture. For the simplest nonequilibrium Baker N2 Map these three approaches can give different results. A pedagogical exploration and explanation of this situation is the 2021 Ian Snook Prize Problem.

Key words: fractals, maps, differential equations, molecular dynamics, Snook Prizes

I. THE 2021 SNOOK PRIZES

Time-reversible deterministic maps provide important statistical models for the fractal distributions characterizing nonequibrium stationary states studied with molecular dynamics. Such a state is the two-body stationary viscous flow generated with driving forces from shearing boundary conditions and thermostating forces extracting energy so as to impose a time-averaged temperature on the dynamics. Though time-reversible such studies lead to fractal phase-space distributions obeying the irreversible Second Law of thermodynamics.

The two Snook Prizes, $500 from the Hoovers and $500 from CMST, will be awarded to the author(s) of the best paper addressing the differences among three methods for calculating the information dimension(s!) of the ergodic fractals obtained by pointwise or areawise iteration of the simple Baker Map illustrated in Fig. 1 and taken from Ref. 1. Pointwise iteration of the linear N2 Map produces the ergodic fractal shown in Fig. 2 and taken from Ref. 2.

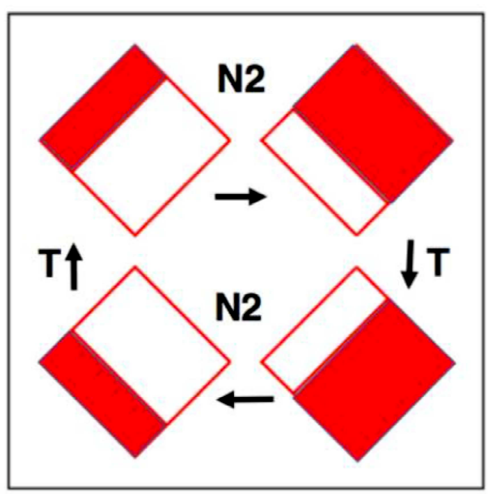

Fig. 1. The compressible Baker Map "N2" expands areas twofold in the red region and contracts twofold in the white. The time-reversal map "T" changes the sign of the vertical "momentum". The clockwise cycle of operations N2 × T × N2 × T shows the time-reversibility of the N2 map

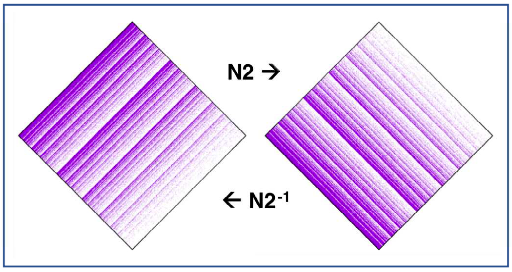

Fig. 2. A million-point attractive fractal and its time-reversed repellor source as obtained from pointwise iteration of the time-reversible Baker Map N2 and its inverse $N2^{-1} = T \times N2 \times T$

Analyses with as many as a trillion points suggest that the bin-based information dimension,

$$D_I = \sum P \ln(P)/\ln(\delta) \text{ with } \delta \to 0 \ ,$$

for bin widths $\delta = (1/2)^n$ and $(1/3)^n$ may well be different with the two mesh families suggesting dimensionalities D_I(points) of 1.734 and 1.741 respectively.

An area-based analysis, likewise with equal-sized bins and an initial condition of uniform density gives a different information dimension! Dividing a unit square into 9 bins of area (1/9) a single iteration of the N2 map gives 3 bins of probability (2/9) and 6 bins of probability (1/18). The resulting information dimension is:

$$D_I(\text{area}) = [3(2/9)\ln(2/9) + 6(1/18)\ln(1/18)]/\ln(1/3) = \\ = (1.00272 + 0.96346)/1.09861 = 1.7897 \\ (\text{from Area Mapping}).$$

Further theoretical analysis, in agreement with numerical evaluations, shows that continued iterations of the area mapping with bins of size $(1/3)^n$ leaves the information dimension unchanged, in clear disagreement with the results of pointwise mappings as well as the Kaplan-Yorke prediction (from the two Lyapunov exponents of the N2 mapping):

$$\lambda_1 = (2/3)\ln(3/2) + (1/3)\ln(3/1) = +0.63651 \ , \\ \lambda_2 = (2/3)\ln(1/3) + (1/3)\ln(2/3) = -0.86756 \ , \\ D_I(KY) = 1 - (\lambda_1/\lambda_2) = 1.7337 \ .$$

Disagreement with the Kaplan-Yorke conjecture using $\delta = = (1/3)^n$ is surprising for such a simple map. An explanation of the difference between pointwise and areawise dimensions is likewise surprising. We would be grateful for cogent explanations of either or both of these two puzzles and are happy to accept Snook Prize contributions dealing with them up until the end of this year 2021. Entries can be submitted to the CMST website[1] or to hooverwilliam@yahoo.com.

References

[1] W.G. Hoover, C.G. Hoover, *2020 Ian Snook Prize Problem: Three Routes to the Information Dimensions for One-Dimensional Stochastic Random Walks and Their Equivalent Two-Dimensional Baker Maps*, Computational Methods in Science and Technology **25**, 153–159 (2019).

[2] W.G. Hoover, C.G. Hoover, *Nonequilibrium Molecular Dynamics, Fractal Phase-Space Distributions, the Cantor Set, and Puzzles Involving Information Dimensions for Two Compressible Baker Maps*, Regular and Chaotic Dynamics **25**, 412–423 (2020).

Bill and Carol Hoover moved from California to Nevada in 2005, some 43 years after Bill began working at the Lawrence Livermore Laboratory in 1962 and as a Professor at the University of California at Davis soon after. Carol was one of Bill's best students and earned her PhD in plasma physics from the University of California Davis Campus Department of Applied Science in Livermore. They have completed about 100 publications in statistical and computational physics. Their most recent book, "Microscopic and Macroscopic Simulation Techniques: Kharagpur Lectures", was published in 2018. The Hoovers enjoy living in the ranching community of Ruby Valley, where the phonebook is a single page and "town" is an hour away. Photo courtesy of Christopher Wilson.

[1] Source: http://www.cmst.eu