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The inverse Riemann zeta function

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Abstract: In this article, we develop a formula for an inverse Riemann zeta function such that for $w = \zeta(s)$ we have $s = \zeta^{-1}(w)$ for real and complex domains s and w. The presented work is based on extending the analytical recurrence formulas for trivial and non-trivial zeros to solve an equation $\zeta(s) - w = 0$ for a given w-domain using logarithmic differentiation and zeta recursive root extraction methods. We further explore formulas for trivial and non-trivial zeros of the Riemann zeta function in greater detail, and next, we introduce an expansion of the inverse zeta function by its singularities, study its properties and develop many identities that emerge from them. In the last part we extend the presented results as a general method for finding zeros and inverses of many other functions, such as the gamma function, the Bessel function of the first kind, or finite/infinite degree polynomials and rational functions, etc. We further compute all the presented formulas numerically to high precision and show that these formulas do indeed converge to the inverse of the Riemann zeta function and the related results. We also develop a fast algorithm to compute $\zeta^{-1}(w)$ for complex w.

Key words: inverse Riemann zeta function, Euler prime product, non-trivial zero formula, Euler-Mascheroni and Stieltjes constants

I. INTRODUCTION

The Riemann zeta function is classically defined by an infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\ , \tag{1}$$

which is absolutely convergent $\Re(s) > 1$, where $s = \sigma + \mathrm{i}t$ is a complex variable. The values for the first few special cases are:

$$\begin{aligned}
&\zeta(1) \sim \sum_{n=1}^{k} \frac{1}{n} \sim \gamma + \log(k) \quad \text{as} \quad k \to \infty,\\
&\zeta(2) = \frac{\pi^2}{6},\\
&\zeta(3) = 1.20205690315959\ldots,\\
&\zeta(4) = \frac{\pi^4}{90},\\
&\zeta(5) = 1.03692775514337\ldots,
\end{aligned} \tag{2}$$

and so on. For $s = 1$, the series diverges asymptotically as $\gamma + \log(k)$, where $\gamma = 0.5772156649\ldots$ is the Euler-Mascheroni constant. The special values for an even positive integer argument are generated by the Euler's formula

$$\zeta(2k) = \frac{|\ B_{2k}\ |}{2(2k)!}(2\pi)^{2k}, \tag{3}$$

for which the value is expressed as a rational multiple of π^{2k} where the constants B_{2k} are Bernoulli numbers defined such that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$ and so on. For an odd positive integer argument, the values of $\zeta(s)$ converge to unique constants, which are not known to be expressed as a rational multiple of π^{2k+1} as occurs in the even positive integer case. For $s = 3$, the value is commonly known as Apéry's constant [1]. The key connection to prime numbers is by means of Euler's infinite product formula

$$\zeta(s) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1}\ , \tag{4}$$

where $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on, denote the prime number sequence. These prime numbers can be recursively extracted from the Euler product by the Golomb's formula [2]. Hence, if we define a partial Euler product up to the n^{th} order as

$$Q_n(s) = \prod_{k=1}^{n} \left(1 - \frac{1}{p_k^s}\right)^{-1} , \tag{5}$$

for $n > 1$ and $Q_0(s) = 1$, then we obtain a recurrence relation for the p_{n+1} prime

$$p_{n+1} = \lim_{s\to\infty} \left(1 - \frac{Q_n(s)}{\zeta(s)}\right)^{-1/s} . \tag{6}$$

This leads to representation of primes by the following limit identities

$$\begin{aligned} p_1 &= \lim_{s\to\infty} \left[1 - \frac{1}{\zeta(s)}\right]^{-1/s} , \\ p_2 &= \lim_{s\to\infty} \left[1 - \frac{(1-\frac{1}{2^s})^{-1}}{\zeta(s)}\right]^{-1/s} , \\ p_3 &= \lim_{s\to\infty} \left[1 - \frac{(1-\frac{1}{2^s})^{-1}(1-\frac{1}{3^s})^{-1}}{\zeta(s)}\right]^{-1/s} , \end{aligned} \tag{7}$$

and so on, whereby all the previous primes are used to excite the Riemann zeta function in a such a way as to extract the next prime. A detailed proof and numerical computation is shown in [3, 4]. We will find that this recursive structure (when taken in the limit as $s \to \infty$) is a basis for the rest of this article and will lead to formulas for trivial and non-trivial zeros and the inverse Riemann zeta function.

Furthermore, the Riemann zeta series (1) induces a general Weierstrass factorization of the form

$$\begin{aligned} \zeta(s) = &\frac{e^{(\log(2\pi)-1)s}}{2(s-1)} \times \\ &\times \prod_{n=1}^{\infty} \left(1 - \frac{s}{\rho_{t,n}}\right) e^{\frac{s}{\rho_{t,n}}} \prod_{\rho_{nt}} \left(1 - \frac{s}{\rho_{nt}}\right) e^{\frac{s}{\rho_{nt}}} , \end{aligned} \tag{8}$$

where it analytically extends the zeta function to the whole complex plane and reveals its full structure of the poles and zeros [5, p. 807]. Only a simple pole exists at $s = 1$, hence $\zeta(s)$ is convergent everywhere else in the complex plane, i.e., the set $\mathbb{C}\backslash 1$. Moreover, there are two kinds of zeros classified as the trivial zeros ρ_t and non-trivial zeros ρ_{nt}. The first infinite product term of (8) encodes the factorization due to trivial zeros

$$\rho_{t,n} = -2n, \tag{9}$$

for $n \geq 1$ which occur at negative even integers $-2, -4, -6\ldots$, where n is the index variable for the n^{th} zero. The second infinite product term of (8) encodes the factorization due to non-trivial zeros, which are complex numbers of the form

$$\rho_{nt,n} = \sigma_n + \mathrm{i}t_n , \tag{10}$$

and, as before, n is the index variable for the n^{th} zero (this convention is straightforward if the zeros are on the critical line). But, in general, the real components of non-trivial zeros are known to be constrained to lie in a critical strip in a region where $0 < \sigma_n < 1$. It is also known that there is an infinity of zeros located on the critical line at $\sigma = \frac{1}{2}$, but it is not known whether there are any zeros off of the critical line, a problem of the Riemann hypothesis (RH), which proposes that all zeros should lie on the critical line. The first few zeros on the critical line at $\sigma_n = \frac{1}{2}$ have imaginary components $t_1 = 14.13472514...$, $t_2 = 21.02203964...$, $t_3 = 25.01085758...$, and so on, which were computed by an analytical recurrence formula as

$$\begin{aligned} t_{n+1} = \lim_{m\to\infty} \Bigg[&\frac{(-1)^m}{2} \Big(2^{2m} - \frac{1}{(2m-1)!} \log(|\zeta|)^{(2m)}\big(\frac{1}{2}\big) + \\ &- \frac{1}{2^{2m}} \zeta\big(2m, \frac{5}{4}\big) \Big) - \sum_{k=1}^{n} \frac{1}{t_k^{2m}} \Bigg]^{-\frac{1}{2m}} , \end{aligned} \tag{11}$$

as we have shown in [4, 6] assuming (RH). The key component of this representation is a $2m^{\text{th}}$ derivative of $\log[\zeta(s)]$ evaluated at $s = \frac{1}{2}$. Also, $\zeta(s, a)$ is the Hurwitz zeta function

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} , \tag{12}$$

which is a shifted version of (1) by an arbitrary parameter $a > 0$. Now, substituting the Weierstrass infinite product (8) into the Golomb's formula (6) can be used to generate primes directly from non-trivial zeros. In later sections we will show that non-trivial zeros can also be generated directly from primes.

Furthermore, the Riemann zeta function can have many points s_n such that $w = \zeta(s_n)$ can map to the same w value. In Fig. 1, we plot $\zeta(s)$ for real s and w, and note that for $s > 1$ the function is monotonically decreasing from $+\infty$ and tends $O(1)$ as $s \to \infty$, and for the domain $-2.7172628292\ldots < s < 1$ it is monotonically decreasing from $0.0091598901\ldots$ to $-\infty$, as also shown in Fig. 2. And for $s < -2.7172628292\ldots$, it becomes oscillatory where there are many s_n solutions. For example, the first two s values

$$\zeta(-2.47270347\ldots) = \zeta(-3) = \frac{1}{120} , \tag{13}$$

map to the same w value as shown in Fig. 2. It is usually customary to report $\zeta(-3) = \frac{1}{120}$, but is actually the second solution s_2, the first solution, or the principal solution s_1 is the value $-2.47270347\ldots$

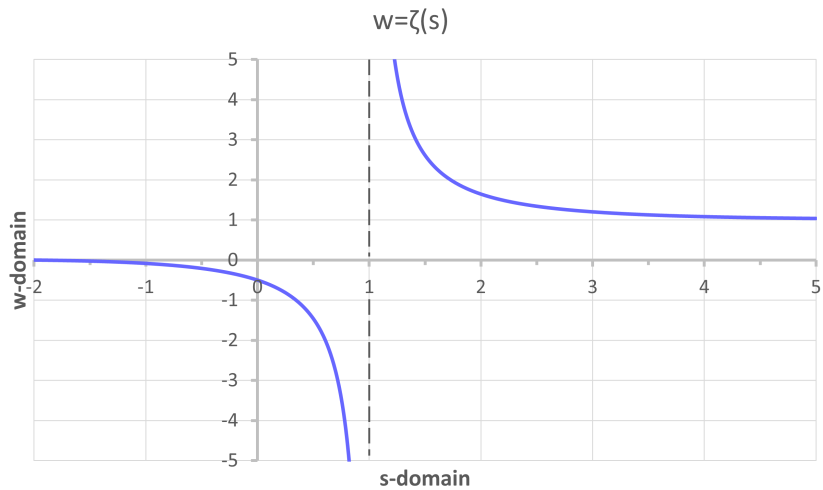

Fig. 1. A plot of $w = \zeta(s)$ for $s \in (-2, 5)$

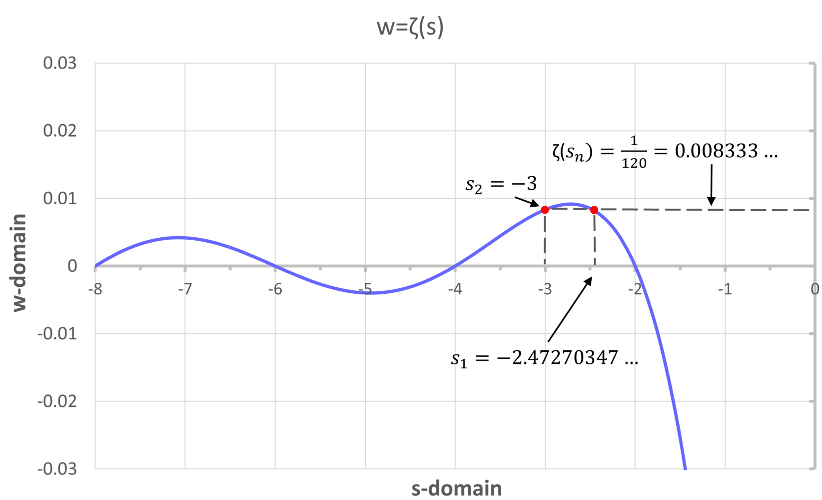

Fig. 2. A zoomed-in plot of $w = \zeta(s)$ for $s \in (-8, 0)$ showing oscillatory behavior

In this article we seek to develop a formula for an inverse Riemann zeta function $s = \zeta^{-1}(w)$. In general, the existence of an inverse is established by an inverse function theorem as shown in [7, p. 135], where for any holomorphic function $f(z)$ an inverse exists provided that $f'(z) \neq 0$ in the z-domain. Hence, for

$$w = \zeta(s), \tag{14}$$

there is an inverse function

$$s = \zeta^{-1}(w), \tag{15}$$

which implies that

$$\zeta^{-1}(\zeta(s)) = s, \tag{16}$$

and

$$\zeta(\zeta^{-1}(w)) = w, \tag{17}$$

for some real and complex domains w and s, assuming that $\zeta'(s) \neq 0$ in the s-domain. Also, [8] discusses additional theoretical basis behind solutions to (14), also known as a-points, which we refer to as s_n. The presented method can also recursively compute these multiple solutions of $s_n = \zeta^{-1}(w)$ but, as we will find, the computational requirements become very high and start exceeding the limitations of the test computer. Therefore, we will primarily focus on the principal solution s_1, which, as we will find, will cover almost the entire complex plane.

We develop a recursive formula for an inverse Riemann zeta function as:

$$s_{n+1} = \zeta^{-1}(w) = \lim_{m\to\infty} \pm\Bigg[-\frac{1}{(2m-1)!} \times \frac{d^{(2m)}}{ds^{(2m)}} \log\Big[(\zeta(s)-w)(s-1)\Big]\Big|_{s\to 0} - \sum_{k=1}^{n} \frac{1}{s_k^{2m}} \Bigg]^{-\frac{1}{2m}}, \tag{18}$$

where s_n is the value for which $w = \zeta(s_n)$. We do not know much about the behavior at higher branches but, roughly, if s_n are real Eq. (18) will generate solutions s_n for a given w-domain. The formula for the principal branch is

$$s_1 = \zeta^{-1}(w) = \lim_{m\to\infty} \pm \Bigg[-\frac{1}{(m-1)!} \times \\ \times \frac{d^m}{ds^m} \log\left[(\zeta(s) - w)(s-1)\right]\Big|_{s\to 0} \Bigg]^{-\frac{1}{m}} , \tag{19}$$

and it is valid for all complex w-domain except in a small strip region $\Re(w) \in (j_1, 1) \cup \{\Im(w) = 0\}$ that we determined experimentally, and $j_1 = 0.00915989\ldots$ is a constant. As we will show in more detail in later sections, this formula can easily invert the Basel problem

$$\zeta^{-1}\left(\frac{\pi^2}{6}\right) = 2, \tag{20}$$

or the Apéry's constant

$$\zeta^{-1}(1.20205690315959428\ldots) = 3, \tag{21}$$

and essentially the entire complex domain $w \in \mathbb{C}$ with an exception of a strip region $\Re(w) \in (j_1, 1) \cup \{\Im(w) = 0\}$ where there lie (possibly) an infinite number of singularities.

The way in which we will arrive at the presented formula is by connecting two simple Theorems. Theorem 1, as presented in Sec. 2, is the m^{th} log-derivative formula for obtaining a generalized zeta series over the zeros of a function. Such a method appears in the literature from time to time and can be traced back to Euler who, according to a reference in [9, p. 500], used it to devise a means of computing several zeros of the Bessel function of the first kind by solving a system of equations generated by the log-derivative formula. In the works of Voros, Lehmer, and Matsuoka, it has been used to find a closed-form formula for the secondary zeta functions [10–13]. Then, Theorem 2, as given in Sec. 3, develops a method for finding a zeta recurrence formula for the n^{th}+1 term of a generalized zeta series, i.e., all terms of a generalized zeta series must be known in order to generate the n^{th}+1 term, as we have shown in our previous work [4, 6]. In Sec. 4 we connect these two theorems and find formulas for trivial and non-trivial zeros of the Riemann zeta function and explore their properties in greater detail. We then develop a formula for an inverse zeta function and study its properties, such as the singularity expansion that emerges from these results. We empirically observe that there are infinitely many singularities of the inverse zeta that are spread out along a narrow strip $(j_1, 1)$ forming a linear singularity. In the final part we briefly extend the presented results to find zeros (and inverses) of many common functions, such as the gamma function, the Bessel function of the first kind, the trigonometric functions, Lambert-W function, and any entire function or finite/infinite degree polynomial or rational function (provided that the function fits the constraints of this method).

Throughout this article we numerically compute the presented formulas to high precision in PARI/GP software package [14], as it is an excellent platform for performing arbitrary precision computations, and show that these formulas do indeed converge to the inverse of the Riemann zeta function to high precision. We note that when running the script in PARI, the precision has to be set very high (we generally set precision to 1000 decimal places). Also, the Wolfram Mathematica software package [15] was instrumental in developing this article.

II. LOGARITHMIC DIFFERENTIATION

In this section we outline the zeta m^{th} log-derivative method. We recall the argument principle that for an analytic function $f(z)$ we have

$$\frac{1}{2\pi\mathrm{i}} \oint_{\Omega} \frac{f'(z)}{f(z)} dz = N_z - N_p , \tag{22}$$

which equals the number of zeros N_z minus the number of poles N_p (counting multiplicity) which are enclosed in a simple contour Ω. But now, instead of working with the number of zeros or poles, the aim of the m^{th} log-derivative method is to find a generalized zeta series over the zeros and poles of the function $f(z)$ in question. Hence if $Z = \{z_1, z_2, z_3, \ldots, z_{N_z}\}$ is a set of all zeros of $f(z)$ and $P = \{p_1, p_2, p_3, \ldots p_{N_p}\}$ is a set of all poles of $f(z)$ in the whole complex plane, then the generalized zeta series are

$$Z(s) = \sum_{n=1}^{N_z} \frac{1}{z_n^s} , \tag{23}$$

and

$$P(s) = \sum_{n=1}^{N_p} \frac{1}{p_n^s} . \tag{24}$$

The number of zeros or poles may be finite or infinite, but in the latter case the convergence of a generalized zeta series to the right-half side of the line $\Re(s) > \mu$ may depend on the distribution of its terms. If the number of zeros and poles is finite, then one could count them by evaluating $Z(0) - P(0) = N_z - N_p$, which reduces to the argument principle (22). The argument principle may be modified further by introducing another function $h(z)$ as such

$$\frac{1}{2\pi\mathrm{i}} \oint_{\Omega} \frac{f'(z)}{f(z)} h(z) dz = \sum_{n=1}^{N_z} h(z_n) - \sum_{n=1}^{N_p} h(p_n). \tag{25}$$

The proof is a slight modification of a standard proof of (22) using the Residue theorem, and if we let $h(z) = z^{-s}$, then we have

$$\frac{1}{2\pi\mathrm{i}} \oint_{\Omega} \frac{f'(z)}{f(z)} \frac{1}{z^s} dz = Z(s) - P(s), \tag{26}$$

but the contour has to be deformed as to not encircle the origin.

However, in the following sections we will not pursue (26) as the integral makes the study and computation more difficult, as well as being dependent on the contour. There is a simpler variation of (26) which is better suited for our study. The main formula of the zeta m^{th} log-derivative method is:

Theorem 1.

$$-\frac{1}{(m-1)!}\frac{d^m}{dz^m}\log\left[f(z)\right]\Big|_{z\to 0} = Z(m) - P(m), \quad (27)$$

valid for a positive integer variable $m \geq 1$.

This formula generates $Z(m) - P(m)$ over all zeros and poles in the whole complex plane, as opposed to being enclosed in some contour by (26). We now outline a basic proof of Theorem 1.

Proof. If we model an analytic function $f(z)$ having simple zeros and poles by admitting a factorization of the form

$$f(z) = g(z)\prod_{n=1}^{N_z}\left(1-\frac{z}{z_n}\right)\prod_{n=1}^{N_p}\left(1-\frac{z}{p_n}\right)^{-1}, \quad (28)$$

where $g(z)$ is a component not having any zeros or poles, then

$$\log[f(z)] = \log[g(z)] + \sum_{n=1}^{N_z}\log\left(1-\frac{z}{z_n}\right) - \sum_{n=1}^{N_p}\log\left(1-\frac{z}{p_n}\right). \quad (29)$$

Now, using the Taylor series expansion for the logarithm as

$$\log(1-z) = -\sum_{k=1}^{\infty}\frac{z^k}{k} = -z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots, \quad (30)$$

for $|z| < 1$, we obtain

$$\log[f(z)] = \log[g(z)] - \sum_{n=1}^{N_z}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z}{z_n}\right)^k + \sum_{n=1}^{N_p}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z}{p_n}\right)^k. \quad (31)$$

Now interchanging the order of summation yields

$$\log[f(z)] = \log[g(z)] - \sum_{k=1}^{\infty}\frac{z^k}{k}\left[\sum_{n=1}^{N_z}\frac{1}{z_n^k}\right] + \sum_{k=1}^{\infty}\frac{z^k}{k}\left[\sum_{n=1}^{N_p}\frac{1}{p_n^k}\right], \quad (32)$$

and hence, by recognizing the inner sum as a generalized zeta series yields

$$\log[f(z)] = \log[g(z)] - \sum_{k=1}^{\infty}Z(k)\frac{z^k}{k} + \sum_{k=1}^{\infty}P(k)\frac{z^k}{k}. \quad (33)$$

From this form, we can now extract $Z(m) - P(m)$ by the m^{th} order differentiation as

$$-\frac{1}{(m-1)!}\frac{d^m}{dz^m}\log\left[\frac{f(z)}{g(z)}\right]\Big|_{z\to 0} = Z(m) - P(m), \quad (34)$$

evaluated as $z \to 0$ in the limit.

□

We can also obtain an integral representation using the Cauchy integral formula applied to coefficients of Taylor expansion (34) as

$$\lim_{z_0\to 0}\left\{-\frac{m}{2\pi \mathrm{i}}\oint_{\Omega_0}\frac{1}{(z-z_0)^{m+1}}\log\left(\frac{f(z)}{g(z)}\right)dz\right\} = = Z(m) - P(m), \quad (35)$$

where Ω_0 is a simple contour encircling the origin but, unlike (26), is not enclosing zeros or poles. In order to apply (34) and (35) successfully, one has to judiciously choose $g(z)$ as to remove it from $f(z)$ so that the m^{th} log-differentiation will not produce unwanted artifacts due to $g(z)$.

III. THE ZETA RECURRENCE FORMULA

We now outline a method to extract the terms of a generalized zeta series by means of a recurrence formula satisfied by the terms of such series. Hence, if the terms of a generalized zeta series are to be zeros of a function, then such a method effectively gives a recurrence formula satisfied by the zeros, which in turn can be used to compute the zeros. And similarly, the same holds if the terms of a generalized zeta series are poles of a function. In fact, any quantities represented by the terms of a generalized zeta series can be recursively found. In [4], we developed a formula for the n^{th}+1 prime based on the prime zeta function. However, for the purpose of this paper let us consider the generalized zeta series over zeros z_n of a function

$$Z(s) = \sum_{n=1}^{N_z}\frac{1}{z_n^s} = \frac{1}{z_1^s} + \frac{1}{z_2^s} + \frac{1}{z_3^s} + \ldots, \quad (36)$$

and let us also assume that the zeros are positive, real, and ordered from smallest to largest such that $0 < z_1 < z_2 < < z_3 < \ldots < z_n$, then the asymptotic relationship holds

$$\frac{1}{z_n^s} \gg \frac{1}{z_{n+1}^s}, \quad (37)$$

as $s \to \infty$. To illustrate how fast the terms decay, let us take $z_1 = 2$ and z_2=3. Then for $s = 10$ we compute

$$\frac{1}{2^s} = 9.7656\ldots \times 10^{-4}, \qquad \frac{1}{3^s} = 1.6935\ldots \times 10^{-5}, \quad (38)$$

where we roughly see an order of magnitude difference. But for $s = 100$ we compute

$$\frac{1}{2^s} = 7.8886\ldots \times 10^{-31}, \qquad \frac{1}{3^s} = 1.9403\ldots \times 10^{-48}, \quad (39)$$

where we see a difference by 17 orders of magnitude. Hence, as $s \to \infty$, then

$$O\left(z_n^{-s}\right) \gg O\left(z_{n+1}^{-s}\right), \tag{40}$$

as the z_{n+1}^{-s} term completely vanishes in relation to z_n^{-s}, and so the z_n^{-s} term dominates the limit. As a result, we write

$$z_1^{-s} \gg z_2^{-s} \gg z_3^{-s} \gg \ldots \gg z_n^{-s}\,. \tag{41}$$

From this we have

$$Z(s) \sim O\left(z_1^{-s}\right), \tag{42}$$

as $s \to \infty$ where the lowest order term dominates, and we refer to it as the principal term, or in the case where the zeta series are considered to be zeros of a function, the principal zero. This rapid decay of higher order zeta terms (41) opens a possibility for a recursive root extraction as shown by Theorem 2 next.

Theorem 2. If $\{z_n\}$ is a set of positive real numbers ordered such that $0 < z_1 < z_2 < z_3 < \ldots < z_n$, and so on, then the recurrence relation for the n^{th}+1 term is

$$z_{n+1} = \lim_{s\to\infty}\left(Z(s) - \sum_{k=1}^{n}\frac{1}{z_k^s}\right)^{-1/s}. \tag{43}$$

Proof. First we begin solving for z_1 in (36) to obtain

$$\frac{1}{z_1^s} = Z(s) - \frac{1}{z_2^s} - \frac{1}{z_3^s} - \ldots, \tag{44}$$

and then we get

$$z_1 = \left(Z(s) - \frac{1}{z_2^s} - \frac{1}{z_3^s} - \ldots\right)^{-1/s}. \tag{45}$$

If we then consider the limit

$$z_1 = \lim_{s\to\infty}\left(Z(s) - \frac{1}{z_2^s} - \frac{1}{z_3^s} - \ldots\right)^{-1/s}, \tag{46}$$

then because the series is convergent and since $z_1^{-s} \gg z_2^{-s}$, then $O[Z(s)] \sim O(z_1^{-s})$ as $s \to \infty$, and so the higher order zeros decay as $O(z_2^{-s})$ faster than $Z(s)$, and so $Z(s)$ dominates the limit, hence the formula for the principal zero is:

$$z_1 = \lim_{s\to\infty}\left[Z(s)\right]^{-1/s}. \tag{47}$$

The next zero is found the same way, by solving for z_2 in (36) we get

$$z_2 = \lim_{s\to\infty}\left(Z(s) - \frac{1}{z_1^s} - \frac{1}{z_3^s} - \ldots\right)^{-1/s}, \tag{48}$$

and since the higher order zeros decay as z_3^{-s} faster than $Z(s) - z_1^{-s}$, we then have

$$z_2 = \lim_{s\to\infty}\left(Z(s) - \frac{1}{z_1^s}\right)^{-1/s}. \tag{49}$$

And continuing on to next zero, by solving for z_3 in (36) and by removing the next dominant terms, we obtain

$$z_3 = \lim_{s\to\infty}\left(Z(s) - \frac{1}{z_1^s} - \frac{1}{z_2^s}\right)^{-1/s}, \tag{50}$$

and the process continues for the next zero. Hence, in general, the recurrence formula for the n^{th}+1 zero is

$$z_{n+1} = \lim_{s\to\infty}\left(Z(s) - \sum_{k=1}^{n}\frac{1}{z_k^s}\right)^{-1/s}, \tag{51}$$

thus all zeros up to the n^{th} order must be known in order to generate the n^{th}+1 zero. □

Let us next give an example of how Theorem 1 and Theorem 2 are applied to find a formula for zeros for a function. Suppose that we wish to find zeros of the function

$$f(s) = \frac{\sin(\pi s)}{\pi s} = 0. \tag{52}$$

We know in advance that the zeros are just integer multiples: $z_n = \pm n$ (for any non-zero integer $n = 1, 2, 3\ldots$). But now, if we apply the m^{th} log-derivative formula to (52), then we get a generalized zeta series of over the zeros as

$$Z(m) = -\frac{1}{(m-1)!}\frac{d^m}{ds^m}\log\left[\frac{\sin(\pi s)}{\pi s}\right]\Big|_{s\to 0}, \tag{53}$$

which is equal to the zeta series over all zeros (including the negative ones) as such

$$\begin{aligned} Z(m) = \ldots + \frac{1}{(-3)^m} + \frac{1}{(-2)^m} + \frac{1}{(-1)^m} + \\ + \frac{1}{(1)^m} + \frac{1}{(2)^m} + \frac{1}{(3)^m} + \ldots \end{aligned} \tag{54}$$

From this we can deduce that even values become double of a half the side of zeros (which in this example is equivalent to $\zeta(s)$) as

$$Z(2m) = 2\zeta(2m), \tag{55}$$

and the odd values cancel

$$Z(2m+1) = 0. \tag{56}$$

In view of this, the Euler's formula (3) is

$$\zeta(2k) = \frac{|\,B_{2k}\,|}{2(2k)!}(2\pi)^{2k}, \tag{57}$$

and is an example of a closed-form representation of $\zeta(2k)$ that does not involve zeros directly, i.e., the positive integers. But such a formula may not always be available, so we will just use the m^{th} log-derivative formula (53) as the main closed-form representation of $Z(s)$. Also, on a side note, the Bernoulli numbers are expansions coefficients of another function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\,, \tag{58}$$

where they can be similarly obtained by m^{th} differentiation:

$$B_m=\lim_{x\to 0}\left\{\frac{d^m}{dx^m}\frac{x}{e^x-1}\right\}. \tag{59}$$

Hence, in essence, the Euler's formula (57) is just a transformed version of (53) but it just happens that Bernoulli numbers are rational constants. And so, when putting this together, we obtain a full solution to (52) as a recurrence relation

$$z_{n+1}=\lim_{m\to\infty}\left[-\frac{1}{2(2m-1)!}\frac{d^{(2m)}}{ds^{(2m)}}\log\left[\frac{\sin(\pi s)}{\pi s}\right]\Big|_{s\to 0}+\right.\\ \left.-\sum_{k=1}^{n}\frac{1}{z_k^{2m}}\right]^{-\frac{1}{2m}}, \tag{60}$$

where a $2m$ limit value ensures that it is even, and so, an additional factor of $\frac{1}{2}$ is added due to (54). The principal zero is:

$$z_1=\lim_{m\to\infty}\left[-\frac{1}{2(2m-1)!}\frac{d^{(2m)}}{ds^{(2m)}}\log\left[\frac{\sin(\pi s)}{\pi s}\right]\Big|_{s\to 0}\right]^{-\frac{1}{2m}}, \tag{61}$$

and a numerical computation for $m=20$ yields

$$z_1=0.9999999999999\underline{9}7726263\ldots, \tag{62}$$

which is accurate to 13 digits after the decimal place, and the script to compute it in PARI is presented in Algorithm 1. The key aspect of the script is the **derivnum** function for computing the m^{th} derivative very accurately, which will be very useful for the rest of this article. The next zero is recursively found as

$$z_2=\lim_{m\to\infty}\left[-\frac{1}{2(2m-1)!}\frac{d^{(2m)}}{ds^{(2m)}}\log\left[\frac{\sin(\pi s)}{\pi s}\right]\Big|_{s\to 0}+\right.\\ \left.-\frac{1}{1^{2m}}\right]^{-\frac{1}{2m}}, \tag{63}$$

but we must know the first zero in advance in order to compute the next zero. A numerical computation for $m=20$ yields

$$z_2=1.99999999\underline{9}547806838689\ldots\,, \tag{64}$$

which is accurate to 8 digits after the decimal place. Then, the next zero in the sequence is

$$z_3=\lim_{m\to\infty}\left[-\frac{1}{2(2m-1)!}\frac{d^{(2m)}}{ds^{(2m)}}\log\left[\frac{\sin(\pi s)}{\pi s}\right]\Big|_{s\to 0}+\right.\\ \left.-\frac{1}{1^{2m}}-\frac{1}{2^{2m}}\right]^{-\frac{1}{2m}}, \tag{65}$$

but we must know the first two zeros in advance in order to compute the next zero. A numerical computation for $m=20$ yields

$$z_3=2.999999\underline{2}4565967669286\ldots\,, \tag{66}$$

which is accurate to 6 digits after the decimal place, and so on. We see that the accuracy becomes lesser for higher zeros, and so the limit variable m has to be increased to get better accuracy. One can then continue this process and extract the z_{n+1} zero.

As a second example, we find a formula for zeros of the Bessel function of the first kind

$$J_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\nu+1)n!}\left(\frac{x}{2}\right)^{2n+2}, \tag{67}$$

for all $\nu>-1$ real orders. We denote the zeros of (67) as $x_{\nu,n}$ where n is the index variable for the n^{th} zero for ν order of the Bessel function. The Weierstrass product representation of (67) is

$$J_\nu(x)=\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}\prod_{n=1}^{\infty}\left(1-\frac{x^2}{x_{\nu,n}^2}\right), \tag{68}$$

for $\nu>-1$ involving the zeros directly [5, p. 370]. To find the roots of this function we apply Theorem 1 to obtain a generalized zeta series over the Bessel zeros

$$Z_\nu(2m)=-\frac{1}{2(2m-1)!}\frac{d^{(2m)}}{dx^{(2m)}}\log\left[\frac{J_\nu(x)}{x^\nu}\right]\Big|_{x\to 0}=\\ =\sum_{n=1}^{\infty}\frac{1}{x_{\nu,n}^{2m}}, \tag{69}$$

which is essentially the $2m^{\text{th}}$ derivative of $\log[J_\nu(x)/x^\nu]$ evaluated at $x=0$, where it is taken in a limiting sense $x\to 0$ as to avoid division by zero. Here we choose $g(x)==x^\nu$ to cancel it from $J_\nu(x)$ as shown in the previous section. This will prevent any artifacts of x^ν in (68) from being generated by the log-differentiation. Furthermore, the Bessel function is even; hence we consider the $2m$ values.

The first few special values of $Z_\nu(2m)$ for even orders are:

$$\begin{aligned}
Z_\nu(2) &= \frac{1}{4(\nu+1)},\\
Z_\nu(4) &= \frac{1}{16(\nu+1)^2(\nu+2)},\\
Z_\nu(6) &= \frac{1}{32(\nu+1)^3(\nu+2)(\nu+3)},\\
Z_\nu(8) &= \frac{5\nu+11}{256(\nu+1)^4(\nu^2+2)^2(\nu+3)(\nu+4)},\\
Z_\nu(10) &= \frac{7\nu+19}{512(\nu+1)^5(\nu^2+2)^2(\nu+3)(\nu+4)(\nu+5)},\\
Z_\nu(12) &= \frac{21\nu^3+181\nu^2+513\nu+473}{2048(\nu+1)^6(\nu^2+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)},
\end{aligned} \tag{70}$$

and so on. These generated values are rational functions of the Bessel order $\nu > -1$ for even orders, and this implies that if $\nu > -1$ is rational, so is $Z_\nu(2m)$. These formulas for $Z_\nu(2m)$ were generated using another recurrence relation found in Sneddon [16] as an alternative to (69), which is given in Appendix B. In Tab. 1 we give the values of $Z_\nu(2m)$ for different m and ν.

Tab. 1. Generated values of $Z_\nu(m)$ for different m and ν

m	$Z_0(m)$	$Z_1(m)$	$Z_2(m)$	$Z_3(m)$
2	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$
4	$\frac{1}{32}$	$\frac{1}{192}$	$\frac{1}{576}$	$\frac{1}{1280}$
6	$\frac{1}{192}$	$\frac{1}{3072}$	$\frac{1}{17280}$	$\frac{1}{61440}$
8	$\frac{11}{12288}$	$\frac{1}{46080}$	$\frac{7}{3317760}$	$\frac{13}{34406400}$
10	$\frac{19}{122880}$	$\frac{13}{8847360}$	$\frac{11}{139345920}$	$\frac{1}{110100480}$
12	$\frac{473}{17694720}$	$\frac{11}{110100480}$	$\frac{797}{267544166400}$	$\frac{263}{1189085184000}$

And now, by applying Theorem 2 we obtain a full recurrence formula satisfied by the Bessel zeros:

$$x_{\nu,n+1} = \lim_{s\to\infty}\left(Z_\nu(s) - \sum_{k=1}^{n}\frac{1}{x_{\nu,k}^s}\right)^{-1/s}. \tag{71}$$

To verify (71) numerically we compute the principal zero using (69), since it is more efficient than (70), for the limit variable $m = 250$ and $\nu = 0$ which results in

$$\begin{aligned}
x_{0,1} &= \lim_{m\to\infty}[Z_0(2m)]^{-\frac{1}{2m}} =\\
&= 2.404825557695772768621631879326\ldots,
\end{aligned} \tag{72}$$

and numerical results is accurate to 181 decimal places (we are showing just the first 30 digits). In [9, p. 500–503], Rayleigh-Cayley generated values for $Z_\nu(2m)$ as shown in (70), extended Euler's original work and computed the smallest Bessel zero using this method in papers dating back to year 1874. Moving on, the next zero is found the same way, we compute

$$\begin{aligned}
x_{0,2} &= \lim_{m\to\infty}\left[Z_0(2m) - \frac{1}{x_{0,1}^{2m}}\right]^{-\frac{1}{2m}} =\\
&= 5.520078110286310649596604112813\ldots,
\end{aligned} \tag{73}$$

for $m = 250$, and it is accurate to 99 decimal places but in order to ensure convergence the first zero $x_{0,1}$ has to be known to high enough precision (usually much higher than can be efficiently computed using this method as we did above). Henceforth, as a numerical experiment we take $x_{0,1}$ that was already pre-computed to high enough precision using more efficient means to 1000 decimal places using the standard equation solver found on mathematical software packages (such root finding algorithms are very effective but must assume an initial condition), rather than taking the zero computed above with less accuracy. And similarly, the third zero is computed as

$$\begin{aligned}
x_{0,3} &= \lim_{m\to\infty}\left[Z_0(2m) - \frac{1}{x_{0,1}^{2m}} - \frac{1}{x_{0,2}^{2m}}\right]^{-\frac{1}{2m}} =\\
&= 8.653727912911012216954198712660\ldots,
\end{aligned} \tag{74}$$

which is accurate to 68 decimal places and that it was assumed $x_{0,1}$ and $x_{0,2}$ was already pre-computed to high enough precision (1000 decimal places using the standard equation solver) in order to ensure convergence. Hence, in general, one can continue and keep removing all the known zeros up to the n^{th} order in order to compute the n^{th}+1 zero. In numerical computations the key is that the accuracy of the previous zeros must be much higher than the next zero in order to ensure convergence, i.e., $x_{\nu,n}^{-s} \gg x_{\nu,n+1}^{-s}$, and also one cannot use the same limit variable to compute the next zero based on the previous zero as it will cause self-cancelation in the formula. Numerically, there is a fine balance as to how many accurate digits are available and the magnitude of the

Algorithm 1 PARI script for computing the first zero (61)

```
{
    m = 20; // set limit variable
    delta = 10^(−100); // set s->0 limit for deriv

    // compute generalized zeta series
    A = −derivnum(s = delta,log(sin(Pi∗s)/(Pi∗s)),2∗m);
    B = 1/factorial(2∗m−1);
    Z = A∗B/2;

    // compute the first zero
    z1 = (Z)^(−1/(2∗m));
    print(z1);
}
```

limit variable m used to compute the next zero. We also note that this method is not numerically an efficient method to compute zeros but it allows to have a true closed-form representation of the zeros, and also that one does not need to make an initial guess for the zero, as is generally the case for many root finding algorithms.

We also remarked that the generalized zeta series will be rational for rational Bessel order $\nu > -1$. Since the first zero can be written as

$$x_{\nu,1} = \lim_{m\to\infty} [Z_\nu(2m)]^{-\frac{1}{2m}}, \tag{75}$$

and that implies that we have a $2m^{\text{th}}$ root of a rational number, which is irrational. Hence, for most purposes the sequence converging to the first Bessel zero for any rational $\nu > -1$ order will be irrational up to the limit variable m. For example, for $Z_\nu(2m)$ for $m = 6$ in (70) we have an approximation to converging to the first zero

$$x_{\nu,1} \approx \left[\frac{21\nu^3 + 181\nu^2 + 513\nu + 473}{2048(\nu+1)^6(\nu^2+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)}\right]^{-\frac{1}{12}}, \tag{76}$$

which is irrational for any rational $\nu > -1$. We remark that this is not a definite proof of the irrationality of the Bessel zero but rather a condition where one can set m arbitrarily high as $m \to \infty$, and the sequence converging to the first Bessel zero will be irrational.

The presented methods by Theorem 1 and Theorem 2 can be effectively used to find zeros of many different functions, such as the digamma function, Bessel functions, the Airy function, and many other finite and infinite degree polynomials (provided that the zeros are loosely well-behaved), and in the next section we will apply this method to find the trivial and non-trivial zeros of the Riemann zeta function.

IV. FORMULAS FOR THE RIEMANN ZEROS

As described in the Introduction, the Riemann zeta function consists of trivial zeros ρ_t and non-trivial zeros ρ_{nt}, so that the full generalized zeta series over all the zeros is

$$Z(s) = Z_t(s) + Z_{nt}(s), \tag{77}$$

where

$$Z_t(s) = \sum_{n=1}^{\infty} \frac{1}{\rho_{t,n}^s}, \tag{78}$$

and

$$Z_{nt}(s) = \sum_{n=1}^{\infty} \left(\frac{1}{\rho_{nt,n}^s} + \frac{1}{\bar\rho_{nt,n}^s}\right), \tag{79}$$

are the trivial and non-trivial components, where they are taken in conjugate-pairs. Now, when applying the root-extraction to (77) directly is not straightforward. First we observe that

$$O[Z_t(s)] \gg O[Z_{nt}(s)], \tag{80}$$

as $s \to \infty$, since

$$\frac{1}{2^s} \gg \left|\frac{1}{(\frac{1}{2}+\mathrm{i}t_1)^s} + \frac{1}{(\frac{1}{2}-\mathrm{i}t_1)^s}\right|, \tag{81}$$

or roughly

$$\frac{1}{2^s} \gg \frac{1}{t_1^s}. \tag{82}$$

Hence, $Z_t(s)$ dominates the limit in relation to $Z_{nt}(s)$.

Next, we develop a formula for trivial zeros using the root-extraction method. It is first convenient to remove the pole of $\zeta(s)$ by inspecting the Weierstrass infinite product (8) to consider the function

$$f(s) = \zeta(s)(s-1), \tag{83}$$

then the m^{th} log-derivative gives the generalized zeta series over all zeros as

$$\begin{aligned} Z(s) &= Z_t(m) + Z_{nt}(m) = \\ &= -\frac{1}{(m-1)!}\frac{d^m}{ds^m}\log\left[\zeta(s)(s-1)\right]\Big|_{s\to 0}. \end{aligned} \tag{84}$$

As a result, the recurrence formula for trivial zeros is:

$$\rho_{t,n+1} = -\lim_{m\to\infty}\left[-\frac{1}{(2m-1)!}\frac{d^{(2m)}}{ds^{(2m)}}\log\left[\zeta(s)(s-1)\right]\Big|_{s\to 0} + \\ -\sum_{k=1}^{n}\frac{1}{\rho_{t,k}^{2m}} - \sum_{k=1}^{\infty}\left(\frac{1}{\rho_{nt,k}^{2m}} + \frac{1}{\bar\rho_{nt,k}^{2m}}\right)\right]^{-\frac{1}{2m}}. \tag{85}$$

We first note that we have used a $2m$ limiting value but in this case one could also use an odd limit value. However, then an alternating sign $(-1)^m$ is needed in the recurrence formula to account for positive and negative terms, but we wish to omit that. Secondly, we have also added a negative sign in front to account for a negative branch in s-domain restricted to $-0.5 < \Re(s) < \{0.0091598901\ldots \cup \Im(s) = 0\}$, so that trivial zeros will come out negative. This sign change will be more apparent in later sections. Thirdly, there is a contribution due to the conjugate-pairs of non-trivial zeros. Initially, the $Z_t(s)$ is the dominant lowest term; hence the contribution due to non-trivial zeros is negligible and may be dropped but during the course of removing the trivial zeros recursively in order to generate the n^{th}+1 trivial zero we eventually arrive at a point where the first non-trivial zero term dominates the limit, as we will see shortly.

First we begin to verify the trivial zero formula numerically and we compute the principal zero as

$$\rho_{t,1} = -\lim_{m\to\infty} \left[-\frac{1}{(2m-1)!} \frac{d^{(2m)}}{ds^{(2m)}} \log\left[\zeta(s)(s-1)\right]\Big|_{s\to 0} \right]^{-\frac{1}{2m}}, \tag{86}$$

and the script in PARI is shown in Algorithm 2. By running the script for a limit variable $m = 20$ we compute

$$\rho_{t,1} = -1.999999999999\underline{9}54525260798701750\ldots, \tag{87}$$

which is a close approximation to within 13 decimal places. The next zero is found as

$$\rho_{t,2} = -\lim_{m\to\infty} \left[-\frac{1}{(2m-1)!} \frac{d^{(2m)}}{ds^{(2m)}} \log\left[\zeta(s)(s-1)\right]\Big|_{s\to 0} + \right. \\ \left. -\frac{1}{(-2)^{2m}} \right]^{-\frac{1}{2m}}, \tag{88}$$

but this time we must know the first zero in advance, so that we compute

$$\rho_{t,2} = -3.9999999\underline{9}09561367737969464212 01\ldots, \tag{89}$$

accurate to within 8 decimal places, and similarly the third zero is

$$\rho_{t,3} = -\lim_{m\to\infty} \left[-\frac{1}{(2m-1)!} \frac{d^{(2m)}}{ds^{(2m)}} \log\left[\zeta(s)(s-1)\right]\Big|_{s\to 0} + \right. \\ \left. -\frac{1}{(-2)^{2m}} - \frac{1}{(-4)^{2m}} \right]^{-\frac{1}{2m}}, \tag{90}$$

but this time we must know first two previous zeros, so that we compute

$$\rho_{t,3} = -5.9999\underline{9}8491319353326392575769396\ldots, \tag{91}$$

which is accurate to 5 decimal places. We see that the accuracy progressively reduces for higher zeros, so the limit variable m has to be increased to get better accuracy.

Algorithm 2 PARI script for computing first trivial zero using (86)

```
{
    // set limit variable
    m = 20;

    // compute generalized zeta series
    A = −derivnum(s = 0,log(zeta(s)∗(s−1)),2∗m);
    B = 1/factorial(2∗m−1);

    Z = A∗B;

    // compute the first trivial zero
    rho_t_1 = Zˆ(−1/(2∗m));
    print(rho_t_1);
}
```

We keep repeating this but now we re-compute with $m = 200$ to get better accuracy until we get to the 7^{th} trivial zero, which should be -14, so by computing

$$\rho_{t,7} = -\lim_{m\to\infty} \left[\frac{-1}{(2m-1)!} \frac{d^{(2m)}}{ds^{(2m)}} \log\left[\zeta(s)(s-1)\right]\Big|_{s\to 0} + \right. \\ \left. -\sum_{n=1}^{6} \frac{1}{(-2n)^{2m}} \right]^{-\frac{1}{2m}}, \tag{92}$$

yields

$$\rho_{t,7} = -14.000007669086476837019928729271\ldots, \tag{93}$$

where we notice that it is becoming less accurate. So when we compute the next zero

$$\rho_{t,8} = -\lim_{m\to\infty} \left[\frac{-1}{(2m-1)!} \frac{d^{(2m)}}{ds^{(2m)}} \log\left[\zeta(s)(s-1)\right]\Big|_{s\to 0} + \right. \\ \left. -\sum_{n=1}^{7} \frac{1}{(-2n)^{2m}} \right]^{-\frac{1}{2m}}, \tag{94}$$

we get

$$\rho_{t,8} = 14.2975976399\ldots - \mathrm{i}0.1122953782\ldots, \tag{95}$$

where it is seen no longer converging to -16 as expected. However, due to equality (82) the first non-trivial zero term is dominating the limit, so if we incorporate the first non-trivial zero term

$$\rho_{t,8} = -\lim_{m\to\infty} \left[\frac{-1}{(2m-1)!} \frac{d^{(2m)}}{ds^{(2m)}} \log\left[\zeta(s)(s-1)\right]\Big|_{s\to 0} + \right. \\ \left. -\sum_{n=1}^{7} \frac{1}{(-2n)^{2m}} - \frac{1}{(\frac{1}{2} + it_1)^{2m}} - \frac{1}{(\frac{1}{2} - it_1)^{2m}} \right]^{-\frac{1}{2m}}, \tag{96}$$

as to remove its contribution, then we re-compute the trivial zero again

$$\rho_{t,8} = -15.99999999999999999999\underline{9}861627109\ldots, \tag{97}$$

as desired. Hence this process continues for the n^{th}+1 trivial zero but for higher trivial zero terms more non-trivial terms have to be removed in this fashion.

Moving on next, we seek to find a formula for non-trivial zeros but according to (82) the trivial zeros dominate the generalized zeta series, and also that non-trivial zeros are complex will make the root extraction more difficult. So we consider the Weierstrass/Hadamard product (8) again but rewrite it in a simpler form

$$\zeta(s) = \frac{\pi^{s/2}}{2(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho_{nt}} \left(1 - \frac{s}{\rho_{nt}}\right), \qquad (98)$$

as to compress the trivial zeros by the gamma function which has the Weierstrass product

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{\frac{s}{n}}, \qquad (99)$$

and $\Gamma(s)$ is also known to have many representations making it a useful function. We now consider the Riemann ξ function

$$\xi(s) = \frac{(s-1)\Gamma(1+\frac{s}{2})}{\pi^{s/2}} \zeta(s) = \frac{1}{2} \prod_{\rho_{nt}} \left(1 - \frac{s}{\rho_{nt}}\right), \quad (100)$$

as to remove all trivial zero terms (and any other remaining terms) in order to obtain an exclusive access to non-trivial zeros. Now, when applying the m^{th} log-derivative to $\xi(s)$ we get

$$Z_{nt}(m) = -\frac{1}{(m-1)!} \frac{d^m}{ds^m} \log\left[\frac{(s-1)\Gamma(1+\frac{s}{2})}{\pi^{s/2}} \zeta(s)\right]\Big|_{s\to 0} = \\ = \sum_{n=1}^{\infty} \left[\frac{1}{(\sigma_n + \mathrm{i}t_n)^m} + \frac{1}{(\sigma_n - \mathrm{i}t_n)^m}\right], \qquad (101)$$

which is valid for $m \geq 1$. The first few special values of this series are:

$$\begin{aligned} Z_{nt}(1) &= 1 - \frac{1}{2}\eta_0 - \frac{1}{2}\log(4\pi) = \\ &= 1 + \frac{1}{2}\gamma - \frac{1}{2}\log(4\pi) = \\ &= 0.0230957089661210338143102479 06\ldots, \\ Z_{nt}(2) &= 1 + \eta_1 - \frac{1}{8}\pi^2 = \\ &= 1 + \gamma^2 + 2\gamma_1 - \frac{1}{8}\pi^2 = \\ &= -0.0461543172958046027571079903 79\ldots, \\ Z_{nt}(3) &= 1 - \eta_2 - \frac{7}{8}\zeta(3) = \\ &= 1 + \gamma^3 + 3\gamma\gamma_1 + \frac{3}{2}\gamma_2 - \frac{7}{8}\zeta(3) = \\ &= -0.0001111582314521059227626682 38\ldots, \\ Z_{nt}(4) &= 1 + \eta_3 - \frac{1}{96}\pi^4 = \\ &= 1 + \gamma^4 + 4\gamma^2\gamma_1 + 2\gamma_1^2 + 2\gamma\gamma_2 + \frac{2}{3}\gamma_3 - \frac{1}{96}\pi^4 = \\ &= 0.0000736272212616895183267713 07\ldots, \end{aligned}$$

$$\begin{aligned} Z_{nt}(5) &= 1 - \eta_4 - \frac{31}{32}\zeta(5) = \\ &= 1 + \gamma^5 + 5\gamma^3\gamma_1 + \frac{5}{2}\gamma^2\gamma_2 + \frac{5}{2}\gamma_1\gamma_2 + 5\gamma\gamma_1^2 + \\ &\quad + \frac{5}{6}\gamma\gamma_3 + \frac{5}{24}\gamma_4 - \frac{31}{32}\zeta(5) = \\ &= 0.0000007150933557626077358010 93\ldots \end{aligned} \qquad (102)$$

The value for $Z_{nt}(1)$ is commonly known throughout the literature [17], and values for $Z_{nt}(m)$ for $m > 1$ also have a closed-form formula

$$Z_{nt}(m) = 1 - (-1)^m 2^{-m} \zeta(m) - \frac{\log(|\zeta|)^{(m)}(0)}{(m-1)!}, \qquad (103)$$

valid for $m > 1$ and is given by Matsuoka [11, p. 249], Lehmer [10, p. 23], and Voros in [13, p. 73]. This formula is valid for even and odd index variable m. Another representation of (101) is given by

$$Z_{nt}(m) = 1 - (1 - 2^{-m})\zeta(m) + (-1)^m \eta_{m-1}, \qquad (104)$$

for $m > 1$ where η_n are the Laurent expansion coefficients of the series

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \eta_n (s-1)^n. \qquad (105)$$

The first few values are:

$$\begin{aligned} \eta_0 &= -0.5772156649015328606 1\ldots, \\ \eta_1 &= 0.1875462328403652246 0\ldots, \\ \eta_2 &= -0.0516886320331928938 02\ldots, \\ \eta_3 &= 0.0147516588254537440 65\ldots, \\ \eta_4 &= -0.0045244778884953787 412\ldots \end{aligned} \qquad (106)$$

These eta constants are probably less familiar than the Stieltjes constants γ_n, and one has $-\eta_0 = \gamma_0 = \gamma$, but its relation to Stieltjes constants will be discussed later, as our immediate goal is to express concisely

$$\eta_n = \frac{(-1)^n}{n!} \lim_{k\to\infty} \left\{\sum_{l=1}^{k} \Lambda(l) \frac{\log^n(l)}{l} - \frac{\log^{n+1}(k)}{n+1}\right\}, \qquad (107)$$

as found in [13, p. 25], where the von Mangoldt's function is defined as

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime and integer } k \geq 1, \\ 0, & \text{otherwise}, \end{cases} \qquad (108)$$

which is purely in terms of primes. Hence the expansion coefficients η_n are written as a function of primes, and then it follows that the generalized zeta series $Z_{nt}(m)$ can also be represented in terms of primes. We note, however, that the limit identity (107) is extremely slow to converge, requiring billions of prime terms to compute to only a few digits, making it very impractical.

Moreover, we note that the generalized zeta series $Z_{nt}(s)$ is over all zeros but in order to extract the non-trivial zeros we will have to assume (RH) so as to remove the real part of $\frac{1}{2}$. It is not readily possible to separate the reciprocal of conjugate-pairs of non-trivial zeros from (101) but what we can do is to consider a secondary zeta function over the complex magnitude, or modulus, squared of non-trivial zeros as

$$Z_{|nt|}(s) = \sum_{n=1}^{\infty} \frac{1}{|\rho_{nt,n}|^{2s}} = \sum_{n=1}^{\infty} \frac{1}{(\frac{1}{4}+t_n^2)^s} , \tag{109}$$

and then by applying Theorem 2 we can obtain non-trivial zeros

$$t_{n+1} = \lim_{m\to\infty} \left[\left(Z_{|nt|}(m) - \sum_{k=1}^{n} \frac{1}{(\frac{1}{4}+t_k^2)^m} \right)^{-1/m} - \frac{1}{4} \right]^{1/2} , \tag{110}$$

recursively. We now need a closed-form formula for $Z_{|nt|}(m)$ which we find can be related to $Z_{nt}(m)$ in several ways. The first way is by an asymptotic formula

$$Z_{|nt|}(m) \sim \frac{1}{2}[Z_{nt}^2(m) - Z_{nt}(2m)] \tag{111}$$

as $m \to \infty$. This immediately leads to a formula for the principal zero

Tab. 2. The computation of t_1 by Eq. (112) for different m

m	t_1 (First 30 Digits)	Significant Digits
2	5.561891787634141032446012810136	0
3	13.757670503723662711511861003244	0
4	12.161258748655529488677538477512	0
5	14.075935317783371421926582853327	0
6	13.579175424560852302300158195372	0
7	$14.\underline{1}16625853057249358432588137893$	1
8	13.961182494234115467191058505224	0
9	$14.\underline{1}26913415083941105873032355837$	1
10	14.077114859427980275510456957007	0
15	$14.1\underline{3}3795710050725394699252528681$	2
20	$14.13\underline{4}370485636531946259958638820$	3
25	$14.134\underline{7}00629574414322701677282886$	4
50	$14.13472514\underline{1}835685792188021492482$	9
100	$14.134725141734693\underline{7}89329888107217$	16

$$t_1 = \lim_{m\to\infty} \left[\left(\frac{1}{2}Z_{nt}^2(m) - \frac{1}{2}Z_{nt}(2m) \right)^{-1/m} - \frac{1}{4} \right]^{1/2} , \tag{112}$$

and a full recurrence formula

$$t_{n+1} = \lim_{m\to\infty} \left[\left(\frac{1}{2}Z_{nt}^2(m) - \frac{1}{2}Z_{nt}(2m) + \right.\right. \\ \left.\left. - \sum_{k=1}^{n} \frac{1}{(\frac{1}{4}+t_k^2)^m} \right)^{-1/m} - \frac{1}{4} \right]^{1/2} , \tag{113}$$

for non-trivial zeros as we have shown in [6, p. 9–14] and Matsuoka in [11], provided that all the zeros are assumed to lie on the critical line. A detailed numerical computation of t_1 by Eq. (112) is shown in Tab. 2 and a script in PARI in Algorithm 3, where we can observe convergence to t_1 as the limit variable m increases from low to high, and at $m = 100$ we get over 16 decimal places. A detailed numerical computation for higher m is summarized in [6].

Algorithm 3 PARI script for computing Eq. (112)

```
{
    // set limit variable
    m1 = 250;
    m2 = 2*m1;

    // compute parameters A1 to C1 for Z1
    A1 = derivnum(x = 0,log(abs(zeta(x))),m1);
    B1 = 1/factorial(m1−1);
    C1 = 1−(−1)^m1*2^(−m1)*zeta(m1);
    Z1 = C1−A1*B1;

    // compute parameters A2 to C2 for Z2
    A2 = derivnum(x = 0,log(abs(zeta(x))),m2);
    B2 = 1/factorial(m2−1);
    C2 = 1−(−1)^m2*2^(−m2)*zeta(m2);
    Z2 = C2−A2*B2;

    // compute t1 zero
    t1 = (((Z1^2−Z2)/2)^(−1/m1)−1/4)^(1/2);
    print(t1);
}
```

The next higher order zeros are recursively found as

$$t_2 = \lim_{m\to\infty} \left[\left(\frac{1}{2}Z_{nt}^2(m) - \frac{1}{2}Z_{nt}(2m) + \right.\right. \\ \left.\left. - \frac{1}{(\frac{1}{4}+t_1^2)^m} \right)^{-1/m} - \frac{1}{4} \right]^{\frac{1}{2}} , \tag{114}$$

and the next is

$$t_3 = \lim_{m\to\infty} \left[\left(\frac{1}{2}Z_{nt}^2(m) - \frac{1}{2}Z_{nt}(2m) + \right.\right. \\ \left.\left. - \frac{1}{(\frac{1}{4}+t_1^2)^m} - \frac{1}{(\frac{1}{4}+t_2^2)^m} \right)^{-1/m} - \frac{1}{4} \right]^{\frac{1}{2}} , \tag{115}$$

and so on, but the numerical computation is even more difficult, so the limit variable m has to be increased to a very large value. We can now express the non-trivial zeros in terms of other constants. By substituting the eta constants (104) to (112) we obtain the first zero:

$$t_1 = \lim_{m\to\infty}\left[\left(\frac{1}{2}\Big(1-(1-2^{-m})\zeta(m)+(-1)^m\eta_{m-1}\Big)^2+ -\frac{1}{2}\Big(1-(1-2^{-2m})\zeta(2m)+\eta_{2m-1}\Big)\right)^{-1/m}-\frac{1}{4}\right]^{\frac{1}{2}}. \tag{116}$$

For example, if we let $m = 10$ then we can generate an approximation converging to t_1 as

$$t_1 \approx \left[\left(-\frac{31}{2903040}\pi^{10}(1+\eta_9)+\eta_9+\frac{1}{2}\eta_9^2-\frac{1}{2}\eta_{19}+ +\frac{10568303}{92681981263872000}\pi^{20}\right)^{-\frac{1}{10}}-\frac{1}{4}\right]^{\frac{1}{2}} \approx \approx 14.07711485942798027551\ldots, \tag{117}$$

where it is seen converging to t_1. For this computation we compute the eta constants:

$$\eta_9 = 0.00001704135704711064 1032\ldots, \quad \eta_{19} = 0.000000000286807697455596\ldots, \tag{118}$$

using m^{th} differentiation of (105). As mentioned before, one could alternatively compute these eta constants using primes by (107) but the number of primes required now would be in trillions (making it very impractical to compute on a standard workstation). The main point, however, is that the non-trivial zeros can be expressed in terms of primes, namely, by Eqs. (107), (105) and (112).

Furthermore, a recurrence relation for the eta constants in terms of Stieltjes constants is

$$\eta_n = (-1)^{n+1}\left[\frac{n+1}{n!}\gamma_n+\sum_{k=0}^{n-1}\frac{(-1)^{k-1}}{(n-k-1)!}\eta_k\gamma_{n-k-1}\right], \tag{119}$$

found in Coffey [18, p. 532]. Using these relations, the non-trivial zeros can be written in terms of Stieltjes constants. For the first zero t_1 and $m = 2$, we obtain an expansion:

$$t_1 \approx \left[\left(2\gamma_1-\frac{\pi^2\gamma_1}{4}+\gamma_1^2-\gamma\gamma_2-\frac{\gamma_3}{3}+\gamma^2-\frac{\pi^2}{8}-\frac{\gamma^2\pi^2}{8}+\frac{5\pi^4}{384}\right)^{-\frac{1}{2}}-\frac{1}{4}\right]^{\frac{1}{2}} \approx \approx 5.561891787634141032446012810136\ldots \tag{120}$$

For $m = 3$ we obtain an expansion:

$$t_1 \approx \left[\left(\gamma^3-\frac{21}{8}\gamma\gamma_1\zeta(3)-\frac{21}{16}\gamma_2\zeta(3)+3\gamma\gamma_1-\gamma_1^3+\frac{3}{2}\gamma_2+\frac{3}{2}\gamma\gamma_1\gamma_2+\frac{3}{4}\gamma_2^2-\frac{1}{2}\gamma^2\gamma_3-\frac{1}{2}\gamma_1\gamma_3+ -\frac{1}{8}\gamma\gamma_4-\frac{1}{40}\gamma_5-\frac{7}{8}\zeta(3)-\frac{7}{8}\gamma^3\zeta(3)+\frac{49}{128}\zeta(3)^2+\frac{1}{1920}\pi^6\right)^{-\frac{1}{3}}-\frac{1}{4}\right]^{\frac{1}{2}} \approx \approx 13.757670503723662711511861003244\ldots \tag{121}$$

For $m = 4$ we obtain an expansion:

$$t_1 \approx \left[\left(4\gamma^2\gamma_1-\frac{1}{24}\gamma^2\gamma_1\pi^4+2\gamma_1^2-\frac{1}{48}\pi^4\gamma_1^2+\gamma_1^4+2\gamma\gamma_2-\frac{1}{48}\gamma\gamma_2\pi^4-2\gamma\gamma_1^2\gamma_2+\frac{1}{2}\gamma^2\gamma_2^2-\gamma_1\gamma_2^2+\frac{2}{3}\gamma_3+ -\frac{1}{144}\pi^4\gamma_3+\frac{2}{3}\gamma^2\gamma_1\gamma_3+\frac{2}{3}\gamma_1^2\gamma_3+\frac{2}{3}\gamma\gamma_2\gamma_3+\frac{1}{6}\gamma_3^2-\frac{1}{6}\gamma^3\gamma_4-\frac{1}{3}\gamma\gamma_1\gamma_4-\frac{1}{12}\gamma_2\gamma_4-\frac{1}{30}\gamma^2\gamma_5-\frac{1}{180}\gamma\gamma_6+ -\frac{1}{1260}\gamma_7+\gamma^4-\frac{\pi^4}{96}-\frac{1}{96}\pi^4\gamma^4+\frac{23}{215040}\pi^8\right)^{-\frac{1}{4}}-\frac{1}{4}\right]^{\frac{1}{2}} \approx 12.161258748655529488677538477512\ldots \tag{122}$$

Hence, as m increases, the value converges to t_1 as shown in Tab. 2 but the number of Stieltjes constants terms grows very large. In Tab. 2 we see that the accuracy of t_1 for odd m is slightly better than for even m. We recall that the Stieltjes constants γ_n themselves are defined as the Laurent expansion coefficients of the Riemann zeta function about $s = 1$ as

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}(-1)^n \frac{\gamma_n (s-1)^n}{n!}\ , \tag{123}$$

where they are similarly expressed as

$$\gamma_n = \lim_{k\to\infty}\left\{\sum_{l=1}^{k}\frac{\log^n(l)}{l} - \frac{\log^{n+1}(k)}{n+1}\right\}. \tag{124}$$

Also, the $\gamma_0 = \gamma$ is the usual Euler-Mascheroni constant.

There is also another way to compute Stieltjes constants that we developed (which can be ultimately expressed in terms of primes). We observe that γ_n are linear coefficients in the Laurent series (123), hence if we form a system of linear equations as

$$\begin{pmatrix} 1 & -\frac{(s_1-1)}{1!} & \frac{(s_1-1)^2}{2!} & -\frac{(s_1-1)^3}{3!} & \cdots & \frac{(s_1-1)^k}{k!} \\ 1 & -\frac{(s_2-1)}{1!} & \frac{(s_2-1)^2}{2!} & -\frac{(s_2-1)^3}{3!} & \cdots & \frac{(s_2-1)^k}{k!} \\ 1 & -\frac{(s_3-1)}{1!} & \frac{(s_3-1)^2}{2!} & -\frac{(s_3-1)^3}{3!} & \cdots & \frac{(s_3-1)^k}{k!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -\frac{(s_k-1)}{1!} & \frac{(s_k-1)^2}{2!} & -\frac{(s_k-1)^3}{3!} & \cdots & \frac{(s_k-1)^k}{k!} \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_k \end{pmatrix} = \begin{pmatrix} \zeta(s_1) - \frac{1}{s_1-1} \\ \zeta(s_2) - \frac{1}{s_2-1} \\ \zeta(s_3) - \frac{1}{s_3-1} \\ \vdots \\ \zeta(s_k) - \frac{1}{s_k-1} \end{pmatrix}, \tag{125}$$

then for a choice of values for $s_1 = 2$, $s_2 = 3$, $s_3 = 4$ and so on, and using the Cramer's rule (for solving a system of linear equations) and some properties of an Vandermonde matrix we find that Stieltjes constants can be represented by determinant of a certain matrix:

$$\gamma_n = \pm\frac{\det(A_{n+1})}{\det(A)}\ , \tag{126}$$

where the matrix $A_n(k)$ is matrix $A(k)$ but with an n^{th} column swapped with a vector B as given next

$$A(k) = \begin{pmatrix} 1 & -\frac{1}{1!} & \frac{1^2}{2!} & -\frac{1^3}{3!} & \cdots & \frac{(-1)^k}{k!} \\ 1 & -\frac{2}{1!} & \frac{2^2}{2!} & -\frac{2^3}{3!} & \cdots & \frac{(-2)^k}{k!} \\ 1 & -\frac{3}{1!} & \frac{3^2}{2!} & -\frac{3^3}{3!} & \cdots & \frac{(-3)^k}{k!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -\frac{(k+1)}{1!} & \frac{(k+1)^2}{2!} & -\frac{(k+1)^3}{3!} & \cdots & \frac{(-1)^k(k+1)^k}{k!} \end{pmatrix}, \tag{127}$$

and

$$B(k) = \begin{pmatrix} \zeta(2) - 1 \\ \zeta(3) - \frac{1}{2} \\ \zeta(4) - \frac{1}{3} \\ \vdots \\ \zeta(k+1) - \frac{1}{k} \end{pmatrix}. \tag{128}$$

The $\pm$ sign depends on the size of the matrix k but in order to ensure a positive sign the size of k must be a multiple of 4. It can be shown that $\det(A) = 1$, hence the determinant formula for Stieltjes constants becomes

$$\gamma_n = \det(A_{n+1}), \tag{129}$$

and the size of the matrix must be $4k$.

Hence, the first few Stieltjes constants can be represented as:

$$\gamma_0 = \lim_{k\to\infty} \det \begin{pmatrix} \zeta(2)-1 & -\frac{1}{1!} & \frac{1^2}{2!} & -\frac{1^3}{3!} & \dots & \frac{(-1)^k 1^k}{k!} \\ \zeta(3)-\frac{1}{2} & -\frac{2}{1!} & \frac{2^2}{2!} & -\frac{2^3}{3!} & \dots & \frac{(-1)^k 2^k}{k!} \\ \zeta(4)-\frac{1}{3} & -\frac{3}{1!} & \frac{3^2}{2!} & -\frac{3^3}{3!} & \dots & \frac{(-1)^k 3^k}{k!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \zeta(k+1)-\frac{1}{k} & -\frac{(k+1)}{1!} & \frac{(k+1)^2}{2!} & -\frac{(k+1)^3}{3!} & \dots & \frac{(-1)^k (k+1)^k}{k!} \end{pmatrix} = \\ = 0.57721566490153286061\dots, \tag{130}$$

and the next Stieltjes constant is

$$\gamma_1 = \lim_{k\to\infty} \det \begin{pmatrix} 1 & \zeta(2)-1 & \frac{1^2}{2!} & -\frac{1^3}{3!} & \dots & \frac{(-1)^k 1^k}{k!} \\ 1 & \zeta(3)-\frac{1}{2} & \frac{2^2}{2!} & -\frac{2^3}{3!} & \dots & \frac{(-1)^k 2^k}{k!} \\ 1 & \zeta(4)-\frac{1}{3} & \frac{3^2}{2!} & -\frac{3^3}{3!} & \dots & \frac{(-1)^k 3^k}{k!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta(k+1)-\frac{1}{k} & \frac{(k+1)^2}{2!} & -\frac{(k+1)^3}{3!} & \dots & \frac{(-1)^k (k+1)^k}{k!} \end{pmatrix} = \\ = -0.07281584548367672486 1\dots, \tag{131}$$

and the next is

$$\gamma_2 = \lim_{k\to\infty} \det \begin{pmatrix} 1 & -\frac{1}{1!} & \zeta(2)-1 & -\frac{1^3}{3!} & \dots & \frac{(-1)^k 1^k}{k!} \\ 1 & -\frac{2}{1!} & \zeta(3)-\frac{1}{2} & -\frac{2^3}{3!} & \dots & \frac{(-1)^k 2^k}{k!} \\ 1 & -\frac{3}{1!} & \zeta(4)-\frac{1}{3} & -\frac{3^3}{3!} & \dots & \frac{(-1)^k 3^k}{k!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -\frac{(k+1)}{1!} & \zeta(k+1)-\frac{1}{k} & -\frac{(k+1)^3}{3!} & \dots & \frac{(-1)^k (k+1)^k}{k!} \end{pmatrix} = \\ = -0.009690363192872318484 5\dots, \tag{132}$$

---

and so on. And in [4] we performed extensive numerical computation of (129) where it clearly converges to the Stieltjes constants and in essence analytically extends $\zeta(s)$ to the whole complex plane by the Laurent expansion (123) with an only knowledge of $\zeta(s)$ for $s > 1$. Finally, we remark that the vector $B(k)$ can be expressed in terms of primes by substituting the Euler product for $\zeta(s)$ as

$$B(k) = \begin{pmatrix} \prod_{n=1}^{\infty}\left(1-\frac{1}{p_n^2}\right)^{-1} - 1 \\ \prod_{n=1}^{\infty}\left(1-\frac{1}{p_n^3}\right)^{-1} - \frac{1}{2} \\ \prod_{n=1}^{\infty}\left(1-\frac{1}{p_n^4}\right)^{-1} - \frac{1}{3} \\ \vdots \\ \prod_{n=1}^{\infty}\left(1-\frac{1}{p_n^{k+1}}\right)^{-1} - \frac{1}{k} \end{pmatrix}. \tag{133}$$

This in turn leads to computing Stieltjes constants by primes, then $Z_{|nt|}$ by the Stieltjes constants and then the non-trivial zeros by $Z_{|nt|}$. Henceforth, we also obtain a similar formula for the η_n constants by defining a similar vector

$$D(k) = \begin{pmatrix} -\frac{\zeta'(2)}{\zeta(2)} - 1 \\ -\frac{\zeta'(3)}{\zeta(3)} - \frac{1}{2} \\ -\frac{\zeta'(4)}{\zeta(4)} - \frac{1}{3} \\ \vdots \\ -\frac{\zeta'(k+1)}{\zeta(k+1)} - \frac{1}{k}, \end{pmatrix}, \tag{134}$$

and matrix C_n which is matrix A but with an n^{th} column swapped with a vector D, and using the Cramers rule we find that

$$\eta_n = \frac{1}{n!}\det(C_{n+1}), \tag{135}$$

and the size of the matrix must be $4k$ to ensure a positive sign. For example, η_2 would be

$$\eta_2 = \lim_{k\to\infty} \frac{1}{2!} \det \begin{pmatrix} 1 & -\frac{1}{1!} & -\frac{\zeta'(2)}{\zeta(2)} - 1 & -\frac{1^3}{3!} & \dots & \frac{(-1)^k 1^k}{k!} \\ 1 & -\frac{2}{1!} & -\frac{\zeta'(3)}{\zeta(3)} - \frac{1}{2} & -\frac{2^3}{3!} & \dots & \frac{(-1)^k 2^k}{k!} \\ 1 & -\frac{3}{1!} & -\frac{\zeta'(4)}{\zeta(4)} - \frac{1}{3} & -\frac{3^3}{3!} & \dots & \frac{(-1)^k 3^k}{k!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -\frac{(k+1)}{1!} & -\frac{\zeta'(k+1)}{\zeta(k+1)} - \frac{1}{k} & -\frac{(k+1)^3}{3!} & \dots & \frac{(-1)^k (k+1)^k}{k!} \end{pmatrix} = \\ = -0.05168863203319289380 2\dots \tag{136}$$

We remarked that the results presented so far are geared toward computing $Z_{|nt|}$ from Z_{nt} by the asymptotic formula (111). We now investigate two other similar formulas of obtaining $Z_{|nt|}$ given by works of Voros [13]. The first is the *Bologna* formula (family) as

$$Z_{|nt|}(m) = \sum_{n=1}^{m} \binom{2m-n-1}{m-1} Z_{nt}(n), \tag{137}$$

for $m > 1$ (and $t = \frac{1}{2}$) given in [13, p. 84]. And the second (very similar formula) relating to the $Z_{|nt|}$ to the Keiper-Li constants λ_m as

$$Z_{|nt|}(m) = \sum_{n=1}^{m} (-1)^{n+1} \binom{2m}{m-n} \lambda_n \ , \tag{138}$$

where λ_m are defined by

$$\lambda_m = \sum_{\rho_{nt}} \left[1 - \left(1 - \frac{1}{\rho_{nt}}\right)^m \right], \tag{139}$$

which has a closed-form representation as

$$\lambda_m = \frac{1}{(m-1)!} \frac{d^m}{ds^m} \left[s^{m-1} \log \xi(s) \right]_{s\to 1} \ , \tag{140}$$

in terms of logarithmic differentiation of the Riemann xi function, which essentially resembles all our previous results. The first few constants are:

$$\begin{aligned} \lambda_1 &= 0.02309570896612103381\dots, \\ \lambda_2 &= 0.09234573522804667038\dots, \\ \lambda_3 &= 0.20763892055432480379\dots, \\ \lambda_4 &= 0.36879047949224163859\dots, \\ \lambda_5 &= 0.57554271446117745243\dots, \end{aligned} \tag{141}$$

and so on. The Li's Criterion for (RH) is if $\lambda_m > 0$ for all $m \geq 1$, which has been widely studied. Henceforth, the first few special values of $Z_{|nt|}(m)$ in terms of Keiper-Li constants are:

$$\begin{aligned} Z_{|nt|}(1) &= \lambda_1 = \\ &= 0.02309570896612103381431024790 6\dots, \\ Z_{|nt|}(2) &= 4\lambda_1 - \lambda_2 = \\ &= 0.00037100636437464871512505433\dots, \\ Z_{|nt|}(3) &= 15\lambda_1 - 6\lambda_2 + \lambda_3 = \\ &= 0.00000143677860288691774848062\dots, \\ Z_{|nt|}(4) &= 56\lambda_1 - 28\lambda_2 + 8\lambda_3 - \lambda_4 = \\ &= 0.00000000659827914542401152690\dots, \\ Z_{|nt|}(5) &= 210\lambda_1 - 120\lambda_2 + 45\lambda_3 - 10\lambda_4 + \lambda_5 = \\ &= 0.00000000003193891860867324232\dots \end{aligned} \tag{142}$$

Now, when using the previous results we can also compute non-trivial zeros in terms of the Keiper-Li constants. For example, for $m = 10$ we approximate t_1 as

$$t_1 \approx \left[\left(167960\lambda_1 - 125970\lambda_2 + 77520\lambda_3 - 38760\lambda_4 + \right.\right. \\ +15504\lambda_5 - 4845\lambda_6 + 1140\lambda_7 - 190\lambda_8 + \\ \left.\left. +20\lambda_9 - \lambda_{10} \right)^{-\frac{1}{10}} - \frac{1}{4} \right]^{1/2} \approx \\ \approx 14.07711485942798027551\dots \tag{143}$$

by substituting (138) to (111). A numerical computation clearly converges to the correct value. These formulas, together with our previous results, can be used to compute non-trivial zeros and generate a wide variety of representations of non-trivial zeros.

Moving on, another way to obtain the non-trivial zeros is to consider the secondary zeta function

$$Z_1(s) = \sum_{n=1}^{\infty} \frac{1}{t_n^s} \ , \tag{144}$$

over just the imaginary part of non-trivial zeros t_n and apply Theorem 2 directly, where it suffices to find a closed-form representation of $Z_1(s)$. To do this, we consider the Riemann xi function again but this time transform the variable $s = \frac{1}{2} + it$ along the critical line yielding a function $\Xi(t) = \xi(\frac{1}{2} + it)$, so that its zeros are only the imaginary parts of non-trivial zeros t_n. Now, when applying the m^{th} log-derivative formula we get

$$Z_1(2m) = -\frac{1}{2(2m-1)!} \frac{d^{(2m)}}{dt^{(2m)}} \log \Xi(t) \Big|_{t\to 0} = \\ = \sum_{n=1}^{\infty} \frac{1}{t_n^{2m}} \ , \tag{145}$$

for $m \geq 1$, which yields the generalized zeta series over imaginary parts of non-trivial zeros t_n. We note that since $\Xi(t)$ is even, we only consider the $2m$ limiting value and require a factor of $\frac{1}{2}$. The first few special values of this series are:

$$\begin{aligned} Z_1(1) &\sim \sum_{0<t\leq T} \frac{1}{t} \sim H + \frac{1}{4\pi}\log^2\left(\frac{T}{2\pi}\right) \quad (T \to \infty), \\ &H = -0.0171594043070981495\ldots, \\ Z_1(2) &= \frac{1}{2}(\log|\zeta|)^{(2)}\big(\frac{1}{2}\big) + \frac{1}{8}\pi^2 + \beta(2) - 4 = \\ &= 0.023104993115418970788933810430\ldots, \\ Z_1(3) &= 0.000729548272709704215875518569\ldots, \\ Z_1(4) &= -\frac{1}{12}(\log|\zeta|)^{(4)}\big(\frac{1}{2}\big) - \frac{1}{24}\pi^4 - 4\beta(4) + 16 = \\ &= 0.000037172599285269686164866262\ldots, \\ Z_1(5) &= 0.000002231188699502103328640628\ldots \end{aligned} \tag{146}$$

For $s = 1$, the series diverges asymptotically as $H + \frac{1}{4\pi}\log^2(\frac{T}{2\pi})$ where H is a constant as shown above, which is investigated by Hassani [19] and R.P. Brent [20, 21], but its precise computation is very challenging because of a very slow convergence of the series. The presented value was accurately computed to high precision by R.P. Brent [21, p. 6] using 10^{10} non-trivial zeros and remainder estimation techniques, which further improve accuracy to over 19 decimal places. We also remark that the number of non-trivial zeros are to be taken less than or equal to T. The resulting Hassani constant is analogous to the harmonic sum and Euler's constant relation

$$\sum_{n=1}^{k} \frac{1}{n} \sim \gamma + \log(k) \quad (k \to \infty). \tag{147}$$

The even values of (146) given were computed using the Voros's closed-form formula

$$Z_1(2m) = (-1)^m \Bigg[-\frac{1}{2(2m-1)!}(\log|\zeta|)^{(2m)}\big(\frac{1}{2}\big) + \\ -\frac{1}{4}\left[(2^{2m}-1)\zeta(2m) + 2^{2m}\beta(2m)\right] + 2^{2m} \Bigg], \tag{148}$$

assuming (RH). As there is no known formula such as this valid for a positive odd integer argument, the odd values given were computed by an algorithm developed by Arias De Reyna [22] in a Python software package in a library **mpmath**, which roughly works by computing (144) up to several zeros and estimating the remainder to a high degree of accuracy. It would otherwise take billions of non-trivial zeros to compute (144) directly. Also, the function

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}\,, \tag{149}$$

is the Dirichlet beta function. Finally, when applying the root-extraction to (144) by Theorem 2, we find the principal zero as

$$t_1 = \lim_{m\to\infty} [Z_1(2m)]^{-\frac{1}{2m}}\,, \tag{150}$$

and a numerical computation for $m = 250$ yields

$$t_1 = 14.134725141734693790457251983562470270784257115699243175685567460149963429809256764949010\underline{0}212214333747\ldots, \tag{151}$$

using a script in Algorithm 4, which is accurate to 87 decimal places. The second zero is recursively found as

$$t_2 = \lim_{m\to\infty} \left[Z_1(2m) - \frac{1}{t_1^{2m}}\right]^{-\frac{1}{2m}}, \tag{152}$$

and a numerical computation for $m = 250$ yields

$$t_2 = 21.02203963877155499262847959389690277733\underline{3}355195796311\ldots, \tag{153}$$

which is accurate to 38 decimal places but the first zero t_1 used was already pre-computed to 1000 decimal places by other means in order to ensure convergence. We cannot substitute the same t_1 computed in (151) for $m = 250$ to (152) as it will cause self-cancelation, so the accuracy of t_n must be much higher than t_{n+1}. Similarly, the third zero is recursively found as

$$t_3 = \lim_{m\to\infty} \left[Z_1(2m) - \frac{1}{t_1^{2m}} - \frac{1}{t_2^{2m}}\right]^{-\frac{1}{2m}} \tag{154}$$

and a numerical computation for $m = 250$ yields

$$t_3 = 25.010857580145688763213790992562821818659549\underline{6}5846378\ldots \tag{155}$$

which is accurate to 43 decimal places, but the t_1 and t_2 zeros used were already pre-computed to 1000 decimal places by other means in order to ensure convergence. We cannot substitute the same t_1 and t_2 computed in (151) and (153) for $m = 250$ to (154) as it will cause self-cancelation, so the accuracy of t_n must be much higher than t_{n+1}. Hence, a full recurrence formula is

$$t_{n+1} = \lim_{m\to\infty} \left[Z_1(2m) - \sum_{k=1}^{n} \frac{1}{t_k^{2m}}\right]^{-\frac{1}{2m}}. \tag{156}$$

Furthermore, we also have a useful identity

$$\frac{1}{2^s}\zeta\big(s, \frac{5}{4}\big) = \sum_{k=1}^{\infty} \frac{1}{\left(\frac{1}{2}+2k\right)^s} = 2^s\left[\frac{1}{2}\left((1-2^{-s})\zeta(s) + \\ +\beta(s)\right) - 1\right], \tag{157}$$

Algorithm 4 PARI script for computing first the non-trivial zero using Eqs. (145) and (150)

```
// define xi(s)
xi(s) = (s−1)∗gamma(1+s/2)/Pi^(s/2)∗zeta(s)

{
    // set limit variable
    m = 20;

    // compute generalized zeta series
    A = −derivnum(t = 0,log(xi(1/2+I∗t)),2∗m);
    B = 1/factorial(2∗m−1);

    Z = 1/2∗A∗B;

    // compute the first zero
    t1 = Z^(−1/(2∗m));
    print(t1);
}
```

found in [12, p. 681] for which we can express the zeta and beta terms in terms of a Hurwitz zeta function, and then substituting the Voros's closed-form formula (148) into (157) we obtain another formula for non-trivial zeros

$$t_{n+1}=\lim_{m\to\infty}\left[\frac{(-1)^m}{2}\left(2^{2m}-\frac{1}{(2m-1)!}\log(|\zeta|)^{(2m)}\left(\frac{1}{2}\right)+ -\frac{1}{2^{2m}}\zeta\left(2m,\frac{5}{4}\right)\right)-\sum_{k=1}^{n}\frac{1}{t_k^{2m}}\right]^{-\frac{1}{2m}}, \tag{158}$$

as anticipated in (11). Also, an extensive numerical computation of (158) to high precision is summarized in [6] and also for higher order non-trivial zeros.

One limitation for all of these formulas for non-trivial zeros is when $n\to\infty$, then the average gap between zeros gets smaller as $t_{n+1}-t_n\sim\frac{2\pi}{\log(n)}$, making the use of these formulas progressively harder and harder to compute the next zero recursively.

By putting these results together we have two main generalized zeta series. The first series is for the complex magnitude (109) over all zeros (including the hypothetical zeros off of the critical line) as

$$Z_{|nt|}(s)=\frac{1}{(\sigma_1^2+t_1^2)^s}+\frac{1}{(\sigma_2^2+t_2^2)^s}+\frac{1}{(\sigma_3^2+t_3^2)^s}+\ldots, \tag{159}$$

from which we have an asymptotic relationship

$$[Z_{|nt|}(s)]^{-\frac{1}{s}}\sim\sigma_1^2+t_1^2 \quad (s\to\infty). \tag{160}$$

The second formula is for generalized zeta series over the imaginary parts

$$Z_1(2s)=\frac{1}{t_1^{2s}}+\frac{1}{t_2^{2s}}+\frac{1}{t_3^{2s}}+\ldots, \tag{161}$$

from which we have asymptotic relationship

$$[Z_1(2s)]^{-\frac{1}{s}}\sim t_1^2\,. \tag{162}$$

Combining (160) and (162) we obtain a true asymptotic formula for the real part of the first non-trivial zero

$$\Re(\rho_{1,nt})=\sigma_1=\lim_{s\to\infty}\sqrt{[Z_{|nt|}(s)]^{-\frac{1}{s}}-[Z_1(2s)]^{-\frac{1}{s}}} \tag{163}$$

and further, by substituting (104) for $Z_{|nt|}(s)$ and (148) for $Z_1(2s)$ we obtain

$$\Re(\rho_{1,nt})=\sigma_1=\lim_{m\to\infty}\left[\left(\frac{1}{2}Z_{nt}^2(m)-\frac{1}{2}Z_{nt}(2m)\right)^{-\frac{1}{s}}+ -\left(\frac{(-1)^m}{2}\left(2^{2m}-\frac{1}{(2m-1)!}\log(|\zeta|)^{(2m)}\left(\frac{1}{2}\right)+ -\frac{1}{2^{2m}}\zeta\left(2m,\frac{5}{4}\right)\right)\right)^{-\frac{1}{s}}\right]^{\frac{1}{2}}=\frac{1}{2}. \tag{164}$$

Earlier we remarked that the Voros's closed-form formula for $Z_1(2s)$ depends on (RH) and the formula for $Z_{|nt|}(s)$ in terms of the eta constants does not. Hence, if the limit converges to $\frac{1}{2}$, it would imply (RH). The convergence is achieved by a cancelation of t_1 generated by both Eqs. (160) and (162). In Tab. 3 we compute $\Re(\rho_{1,nt})$ by Eq. (164) for various values of the limit variable m from low to high and observe convergence to $\frac{1}{2}$ as m increases.

Tab. 3. The computation of the real part of the first non-trivial zero $\Re(\rho_{1,nt})$ by Eq. (164) (first 30 decimal places)

m	$\Re(\rho_{1,nt})$	Significant Digits
15	0.473092533136919315298424867840	1
20	0.489872906754757867871088167822	1
25	0.499306593693622997849224832930	3
50	0.500000002854988386875132586206	8
100	0.499999999999999968130042946283	16
150	0.500000000000000000000000039540	25
200	0.499999999999999999999999999999	35

V. THE INVERSE RIEMANN ZETA FUNCTION

In the previous section we outlined the full solution set to

$$w=\zeta(s)=0, \tag{165}$$

(assuming RH), which can also be interpreted as an inverse of

$$s=\zeta^{-1}(0), \tag{166}$$

as a set of all points s such that $w = \zeta(s) = 0$. Now, for other values of w-domain of the Riemann zeta function we seek to find s such that

$$s = \zeta^{-1}(w), \tag{167}$$

which implies that

$$\zeta^{-1}(\zeta(s)) = s, \tag{168}$$

and

$$\zeta(\zeta^{-1}(w)) = w, \tag{169}$$

for some domains w and s. Again, the zeta function can have many solutions s_n (just like for the zeros) for any given input value. Hence, we need to solve an equation

$$\zeta(s) - w = 0, \tag{170}$$

as a function of variable w. Then, by employing the m^{th} log-derivative method and the recursive root extraction described earlier we can arrive at a solution to (170). To illustrate this we re-consider the recurrence formula for trivial zeros (85) again as

$$\rho_{t,n+1} = \lim_{m\to\infty} \pm \left[-\frac{1}{(2m-1)!} \frac{d^{(2m)}}{ds^{(2m)}} \log\Big[\zeta(s)(s-1)\Big]\Big|_{s\to 0} + \\ - \sum_{k=1}^{n} \frac{1}{\rho_{t,k}^{2m}} - \sum_{k=1}^{\infty} \left(\frac{1}{\rho_{nt,k}^{2m}} + \frac{1}{\bar{\rho}_{nt,k}^{2m}} \right) \right]^{-\frac{1}{2m}}, \tag{171}$$

(since Z_t is dominating the series) and, comparing (170) with (171), we solve this equation by replacing trivial zeros with s_n as

$$s_{n+1} = \zeta^{-1}(w) = \\ = \lim_{m\to\infty} \pm \left[-\frac{1}{(2m-1)!} \frac{d^{(2m)}}{ds^{(2m)}} \log\Big[(\zeta(s)-w)(s-1)\Big]\Big|_{s\to 0} + \\ - \sum_{k=1}^{n} \frac{1}{s_k^{2m}} - \sum_{k=1}^{\infty} \left(\frac{1}{\rho_{nt,k}^{2m}} + \frac{1}{\bar{\rho}_{nt,k}^{2m}} \right) \right]^{-\frac{1}{2m}}, \tag{172}$$

where s_n is the multi-valued solution of s-domain, as indexed by variable n, and which is extracted from the recurrence relation of (172), where $s = s_1$ is the principal solution. Presently, we do not know whether the contribution in (172) due to non-trivial zeros will become relevant at higher branches (just like for trivial zeros), or whether there are other complex solutions at higher branches, hence we drop the non-trivial zero terms and obtain the form:

$$s_{n+1} = \zeta^{-1}(w) = \lim_{m\to\infty} \pm \left[-\frac{1}{(2m-1)!} \times \\ \times \frac{d^{(2m)}}{ds^{(2m)}} \log\left[(\zeta(s)-w)(s-1)\right]\Big|_{s\to 0} - \sum_{k=1}^{n} \frac{1}{s_k^{2m}} \right]^{-\frac{1}{2m}}, \tag{173}$$

and the principal solution is

$$s = s_1 = \zeta^{-1}(w) = \lim_{m\to\infty} \pm \left[-\frac{1}{(m-1)!} \times \\ \times \frac{d^{(m)}}{ds^{(m)}} \log\left[(\zeta(s)-w)(s-1)\right]\Big|_{s\to 0} \right]^{-\frac{1}{m}}, \tag{174}$$

where we consider an even and odd m and remove the $2m$ for convenience. We next seek to verify this formula by performing a high precision numerical computation of (174) in PARI/GP software package for various test cases. The script that we run is a slight modification of Algorithm 2, as shown in Algorithm 5.

Algorithm 5 PARI script for computing the inverse zeta by Eq. (174)

```
{
    // set limit variable
    m = 40;

    // set a value for w-domain
    w = zeta(2);

    // compute generalized zeta series
    A = −derivnum(s = 0,log((zeta(s)−w)∗(s−1)), m);
    B = 1/factorial(m−1);
    Z = A∗B;

    // compute s-domain
    s = Zˆ(−1/m);
    print(s);
}
```

In the first example we attempt to invert the Basel problem

$$w = \zeta(2) = \frac{\pi^2}{6} = 1.64493406684822643647\ldots, \tag{175}$$

by computing (174) for $m = 40$ and we obtain

$$s = \zeta^{-1}(\frac{\pi^2}{6}) = \\ = 2.0000000000000000000000000000\underline{0}534151435532\ldots, \tag{176}$$

which is accurate to 28 digits after the decimal place. As m increases, the result clearly converges to 2. In the next example we invert the Apéry's constant

$$w = \zeta(3) = 1.20205690315959428539\ldots, \tag{177}$$

then for $m = 40$ we compute

$$s = \zeta^{-1}(\zeta(3)) = \\ = 3.0000000000000000000\underline{0}2214079006164043806 9\ldots, \tag{178}$$

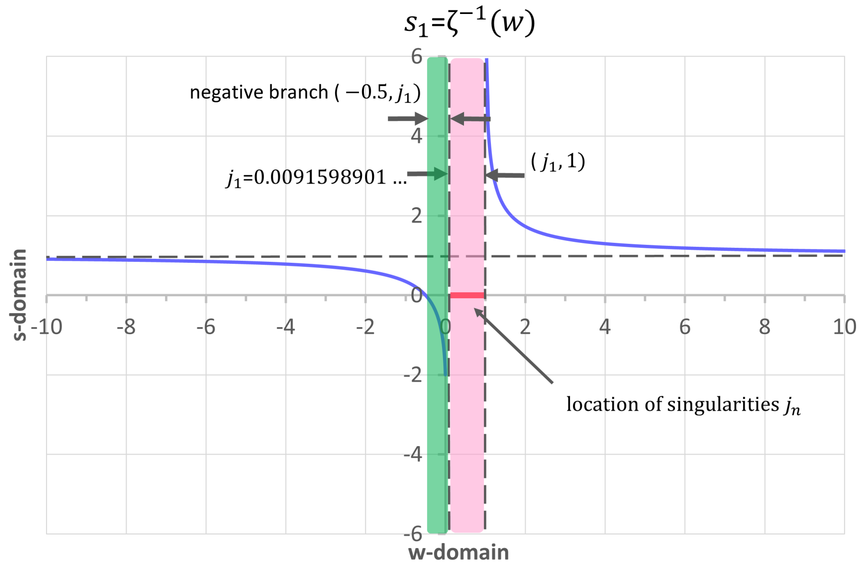

Fig. 3. A plot of $s_1 = \zeta^{-1}(w)$ for $w \in (-10, 10)$ by Eq. (174) showing location of zeros and singularities

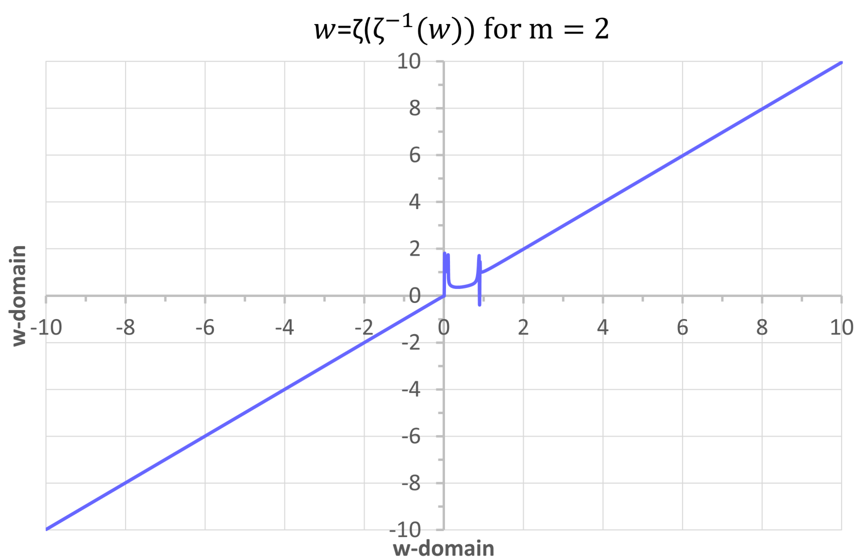

Fig. 4. A plot of $w = \zeta(\zeta^{-1}(w))$ for real $w \in (-10, 10)$ for $m = 2$ by the 2^{nd} order approximation Eq. (185)

accurate to 20 decimal places, where it is seen converging to 3 (even for lower values of limit variable m, the convergence is fast). In Tab. 4 we summarize computations for various other values of w-domain, where we can see the correct convergence to the inverse Riemann zeta function for $m = 20$ every time. For $w = \zeta(0) = -\frac{1}{2}$ there is a singularity at higher derivatives, so we take $\lim_{w \to -\frac{1}{2}} \zeta^{-1}(w)$, and for $\Re(w) \in (-0.5, j_1) \cup \{\Im(w) = 0\}$ where $j_1 = 0.00915989\ldots$ is a constant. There is also a sign change from positive to negative due to this branch so that the output will come out negative as shown by numerical computations in Tab. 4. In general, we find that for $\Re(w) \in (-\infty, -0.5) \cup (1, \infty)$ we consider the positive solution

$$s = s_1 = \zeta^{-1}(w) =$$
$$= \lim_{m \to \infty} + \left[\frac{-1}{(m-1)!} \frac{d^m}{ds^m} \log\left[(\zeta(s) - w)(s-1) \right] \Big|_{s \to 0} \right]^{-\frac{1}{m}}, \tag{179}$$

and otherwise for $\Re(w) \in (-0.5, j_1) \cup \{\Im(w) = 0\}$ we consider the negative solution

$$s = s_1 = \zeta^{-1}(w) =$$
$$= \lim_{m \to \infty} - \left[\frac{-1}{(m-1)!} \frac{d^m}{ds^m} \log\left[(\zeta(s) - w)(s-1) \right] \Big|_{s \to 0} \right]^{-\frac{1}{m}}. \tag{180}$$

We observe that convergence is faster near $w = -0.5$ for both sides and as $w \to -0.5$ we get convergence to 0 as desired. Furthermore, we observe that near both sides of the pole at $s = 1$ we can recover the inverse zeta. Hence, when we compute for higher limit variable m, the values are clearly converging to the inverse of the Riemann zeta function. In Tab. 5 we also compute the inverse zeta for various arbitrary values of w-domain for $m = 100$.

In Fig. 3 we plot $s_1 = \zeta^{-1}(w)$ for the principal solution by Eq. (174). The function reproduces the inverse zeta correctly everywhere except in a region $\Re(w) \in (j_1, 1)$ where the convergence is erroneous due to (possibly) an infinite number of singularities in an interval $\Re(w) \in (j_1, 1)$. That is not to say that $\zeta(s)$ does not have an inverse in this strip, for example, we have $\zeta(-15.48765247\ldots) = 0.5$ so that $\zeta^{-1}(0.5) = -15.48765247\ldots$, but it does not exist on the principal branch s_1.

In the next example, we seek to compute the next branch recursively. Let us first compute an inverse of $\zeta(-3) = \frac{1}{120}$ again for $m = 40$ and obtain

$$s_1 = \zeta^{-1}(\frac{1}{120}) = -2.4727\underline{0}347315140943243\ldots \tag{181}$$

where we consider the negative solution. At first, one might wonder that the result is incorrect but in fact it is only the principal solution. The second solution for $\zeta^{-1}(\frac{1}{120})$ is the value that we anticipate but we recall that the m^{th} log-derivative generates the generalized zeta series of over all zeros of a function. Hence, we can recursively obtain the second solution as

$$\begin{aligned} s_2 = \zeta^{-1}(\frac{1}{120}) = -\lim_{m\to\infty}\Bigg[-\frac{1}{(2m-1)!}\times \\ \times\frac{d^{(2m)}}{dx^{(2m)}}\log\left((\zeta(x)-\frac{1}{120})(x-1)\right)\Big|_{x\to 0}+ \\ -\frac{1}{(-2.4727305901\ldots)^{2m}}\Bigg]^{-\frac{1}{2m}} = \\ = -3.0000000\underline{0}597327044430\ldots \end{aligned} \tag{182}$$

by removing the first solution and for $m = 20$ the computation converges to a value -3 to within 8 decimal places. As we mentioned before, such a computation is getting more difficult because it requires the first branch s_1 to be known to very high precision in order to ensure convergence. Hence, we pre-computed s_1 to 1000 decimal places using the standard root finder in PARI, since it is more efficient than using (174) for higher m. As a result, by knowing s_1 accurately we compute s_2 using the recurrence formula. Hence, using this process we can recursively compute all the solutions which lie on different branches provided that s_n are real. But again, numerical computation becomes difficult as very high arbitrary precision is required.

Moving on, if we take $m = 2$ and expand the inverse zeta formula (174) as

$$\zeta^{-1}(w) \approx \left[\frac{1}{(w+\frac{1}{2})^2}\Bigg(w^2 + w[-2\zeta(0)+\zeta''(0)]+ +\zeta(0)^2+\zeta'(0)^2-\zeta(0)\zeta''(0)\Bigg)\right]^{-\frac{1}{2}}, \tag{183}$$

and using the identities

$$\begin{aligned} \zeta(0) &= -\frac{1}{2}, \\ \zeta'(0) &= -\frac{1}{2}\log(2\pi), \\ \zeta''(0) &= \frac{1}{2}\gamma^2+\gamma_1-\frac{1}{24}\pi^2-\frac{1}{2}\log^2(2\pi), \end{aligned} \tag{184}$$

we then obtain the 2^{nd} order approximation:

$$\zeta^{-1}(w) \approx \pm(w+\frac{1}{2})\Bigg[w^2 + w[1+\frac{1}{2}\gamma^2+\gamma_1-\frac{1}{24}\pi^2+ -\frac{1}{2}\log^2(2\pi)]+\frac{1}{4}+\frac{1}{4}\gamma^2+\frac{1}{2}\gamma_1-\frac{\pi^2}{48}\Bigg]^{-\frac{1}{2}}. \tag{185}$$

We collect these results into expansion coefficients for $m = 2$ as:

$$\begin{aligned} I_0(m) &= \frac{1}{4}+\frac{1}{4}\gamma^2+\frac{1}{2}\gamma_1-\frac{\pi^2}{48} = \\ &= 0.09126979985406300159\ldots, \\ I_1(m) &= 1+\frac{1}{2}\gamma^2+\gamma_1-\frac{1}{24}\pi^2-\frac{1}{2}\log^2(2\pi) = \\ &= -1.00635645590858485121\ldots, \\ I_2(m) &= 1, \end{aligned} \tag{186}$$

and then re-write (174) more conveniently as

$$\zeta^{-1}(w) \approx \pm(w+\frac{1}{2})\Bigg[I_2(2)w^2+I_1(2)w+I_0(2)\Bigg]^{-\frac{1}{2}}. \tag{187}$$

Here we have added a $\pm$ sign which is sensitive to the branch (usually due to the $(w+\frac{1}{2})$ term that must be positive for $w < -0.5$). This second order approximation above is very accurate for a variety of input argument (even complex). For example, for $w = 2$ we compute

$$\zeta^{-1}(2) \approx 1.7340397592898484279\ldots \tag{188}$$

and to verify $\zeta(\zeta^{-1}(2)) \approx 1.9902700570\ldots$ is accurate to 2 significant digits. In Fig. 4 we plotted the function $w = \zeta(\zeta^{-1}(w))$ for the 2^{nd} order approximation and see

Tab. 4. The computation of inverse zeta $s = s_1 = \zeta^{-1}(w)$ for $m = 20$ by Eq. (174) for different values of w. For $w \in (-\infty, -0.5) \cup (1, \infty)$ we consider positive solutions, otherwise for $w \in (-0.5, j_1)$ we consider negative solutions

s	$w = \zeta(s)$	$s = \zeta^{-1}(w)$ (First 15 Digits)	Significant Digits
-5	-0.003968253968253	-1.884741377602060	8
-4	0	-1.999999904603844	7
-3	0.008333333333333	-2.470168918790366	5
-2	0	-1.999999904603844	7
-1.5	-0.025485201889833	-1.499999999998134	11
-1	-0.083333333333333	-1.000000000000000	16
-0.5	-0.207886224977354	-0.499999999999999	23
-0.125	-0.399069668945045	-0.125000000000000	36
-0.001	-0.499082063645236	0.000999999999999	42
0.001	-0.500919942713218	0.000999999999999	42
0.125	-0.632775623498695	0.125000000000000	36
0.5	-1.460354508809586	0.500000000000000	26
0.75	-3.441285386945222	0.749999999999999	22
0.9999	-9999.422791616731466	0.999900000000000	27
1.0001	10000.577222946437629	1.000099999999999	26
1.5	2.612375348685488	1.500000000000000	18
2	1.644934066848226	1.999999999999997	14
2.5	1.341487257250917	2.500000000000706	12
3	1.202056903159594	3.000000000032817	10
4	1.082323233711138	4.000000008467328	8
5	1.036927755143369	5.000001846688341	5

how w is recovered, except in a small region $(j_1, 1)$ where we get an erroneous result. Similarly, for complex argument for $w = 2 + \mathrm{i}$ we compute

$$\zeta^{-1}(2+\mathrm{i}) \approx 1.4690117151\dots - \mathrm{i}0.3470428878\dots, \tag{189}$$

and to verify $\zeta(\zeta^{-1}(2+\mathrm{i})) \approx 1.9886804524\dots + {}+\mathrm{i}0.9958475706\dots$ we recover w correctly also to within 2 significant digits. We will investigate the complex argument in more detail a little later. Furthermore, the 2$^{\text{nd}}$ degree polynomial in (187) can be factored into its zeros as

$$\zeta^{-1}(w) \approx \pm(w + \frac{1}{2})\left[(w - j_1)(w - j_2)\right]^{-\frac{1}{2}}, \tag{190}$$

where $j_1 = 0.1007872126\dots$ is the first zero, and $j_2 = {} = 0.9055692433\dots$ is the second zero (computed by solving a quadratic equation). We note that these are the zeros of a polynomial in (187), and hence they are the singularities of 2$^{\text{nd}}$ order approximation of $\zeta^{-1}(w)$. To investigate the higher order expansion for $\zeta^{-1}(w)$ in terms of these polynomials $I_n(m)$, can be written with coefficients in terms of Stieltjes constants and incomplete Bell polynomials $\mathbf{B}_{n,k}(x_1, x_2, x_3 \dots, x_n)$ due to the Faàdi-Bruno expansion formula for the n^{th} derivative

$$\frac{d^n}{dx^n} f(g(x)) = = \sum_{k=1}^{n} f^{(k)}(g(x))\mathbf{B}_{n,k}(g'(x), g''(x), \dots, g^{n-k+1}(x)), \tag{191}$$

and if we take

$$f(x) = \log(x), \tag{192}$$

and

$$f^{(n)}(x) = (-1)^{n+1}(n-1)!\frac{1}{x^n}\,. \tag{193}$$

Such Bell polynomial expansion will lead to long and complicated expressions for the m^{th} log-derivative so we will not pursue them in this paper. For the moment we will just rely on numerical computations so, based on (187), we deduce the following asymptotic expansion

Tab. 5. The computation of inverse zeta $s = s_1 = \zeta^{-1}(w)$ for $m = 100$ by Eq. (174) for different values of w. For $w \in (-\infty, -0.5) \cup (1, \infty)$ we consider positive solutions, otherwise for $w \in (-0.5, j_1)$ we consider negative solutions. The red color indicates the singularity region where convergence is erroneous

w	$s = \zeta^{-1}(w)$	$w = \zeta(\zeta^{-1}(w))$
−10	0.90539516131918826348	−10.00000000000000000000
−5	0.82027235216804898973	−5.00000000000000000000
−4	0.78075088259313749868	−4.00000000000000000000
−3	0.71881409407526189655	−3.00000000000000000000
−2	0.60752203756637705289	−2.00000000000000000000
−1	0.34537265729115398953	−1.00000000000000000000
−0.5001	0.00010880828067160644	−0.50009999999999999999
−0.4999	−0.00010883413591990730	−0.49989999999999999999
−0.1	−0.90622982899228246768	−0.10000000000000000000
0	−1.99999999999999999999	0
0.001	−2.03407870819025354208	0.00099999999999999999
0.0015	−2.05213532171740716650	0.00149999999999999999
0.0091598	−2.69835815770380622679	0.00915551952718300130
0.01	2.69182425874263410494	1.27522086147958091320
0.02	2.68341537834567817177	1.27769681556903809338
0.1	2.62327826166715651687	1.29626791092230654966
0.5	3.28523402279617101762	1.15403181697782434872
0.8	4.35892653933022255726	1.06086646037035161615
0.999	9.19090684760189275051	1.00175563731403047546
1.001	9.19454270908484711549	1.00175114882142955996
1.01	6.75096988949758004724	1.01000000000000001556
1.1	3.77062121683766280843	1.10000000000000000000
2	1.72864723899818361813	2.00000000000000000000
3	1.41784593578735729296	3.00000000000000000000
4	1.29396150555724361741	4.00000000000000000000
5	1.22693680841631476071	5.00000000000000000000
10	1.10621229947483799036	10.00000000000000000000

$$\left[\frac{\zeta^{-1}(w)}{(w+\frac{1}{2})}\right]^{-m} \sim \sum_{n=0}^{m} I_n(m) w^n, \tag{194}$$

into a m^{th} degree polynomial as $m \to \infty$, where $I_n(m)$ are the expansion coefficients. These coefficients are a function of a limit variable m whose values vary depending on m. In Tab. 6 we compute these coefficients (for several m) for further study and observe the following. For $w = 0$, $\zeta^{-1}(w) = -2$ is the first trivial zero, hence we deduce that

$$I_0(m) \sim (2\rho_{t,1})^{-m} \sim (-1)^m \frac{1}{2^{2m}}\,. \tag{195}$$

From Tab. 6 we also observe

$$I_m(m) \sim 1, \tag{196}$$

and

$$I_{m-1}(m) \sim -\frac{m}{2}\,. \tag{197}$$

As $m \to \infty$, this expansion generates an infinite degree polynomial, which will also have infinite zeros j_n that we will next glimpse numerically. We re-write (194) as factorization

$$\left[\frac{\zeta^{-1}(w)}{(w+\frac{1}{2})}\right]^{-m} \sim \prod_{n=1}^{m} (w - j_n)\,, \tag{198}$$

Tab. 6. The computation of expansion coefficients $I_n(m)$ of Eq. (194) for even m

$I_n(m)$	$m=2$	$m=4$	$m=6$	$m=8$
$n=0$	0.0912697998	0.0042324268	0.0002483703	0.00001532100
$n=1$	−1.0063564559	−0.1967919743	−0.0204091776	−0.00174497183
$n=2$	1	1.1920976317	0.3162826334	0.04840981341
$n=3$		−1.9995171980	−1.5828262271	−0.48669059013
$n=4$		1	3.2866782629	2.21705837605
$n=5$			−3.0000078068	−5.15932314768
$n=6$			1	6.38227467998
$n=7$				−4.00000004611
$n=8$				1

in terms of these zeros and compute them for $m=4$ in Tab. 7 and for $m=10$ in Tab. 8, using a standard polynomial root finder for a generated polynomial in (194). In Appendix A we also give values of j_n for $m=30$ in Tab. A1 as a reference. It is also much better to see j_n's graphically in Fig. 5 (where we plot them in a complex plane for $m=4$, $m=10$, $m=30$ and $m=50$), and observe that they cluster near the endpoints. The exact values of these zeros are numerically spread out and, as more zeros are generated as a function of m as m increases, their accuracy also increases. Interestingly, they are mostly real and cluster roughly in an interval $(0,1)$ but we will narrow it down next, and some zeros are also complex that cluster near $w=1$.

Tab. 7. The computation of j_n singularities for $m=4$

n	$\Re(j_n)$	$\Im(j_n)$
1	0.02519077171287255364	0
2	0.22387780988390681825	0
3	0.75055928996119915729	0
4	0.99988932644430613063	0

Tab. 8. The computation of j_n singularities for $m=10$

n	$\Re(j_n)$	$\Im(j_n)$
1	0.01141939762352641311	0
2	0.03270893154877055459	0
3	0.08725746253768978834	0
4	0.18974173730082442926	0
5	0.35313390831120714095	0
6	0.57365189826222332925	0
7	0.80181268425373759307	0
8	0.95232274935073811513	0
9	0.99897561465713752103	−0.00219195619260189999
10	0.9989756146571375210	0.00219195619260189999 4

With more detailed numerical computation we observe that as $m\to\infty$ they will span the interval (j_1, j_m) where the lower bound is $(m\to\infty)$

$$j_1(m) \to 0.00915989011990346184005603872 8\ldots, \tag{199}$$

is the lowest zero or the principal zero. The value presented was computed numerically to high precision. The upper bound is $(m\to\infty)$

$$j_m(m) \to O(1), \tag{200}$$

due to the pole of $\zeta(1)$.

From Fig. 6 we see that j_1 corresponds to the first local maxima (between $s=-4$ and $s=-2$) in a region where the zeta takes a first turn from being monotonically increasing when going from left to right in s-domain ($s=1$ to $s=-2.7172628292\ldots$) and in w-domain as ($w=-\infty$ to $w=0.0091598901\ldots$), at which point causes a discontinuity for this branch. We observe that j_n are zeros of the expansion (194) and this implies at first that they are poles of $\zeta^{-1}(w)$, but because they are under an m^{th} root which induces m branches and forms an algebraic branch point [7, p. 143]. Hence, the strip $(j_1,1)$ fills the remaining w-domain gap from j_1 to $\zeta(1)$ with these singularities. We conjecture that

Conjecture 1. The principal branch $s_1=\zeta^{-1}(w)$ has an infinite number of real singularities in a strip $(j_1,1)$.

The inverse zeta can be represented by factorization by these singularities. In Fig. 3 we highlighted this singularity strip region in relation to $s_1=\zeta^{-1}(w)$. We will refer to the constants j_n interchangeably as either zeros of (198) or singularities of $s_1=\zeta^{-1}(w)$.

Now, since j_1 is the principal zero of the expansion (194), we can find its formula by solving the infinite degree polynomial equation using Theorem 1 and Theorem 2, and find that

$$j_1 = \lim_{m\to\infty}\left[\frac{m}{(m-1)!}\frac{d^m}{dw^m}\log\left[\frac{\zeta^{-1}(w)}{(w+\frac{1}{2})}\right]\Big|_{w\to 0}\right]^{-\frac{1}{m}}, \tag{201}$$

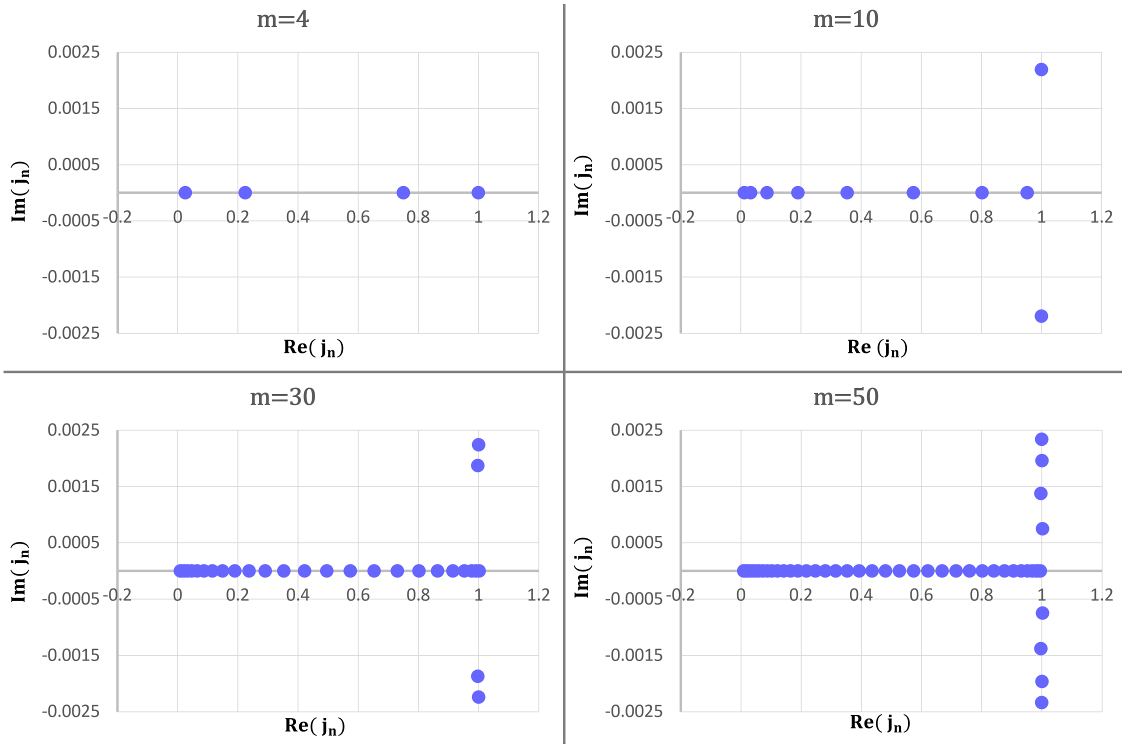

Fig. 5. A plot of locations of singularities j_n in a complex plane generated for various values of limit variable m

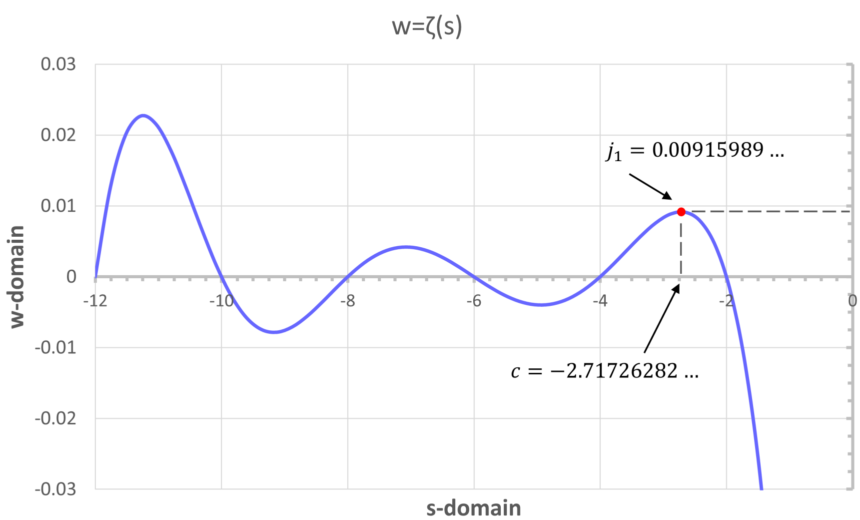

Fig. 6. A plot of $w = \zeta(s)$ for $s \in (-12, 0)$ locating local maxima j_1

and in Tab. 9 compute (201) for a few values of m and observe a slow convergence to j_1. One could also represent the next higher singularities by the recurrence relation

$$j_{n+1} = \lim_{m\to\infty} \left[\frac{2m}{(2m-1)!} \frac{d^{(2m)}}{dw^{(2m)}} \log\left[\frac{\zeta^{-1}(w)}{(w+\frac{1}{2})} \right] \Big|_{w\to 0}^{+} - \sum_{k=1}^{n} \frac{1}{j_k^{2m}} \right]^{-\frac{1}{2m}}, \tag{202}$$

where we consider a $2m$ limit value to avoid an alternating sign in the recurrence, but numerically it is very hard to compute since these singularities are so densely spaced in an interval $(j_1, 1)$. Also, the value for a constant c for which $j_1 = \zeta(c)$ is $c = -2.717262829\ldots$, which is close to e to within 3 decimal places, and from this we obtain a simple approximation to j_1 as

$$j_1 \approx \zeta(-e) = 0.00915\underline{9}877559420231\ldots, \tag{203}$$

which is accurate to within 7 decimal places.

Tab. 9. The computation of j_1 by Eq. (201) for various m from low to high

m	j_1	Significant Digits
10	0.01141936690297939790	1
50	0.00924371071593150307	3
100	0.00916896287172313725	4

These relations allow us to write the inverse Riemann zeta function as factorization into zeros and singularities as

$$s_1 = \zeta^{-1}(w) = \lim_{m\to\infty} \pm 4(w+\frac{1}{2})\prod_{n=1}^{m}\left(1-\frac{w}{j_n}\right)^{-\frac{1}{m}}. \tag{204}$$

These generated singularities are so finely balanced that even for $m = 10$ they can reproduce the inverse zeta function to a great degree of accuracy, as we will see shortly. Also, the factor of 4 in (204) comes from the convergence of

$$\prod_{n=1}^{m}(j_n)^{-\frac{1}{m}} = (j_1 j_2 j_3 \ldots j_m)^{-\frac{1}{m}} \to 4 \quad (m\to\infty), \tag{205}$$

that is related to the first trivial zero by Eq. (195). We remarked earlier that some of the singularities are complex and cluster near 1, as shown in Fig. 5 for higher m. Initially, we were unsure as to whether these complex zeros are real or artifacts of the root finder but we find that they play a central role (in conjunction with the real roots) in computing the product formula (204) and many identities that follow. For example, we have

$$\sum_{n=1}^{m} j_n = j_1 + j_2 + j_3 + \ldots \sim \frac{m}{2} \quad (m\to\infty), \tag{206}$$

obtained based on expanding the coefficients in (204). From this we have the mean value of j_n:

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m} j_n = \frac{1}{m}(j_1+j_2+j_3+\ldots) = \frac{1}{2}, \tag{207}$$

and also from (205) another identity

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\log(j_n) = \\ = \frac{1}{m}(\log(j_1)+\log(j_2)+\log(j_3)+\ldots) = -2\log(2). \tag{208}$$

We observe that as m increases, the number of complex singularities that are generated increases, but their absolute values tends 1. This tendency is also captured by Conjecture 1 above. If true, then it may also be possible that as $m\to\infty$ the complex singularities will disappear.

We next investigate how the inverse zeta function converges for complex argument. As another example, we compute the inverse zeta of

$$s = \zeta^{-1}(2+\mathrm{i}) = \\ = 1.466595797094670\ldots - \mathrm{i}0.343719739467598\ldots, \tag{209}$$

for $m = 10$, and then, when taking the zeta of the inverse zeta

$$w = \zeta(\zeta^{-1}(2+\mathrm{i})) = \\ = 2.000000007384116\ldots + \mathrm{i}0.999999997993535\ldots, \tag{210}$$

we recover the w-domain correctly (we see it is better approximation than the 2nd order Eq. (185)). As another example we take the inverse zeta for large input argument

$$s = \zeta^{-1}(123456789 - \mathrm{i}987654321) = \\ = 1.000000000124615\ldots + \mathrm{i}0.000000000996923\ldots, \tag{211}$$

and then, when taking the zeta of the inverse zeta above, we compute

$$w = \zeta(s) = 123456789.01848\ldots - \mathrm{i}987654321.14785\ldots, \tag{212}$$

where we see correct convergence to within 1 decimal place, but if we re-compute for $m = 20$, then we get

$$w = \zeta(s) = 123456789.00000\ldots - \mathrm{i}987654321.00000\ldots, \tag{213}$$

which is now accurate to 15 digits after the decimal place. In general, we find that for large complex input argument the convergence is very good but that is sometimes not the case for smaller input argument, where in many cases we do not get correct convergence at first. For example, if we evaluate

$$s = \zeta^{-1}(1.5+\mathrm{i}) = \\ = 1.521134764270121\ldots + \mathrm{i}0.417327503093697\ldots, \tag{214}$$

for $m = 10$, and then inverting back

$$w = \zeta(\zeta^{-1}(1.5+\mathrm{i})) = \\ = 1.783854226864277\ldots - \mathrm{i}0.908052465458989\ldots, \tag{215}$$

we get erroneous results. The reason is because the m^{th} root involved in the computation of the inverse zeta actually has m branches. In general, the m^{th} root of a complex number z can be written as

$$z^{\frac{1}{m}} = |z|^{\frac{1}{m}} e^{\mathrm{i}\frac{1}{m}(\arg(z)+2\pi n)}, \tag{216}$$

for $n = 0\ldots m-1$. So far, we have been using the principal root for $n = 0$, which is the standard m^{th} root, but for complex numbers we have to select n for which the solution that we want lies. We do not have an exact criterion for which n solution to use so we have to individually check every solution and find the one that we need. For example, in re-computing (214) we find that the m^{th} root for $n = 9$ gives

$$s = \zeta^{-1}(1.5 + \mathrm{i}) =$$
$$= 1.475922826723574\ldots - \mathrm{i}0.556475538964500\ldots, \tag{217}$$

for $m = 10$, and then

$$w = \zeta(\zeta^{-1}(1.5 + \mathrm{i})) =$$
$$= 1.500000011509227\ldots + \mathrm{i}0.999999987375822\ldots, \tag{218}$$

finally reproduces the correct result. These results lead us to introducing an error function

$$E(w) = |w - \zeta(\zeta^{-1}(w))|, \tag{219}$$

used to quantify how well the inverse zeta is inverting. Essentially, taking $\zeta(\zeta^{-1}(w))$ should reproduce w, so when subtracting w off we should expect

$$E(w) = 0, \tag{220}$$

and when computing it numerically, $E(w)$ will be very small because the convergence of $\zeta^{-1}(w)$ is generally very good. But when $\zeta^{-1}(w)$ is not converging correctly, usually due to the m^{th} root lying on another branch, then $E(w)$ will be very high in relation to a case when $\zeta^{-1}(w)$ is normally converging. This contrast between high convergence rate and no convergence at all allows us to write a simple search algorithm to sweep the branch of the m^{th} root and minimize $E(w)$. There we introduced a reasonable threshold value of $t_x = 10^{-3}$ to minimize $E(w)$ (which may be re-adjusted) and, once the minima has been found, the code exits out of the loop and returns the correct branch. From further numerical study we found that there is only one branch of the m^{th} root giving the correct answer and all other branches give erroneous results, thus making use of this loop very easy. In our code we define a custom m^{th} root function in Algorithm 6, and in Algorithm 7 we modify the inverse zeta function with the new m^{th} root search loop. The second modification to the script we made is that now we load a pre-computed table of j_n's from a text file and evaluate the product formula (204) instead of computing the m^{th} derivative using the **derivnum** function (which is slow for high m).

Algorithm 6 A custom function in PARI to compute an m^{th} root for an n^{th} branch

```
// define mth root function
// s is input argument, m is mth root, n is nth branch
xroot(s,m,n)=
{
    r = abs(s);
    y = rˆ(1/m)∗exp(I∗arg(s)/m+I∗n∗2∗Pi/m);
    return(y);
}
```

Algorithm 7 A new function in PARI for $\zeta^{-1}(w)$ using the m^{th} root search and singularity expansion representation (204)

```
// inverse zeta function valid for complex w argument
izeta(w)=
{
    // set mth root branch threshold
    tx = 1e−3;

    // load singularities from txt file into a vector
    jx = readvec("jx_singularities_m50.txt");

    // compute the length of vector
    m = length(jx);

    // compute product due to singularities
    A = prod(i = 1,m,(w−jx[i]))ˆ(−1);

    // mth root search
    for(i = 0,m−1,

        // compute s-domain
        s = (w+1/2)∗xroot(A,m,i);

        // compute error function
        E = abs(zeta(s)−w);

        // exit out of loop when threshold is met
        if(E<tx, break);
    );
    return(s);
}
```

In Appendix A we provide a Tab. A1 with pre-computed j_n for $m = 30$ for reference. Hence, together with the m^{th} root function, the presented algorithm allows for a very fast evaluation of $s_1 = \zeta^{-1}(w)$ for any complex argument w (in just under several milli-seconds) on a standard workstation. The only requirement is to pre-compute a table of j_n singularities and store them in a file. In contrast, the **derivnum** function takes 60 ms to evaluate one inversion for $m = 10$ on our workstation and over 5–20 minutes for $m = 400$.

When running the new script in Algorithm 7, we can reproduce all the results in this paper, including for the negative branch for the range $(-0.5, j_1)$ we saw earlier, which actually corresponds to an m^{th} root branch at $n = \frac{m}{2}$ that is automatically found by the code. One more example, we invert

$$s = \zeta^{-1}(0.5 + \mathrm{i}) =$$
$$= 0.933314322626762\ldots - \mathrm{i}0.930958378790106\ldots, \tag{221}$$

for $m = 10$ which lies just above the singularities $(j_1, 1)$ using the new script in Algorithm 7, then we get

$$w = \zeta(\zeta^{-1}(0.5+\mathrm{i})) =$$
$$= 0.500000004914683\ldots + \mathrm{i}1.000000012981412\ldots, \quad (222)$$

which inverts s back correctly, which corresponds to the m^{th} root branch of $n = 8$ that is automatically found by the code. To check more complex points we generated a density plot of the error function $E(w)$ from Eq. (219) in Fig. 7 by computing it for a grid of complex points 101×101 which contains 10 201 total points spanning a range $\Re(w) \in (-2, 2)$ and $\Im(w) \in (-2, 2)$ equally spaced for $m = 10$,

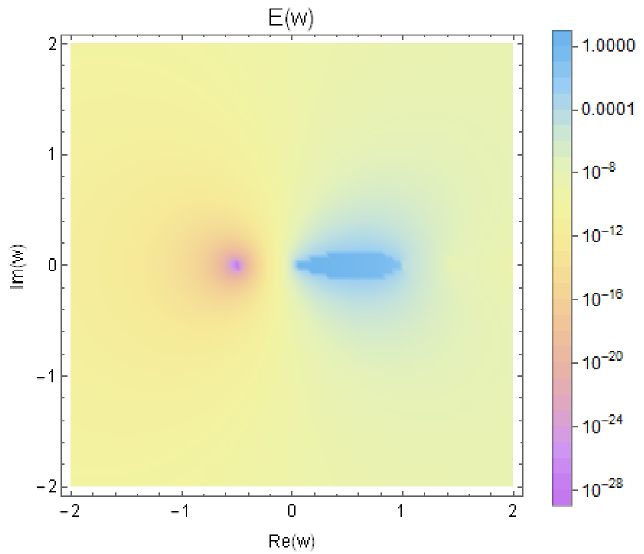

Fig. 7. A density plot of $E(w)$ by Eq. (219) for $m = 10$ in the range of $\Re(w) \in (-2, 2)$ and $\Im(w) \in (-2, 2)$

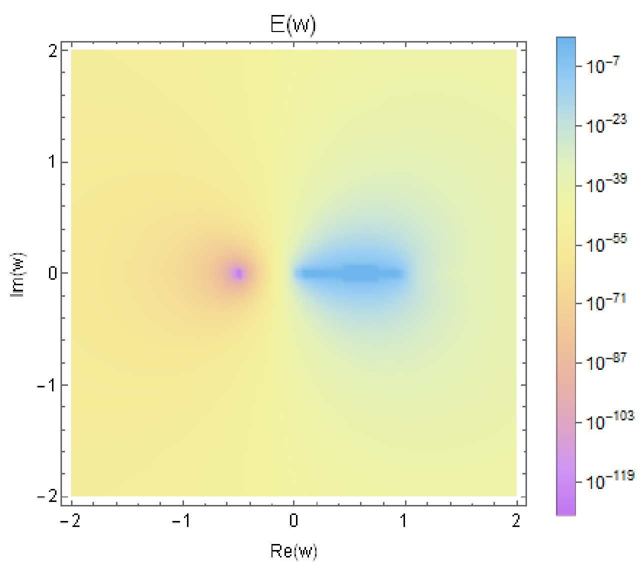

Fig. 8. A density plot of $E(w)$ by Eq. (219) for $m = 50$ in the range of $\Re(w) \in (-2, 2)$ and $\Im(w) \in (-2, 2)$

and using the new code in Algorithm 7 which took a 1–2 minutes to compute all points. We see that generally $E(w) \sim 10^{-8}$ throughout, and when it is close to the zero at $w = -\frac{1}{2}$, then $E(w) \sim 10^{-28}$ which is surprisingly very good. When it is near the singularities in the range $(j_1, 1)$, then $E(w)$ gets worse (as expected) and completely fails at the singularities (blue color). The function still runs in the singularity region because numerically it is very unlikely to hit an exact location of the singularity, causing a $\frac{1}{0}$ division. In Fig. 8 we re-plot again but for $m = 50$, and now see much better convergence over the previous case for $m = 10$, where now we get $E(w) \sim 10^{-55}$ throughout, and $E(w) \sim 10^{-128}$ close to zero, and when it is near the singularity region $E(w) \sim 10^{-7}$.

VI. THE ζ^{-1}(W) NEAR ITS ZERO

The first few terms of Taylor expansion coefficients of $\zeta(s)$ about $s = 0$ are

$$\zeta(s) = -\frac{1}{2} + \zeta'(0)s + \frac{1}{2}\zeta''(0)s^2 + \ldots, \quad (223)$$

(the series has a rather small radius of convergence), and $\zeta'(0) = -\log(\sqrt{2\pi})$. If we write

$$\zeta(\frac{1}{s}) \sim -\frac{1}{2} - O(\frac{1}{s}\log\sqrt{2\pi}) \quad (s \to \infty), \quad (224)$$

and then taking the inverse zeta of both sides, we deduce that

$$\frac{1}{s} \sim \zeta^{-1}\left(-\frac{1}{2} - \frac{1}{s}\log\sqrt{2\pi}\right) \quad (s \to \infty), \quad (225)$$

and now, recalling the inverse zeta factorization formula (204) as

$$\zeta^{-1}(w) = \lim_{m\to\infty} -4(w+\frac{1}{2})\prod_{n=1}^{m}\left(1 - \frac{w}{j_n}\right)^{-\frac{1}{m}}, \quad (226)$$

(and taking the negative branch), then substituting (204) into (226) above the $-\frac{1}{2}$ factor will cancel, and we get

$$\lim_{m\to\infty} 4(\frac{1}{s}\log\sqrt{2\pi})\prod_{n=1}^{m}\left(1 - \frac{-\frac{1}{2} - \frac{1}{s}\log\sqrt{2\pi}}{j_n}\right)^{-\frac{1}{m}} = \frac{1}{s}. \quad (227)$$

The s variable also cancels on both sides, and we get

$$\lim_{m\to\infty} 4(\log\sqrt{2\pi})\prod_{n=1}^{m}\left(1 + \frac{1}{2j_n}\right)^{-\frac{1}{m}} = 1. \quad (228)$$

Since the main asymptote $\frac{1}{s}$ has been canceled, we find that the remaining $\frac{1}{s}$ inside the infinite product term becomes negligible and we obtain a product formula

$$\lim_{m\to\infty}\prod_{n=1}^{m}\left(1 + \frac{1}{2j_n}\right)^{\frac{1}{m}} = 4\log\sqrt{2\pi}. \quad (229)$$

A numerical computation for $m = 50$ yields 3.6757541328 1869090182... which is accurate to 16 decimal places.

VII. THE ASYMPTOTIC RELATIONS OF $\zeta^{-1}(W)$

We first investigate a limit formula for the Euler-Mascheroni constant. From the Laurent expansion of $\zeta(s)$ in (124) we can deduce a limit identity

$$\gamma = \lim_{s\to 1+}\left[\zeta(s) - \frac{1}{s-1}\right], \tag{230}$$

and further by transforming the limit variable $s \to 1 + \frac{1}{s}$ we obtain

$$\gamma = \lim_{s\to\infty}\left[\zeta\left(1+\frac{1}{s}\right) - s\right]. \tag{231}$$

We empirically find a similar relation for the inverse Riemann zeta function by numerically evaluating for $s = 10\,000$ as

$$\zeta^{-1}(10\,000) = 1.0001000057725626 74143\ldots, \tag{232}$$

where we observe a sign of a tailing γ in the digits, which is on the order of $O(s^{-2})$. So we deduce that

$$\zeta^{-1}(s) \sim 1 + \frac{1}{s} + O\left(\gamma\frac{1}{s^2}\right) \quad (s\to\infty), \tag{233}$$

from which we have

$$\gamma = \lim_{s\to\infty}\left[\zeta^{-1}(s) - \left(1+\frac{1}{s}\right)\right]s^2\,. \tag{234}$$

And similarly, we find that

$$\gamma = \lim_{s\to\infty}\left[\zeta^{-1}(-s) - \left(1-\frac{1}{s}\right)\right]s^2\,, \tag{235}$$

from which we conclude that

$$\zeta^{-1}(s) \sim \zeta^{-1}(-s) \to O(1), \tag{236}$$

as it is seen in a graph in Fig. 3. In Tab. 10 we summarize numerical computation of (234) using the inverse zeta formula for $m = 100$ and observe convergence to γ.

We can also obtain a different representation by expanding (234) as

$$\gamma = \lim_{s\to\infty}\left[s^2\zeta^{-1}(s) - (s^2+s)\right], \tag{237}$$

from which we recognize the sum of natural numbers

$$\sum_{n=1}^{k} n = 1+2+3+\ldots k = \frac{k^2}{2} + \frac{k}{2}\,, \tag{238}$$

Tab. 10. The computation of γ by inverse zeta for various s from low to high by Eq. (234) and $\zeta^{-1}(s)$ for $m = 100$

s	γ	Significant Digits
10^1	0.62122994748379903608	0
10^2	0.58130721658646456077	1
10^3	0.57762197248836203702	3
10^4	0.57725626741433442042	4
10^5	0.57721972487058219773	5
10^6	0.57721607089561571393	5
10^7	0.57721570550091292536	6
10^8	0.57721566896147058487	8
10^9	0.57721566530752663021	8
10^{10}	0.57721566494213223753	10
10^{11}	0.57721566490559279829	11
10^{12}	0.57721566490193885437	12

where we obtain

$$\sum_{n=1}^{k} n = 1+2+3+\ldots \sim -\frac{1}{2}\gamma + \frac{1}{2}k^2\zeta^{-1}(k) \quad (k\to\infty), \tag{239}$$

and this is in contrast to the Euler's relation for harmonic sum

$$\sum_{n=1}^{k}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\ldots \sim \gamma + \log(k) \quad (k\to\infty), \tag{240}$$

where the term $\frac{1}{2}k^2\zeta^{-1}(k)$ is dual to $\log(k)$ in the sense of a reflection about the origin $\zeta(s) \leftrightarrow \zeta(-s)$ for series (1), when $s \to 1$.

On a side note, it is often loosely written that

$$\zeta(-1) = \sum_{n=1}^{\infty} n = 1+2+3+\ldots = -\frac{1}{12}\,, \tag{241}$$

in the context of the Riemann zeta function and zeta regularization, where the asymptotic term is omitted. We briefly investigate the asymptotic term of (241) by the Euler-Maclaurin formula, which breaks up the series (1) into a partial sum up to the $k-1$ order, and the remainder starting at k and going to infinity

$$\zeta(s) = \sum_{n=1}^{k-1}\frac{1}{n^s} + \sum_{n=k}^{\infty}\frac{1}{n^s}\,, \tag{242}$$

as shown in [17, p. 114], when the Euler-Maclaurin summation formula is applied to the remainder term we get

$$\zeta(s) = \sum_{n=1}^{k-1}\frac{1}{n^s} - \frac{k^{1-s}}{1-s} + \frac{1}{2}k^{-s} + \frac{B_2}{2}sk^{-s-1} + O(k^{-s-3}), \tag{243}$$

and then substituting $s = 2$ we get

$$\zeta(2) = \sum_{n=1}^{k-1} \frac{1}{n^2} + \frac{1}{k} + \frac{1}{2k^2} + B_2 \frac{1}{k^3} + O\left(\frac{1}{k^5}\right), \quad (244)$$

now, when solving for B_2 we get

$$B_2 = k^3 \left(\zeta(2) - \sum_{n=1}^{k-1} \frac{1}{n^2}\right) - k^2 - \frac{k}{2} - O\left(\frac{1}{k^2}\right), \quad (245)$$

and see that is slowly resembling (241), and we multiply by $-\frac{1}{2}$ yields

$$-\frac{1}{2}B_2 = \frac{1}{2}k^2 + \frac{1}{4}k - \frac{1}{2}k^3 \left(\zeta(2) - \sum_{n=1}^{k-1} \frac{1}{n^2}\right) + O\left(\frac{1}{2k^2}\right). \quad (246)$$

From this we have the full asymptotic relation

$$\sum_{n=1}^{k} n = 1 + 2 + 3 + \ldots - \left[\frac{1}{4}k + \frac{1}{2}k^3 \times \times \left(\zeta(2) - \sum_{n=1}^{k-1} \frac{1}{n^2}\right)\right] = -\frac{1}{12}, \quad (247)$$

as $k \to \infty$ which is the correct version of (241) in the context of the Riemann zeta function (involving its analytical continuation) by including the asymptotic term. Collecting these results, we have two asymptotic representations for the sum of natural numbers in the context of the Riemann zeta function as

$$\sum_{n=1}^{k} n = 1 + 2 + 3 + \ldots - \frac{1}{2}k^2 \zeta^{-1}(k) = -\frac{1}{2}\gamma \quad (k \to \infty), \quad (248)$$

and

$$\sum_{n=1}^{k} n = 1 + 2 + 3 + \ldots - \left[\frac{1}{4}k + \frac{1}{2}k^3 \times \times \left(\frac{\pi^2}{6} - \sum_{n=1}^{k-1} \frac{1}{n^2}\right)\right] = -\frac{1}{12} \quad (k \to \infty). \quad (249)$$

If we drop the asymptotic terms, and summing to infinity, then we casually write

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots = -\frac{1}{2}\gamma, \quad (250)$$

and

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots = -\frac{1}{12}, \quad (251)$$

which are only loosely taken at face value and which is implied in the context Riemann zeta function. The complete asymptotic representations are (248) and (249). Also, comparing the values

$$-\frac{1}{2}\gamma = -0.2886078324\ldots, \qquad -\frac{1}{12} = -0.0833333333\ldots \quad (252)$$

It is often written in the literature that $-\frac{1}{12}$ is the assigned value to the sum of natural numbers and the asymptotic part is discarded. In actuality, one could arbitrarily assign any value to any divergent series by subtracting two such series with the same growth rate and where the difference results in a finite constant and the divergent parts cancel.

VIII. ON THE DERIVATIVES OF $\zeta^{-1}(W)$

We now consider the derivatives of $s_1 = \zeta^{-1}(w)$ as such. By differentiating the inverse function relation

$$\zeta(\zeta^{-1}(w)) = w, \quad (253)$$

we get

$$\zeta'[\zeta^{-1}(w)][\zeta^{-1}(w)]' = 1, \quad (254)$$

by the composition rule. This leads to a simple formula

$$[\zeta^{-1}(w)]' = \frac{1}{\zeta'[\zeta^{-1}(w)]}, \quad (255)$$

provided that $\zeta'(s) \neq 0$. We saw earlier that the constant $s = c = -2.71726282\ldots$ is a zero of $\zeta'(c) = 0$, and for which $j_1 = \zeta(c) = 0.00915989\ldots$ Then, evaluating (255) for $w = 0$ we get

$$[\zeta^{-1}(0)]' = \frac{1}{\zeta'[\zeta^{-1}(0)]} = \frac{1}{\zeta'[-2]}, \quad (256)$$

(taking the principal zero), and using the well-known identity

$$\zeta'(-2n) = \frac{(-1)^n \zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}}, \quad (257)$$

we have

$$[\zeta^{-1}(0)]' = -\frac{4\pi^2}{\zeta(3)}. \quad (258)$$

Recalling the inverse zeta factorization formula again

$$\zeta^{-1}(w) = \lim_{m\to\infty} -4(w + \frac{1}{2}) \prod_{n=1}^{m} \left(1 - \frac{w}{j_n}\right)^{-\frac{1}{m}}, \quad (259)$$

(the negative branch), and taking the first log-derivative gives

$$\frac{[\zeta^{-1}(w)]'}{\zeta^{-1}(w)} = \frac{1}{(w+\frac{1}{2})} + \lim_{m\to\infty} \frac{1}{m}\sum_{n=1}^{m} \frac{1}{j_n}\left(1-\frac{w}{j_n}\right), \tag{260}$$

from which we have an alternate formula in terms of singularities j_n as

$$[\zeta^{-1}(w)]' = \zeta^{-1}(w)\left[\frac{1}{(w+\frac{1}{2})} + \lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\frac{1}{j_n}\left(1-\frac{w}{j_n}\right)\right]. \tag{261}$$

If we let $w = 0$ then we have

$$\frac{[\zeta^{-1}(0)]'}{\zeta^{-1}(0)} = 2 + \lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\frac{1}{j_n}\,. \tag{262}$$

Now relating with (258) we obtain a formula for either the average value of

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\frac{1}{j_n} = 2\left(\frac{\pi^2}{\zeta(3)} - 1\right), \tag{263}$$

or a formula for Apéry's constant

$$\zeta(3) = \pi^2\left(1 + \frac{1}{2}\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\frac{1}{j_n}\right)^{-1}. \tag{264}$$

These relations motivate to obtain the generalized zeta series for $\zeta^{-1}(w)$ by Theorem 1 using the m^{th} logarithmic differentiation to obtain

$$\begin{aligned} Z_j(m) &= \frac{1}{(m-1)!}\frac{d^m}{dw^m}\log\left[\frac{\zeta^{-1}(w)}{w+\frac{1}{2}}\right]\Big|_{w\to 0} = \\ &= \lim_{k\to\infty}\frac{1}{k}\sum_{n=1}^{k}\frac{1}{j_n^m}, \end{aligned} \tag{265}$$

where we specifically canceled the only zero with $w+\frac{1}{2}$. This leads to a generalized zeta series over just the singularities of $s_1 = \zeta^{-1}(w)$, and the first few special values are:

$$\begin{aligned} Z_j(1) &= 2\left(-1+\frac{\pi^2}{\zeta(3)}\right) = \\ &= 14.42119333144247050884\ldots, \\ Z_j(2) &= 4\left(1-\frac{\pi^4}{\zeta(3)^2} - 8\frac{\pi^6}{\zeta(3)^3}\zeta''(-2)\right) = \\ &= 899.16532329931876633541\ldots, \\ Z_j(3) &= 8\left(-1+\frac{\pi^6}{\zeta(3)^3} + (12\zeta''(-2)+8\zeta'''(-2))\frac{\pi^8}{\zeta(3)^4} + \right. \\ &\left. +96\frac{\pi^{10}}{\zeta(3)^5}\zeta''(-2)^2\right) = \\ &= 75463.66774845673072302538\ldots, \end{aligned}$$

$$\begin{aligned} Z_j(4) = 16&\left[1-\frac{\pi^8}{\zeta(3)^4} - \frac{\pi^{10}}{\zeta(3)^5}\left(16\zeta''(-2)+\frac{32}{3}\zeta'''(-2)+\right.\right. \\ &\left. +\frac{16}{3}\zeta''''(-2)\right) - \frac{\pi^{12}}{\zeta(3)^6}\left(160\zeta''(-2)^2+\right. \\ &\left.\left. +\frac{640}{3}\zeta''(-2)\zeta'''(-2)\right) - 1280\frac{\pi^{14}}{\zeta(3)^7}\zeta''(-2)^3\right] = \\ &= 6936470.11903064697027091228\ldots \end{aligned} \tag{266}$$

These closed-form formulas were obtained using (265) in conjunction with differentiating (255) and (261). The values of $Z_j(m)$ are naturally normalized by a factor $\frac{1}{k}$ taken in the limit and they converge to a finite constant. Otherwise, without the $\frac{1}{k}$ factor they would be quickly divergent. Also, as $m\to\infty$ the $Z_j(m)$ diverges but presently we do not know its growth rate.

As we have shown in the previous sections by Eq. (201) that one could obtain a formula for j_1 by the limit

$$j_1 = \lim_{m\to\infty}[Z_j(m)]^{-\frac{1}{m}}, \tag{267}$$

and substituting (265) for $Z_j(m)$ we have

$$j_1 = \lim_{m\to\infty}\left[-\frac{m}{(m-1)!}\frac{d^m}{dw^m}\log\left[\frac{\zeta^{-1}(w)}{w+\frac{1}{2}}\right]\Big|_{w\to 0}\right]^{-\frac{1}{m}}, \tag{268}$$

and a factor of m is needed to cancel the $\frac{1}{m}$ from $Z_j(m)$. Numerical computation of (268) is summarized in Tab. 5 in the previous section.

IX. CONCLUSION

In the presented work we utilized the m^{th} log-derivative formula to obtain a generalized zeta series of the zeros of the Riemann zeta function from which we can recursively extract trivial and non-trivial zeros. We then extended the same methods as to solve an equation $\zeta(s) - w = 0$ in order to obtain an inverse Riemann zeta function $s = \zeta^{-1}(w)$. We introduced a singularity expansion formula of the inverse zeta where the singularities j_n span an interval $(j_1, 1)$. Not much is known about these quantities, yet they can describe the entire $s_1 = \zeta^{-1}(w)$ (the principal branch) and many identities that follow. We further numerically explored these formulas to high precision in PARI/GP software package and show that they do indeed converge to the inverse Riemann zeta function for various test cases. Then we developed an efficient computer code to compute the inverse zeta for complex w-domain.

We remark that the methods presented in this article can be naturally applied to find zeros and inverses of many other functions but each function has to be custom fitted for this method. For example, we obtain an inverse of the gamma function:

$$s = \Gamma^{-1}(w) =$$
$$= \lim_{m\to\infty} \left[-\frac{1}{(m-1)!} \frac{d^m}{ds^m} \log\left[(\Gamma(s) - w)s\right]\Big|_{s\to 0} \right]^{-\frac{1}{m}}, \tag{269}$$

which is the principal solution or the inverse Bessel function of the first kind

$$s = J_{\nu,n}^{-1}(w) =$$
$$= \lim_{m\to\infty} \left[-\frac{1}{2(m-1)!} \frac{d^m}{ds^m} \log\left[J_{\nu,n}(s) - w)\right]\Big|_{s\to 0} \right]^{-\frac{1}{m}}. \tag{270}$$

The Lambert-W function is defined as an inverse of $g(x) = = xe^x$ which we could write as

$$g^{-1}(x) = W(x) =$$
$$= \lim_{m\to\infty} \left[-\frac{1}{(m-1)!} \frac{d^m}{ds^m} \log\left[se^s - x\right]\Big|_{s\to 0} \right]^{-\frac{1}{m}}, \tag{271}$$

but one is no longer limited to $g(x) = xe^x$ and could invert essentially any function, for example, we can invert $h(x) = (x - x^3)e^x$ as

$$h^{-1}(x) =$$
$$= \lim_{m\to\infty} \left[-\frac{1}{(m-1)!} \frac{d^m}{ds^m} \log\left[(s - s^3)e^s - x\right]\Big|_{s\to 0} \right]^{-\frac{1}{m}}. \tag{272}$$

One also has the inversion of trigonometric functions

$$\cos^{-1}(x) =$$
$$= \lim_{m\to\infty} \left[-\frac{1}{2(m-1)!} \frac{d^m}{ds^m} \log\left[\cos(s) - x\right]\Big|_{s\to 0} \right]^{-\frac{1}{m}}, \tag{273}$$

or

$$\cos(x) =$$
$$= \lim_{m\to\infty} \left[-\frac{1}{(m-1)!} \frac{d^m}{ds^m} \log\left[\cos^{-1}(s) - x\right]\Big|_{s\to 0} \right]^{-\frac{1}{m}}. \tag{274}$$

A generalized zeta series can also be inverted, for example, the inverse of the secondary zeta function as

$$Z_1^{-1}(w) = \lim_{m\to\infty} \left[-\frac{1}{(m-1)!} \times \right.$$
$$\left. \times \frac{d^m}{ds^m} \log\left[\frac{1}{t_1^s} + \frac{1}{t_2^s} + \frac{1}{t_3^s} + \ldots - w\right]\Big|_{s\to 0} \right]^{-\frac{1}{m}}, \tag{275}$$

which is over imaginary parts of non-trivial zeros.

Finally, we give an example of how to solve a finite degree polynomial of degree six. First we create a polynomial with prescribed zeros by factorization as

$$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6), \tag{276}$$

which yields the polynomial

$$f(x) = x^6 - 21x^5 + 175x^4 - 735x^3 + 1624x^2 - 1764x + 720, \tag{277}$$

we wish to solve. Then using Theorem 1 we obtain the generalized zeta series over its zeros as

$$Z(m) = -\frac{1}{(m-1)!} \frac{d^m}{dx^m} \log\left[x^6 - 21x^5 + 175x^4 + \right.$$
$$\left. -735x^3 + 1624x^2 - 1764x + 720\right]\Big|_{x\to 0}. \tag{278}$$

The principal zero is computed as

$$z_1 = \lim_{m\to\infty} [Z(m)]^{-\frac{1}{m}}, \tag{279}$$

and a numerical computation for $m = 100$ yields

$$z_1 = 0.99999999999999999999\ldots, \tag{280}$$

accurate to 32 decimal places, at which point we just round to 1. The next zero is recursively found as

$$z_2 = \lim_{m\to\infty} [Z(m) - \frac{1}{1^m}]^{-\frac{1}{m}}, \tag{281}$$

and a numerical computed for $m = 100$ yields

$$z_2 = 1.999999999999999999\underline{9}5\ldots, \tag{282}$$

which is accurate to 19 decimal places, at which point we just round to 2. The next zero is recursively found as

$$z_3 = \lim_{m\to\infty} [Z(m) - \frac{1}{1^m} - \frac{1}{2^m}]^{-\frac{1}{m}}, \tag{283}$$

and a numerical computation for $m = 100$ yields

$$z_3 = 2.9999999999999\underline{9}037839\ldots, \tag{284}$$

which is accurate to 14 decimal places, at which point we just round to 3. We keep repeating this and compute the remaining zeros

$$\begin{aligned} z_4 &= 3.9999999999\underline{9}185185599\ldots, \\ z_5 &= 4.999999999\underline{9}3962663006\ldots, \\ z_6 &= 5.99999999999999999999\ldots \end{aligned} \tag{285}$$

Such methods can be effectively used to solve a finite or infinite degree polynomial and is straightforward if the zeros are positive and real but the root extraction becomes more difficult if the roots are a mix of positive and negative numbers or if they are complex. Usually, some other knowledge about these zeros would be required in order to extract them recursively. The presented methods give analytical formulas for the zeros of functions and their inverses.

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Appendix A

As a reference, we compute a set of j_n for $m = 30$ in Tab. A1 and the identities that follow. The mean value is computed as

$$\mu = \frac{1}{m}\sum_{n=1}^{m} j_n = \frac{1}{m}(j_1 + j_2 + j_3 + \ldots) = \frac{1}{2} \approx \\ \approx 0.500000000000000000000000000000000\underline{0}997753\ldots, \tag{A1}$$

accurate to 34 decimal places. And also the variance is

$$\sigma^2 = \frac{1}{m}\sum_{n=1}^{m}(j_n - \frac{1}{2})^2 = \frac{1}{m}\left[(j_1 - \frac{1}{2})^2 + (j_2 - \frac{1}{2})^2 + (j_3 - \frac{1}{2})^2 + \ldots\right] \approx \\ \approx 0.15443132980306572121\ldots, \tag{A2}$$

and

$$\sigma = 0.39297751819037399128\ldots \tag{A3}$$

The mean value of $\log(j_n)$ is

$$\frac{1}{m}\sum_{n=1}^{m}\log(j_n) = \frac{1}{m}(\log(j_1) + \log(j_2) + \log(j_3) + \ldots) = -2\log(2) \approx \\ \approx -1.38629436\underline{1}08884637110\ldots, \tag{A4}$$

Tab. A1. The computation of j_n singularities for $m = 30$ (first 20 digits)

n	$\Re(j_n)$	$\Im(j_n)$
1	0.00940557776026071124	0
2	0.01141938808870173355	0
3	0.01568670434785971279	0
4	0.02261775560919276321	0
5	0.03270868212775017715	0
6	0.04648619390258016526	0
7	0.06448721566099810248	0
8	0.08725783916047024685	0
9	0.11535480283979529233	0
10	0.14933745452446960794	0
11	0.18974131865445665052	0
12	0.23702542541097595291	0
13	0.29148628794215263842	0
14	0.35313433733493120220	0
15	0.42153661001834201445	0
16	0.49564501219117809436	0
17	0.57365240977960573698	0
18	0.65294264589797238400	0
19	0.73021005275034721000	0
20	0.80179908715805887119	0
21	0.86424567115383859676	0
22	0.91490810371118783718	0
23	0.95250694689898902064	0
24	0.97740075208484199041	0
25	0.99140792923529390418	0
26	1.00203489268489807908	0
27	0.99719503440906385238	−0.00187149506426021143
28	0.99719503440906385238	+0.00187149506426021143
29	1.00058541712636179950	−0.00224107576516750101
30	1.00058541712636179950	+0.00224107576516750101

accurate to 9 decimal places. We also have the generalized zeta series

$$\frac{1}{m}\sum_{n=1}^{m}\frac{1}{j_n} = \frac{1}{m}\left(\frac{1}{j_1}+\frac{1}{j_2}+\frac{1}{j_3}+\ldots\right) = 2\left(-1+\frac{\pi^2}{\zeta(3)}\right) \approx \approx 14.42119\underline{3}28698625644727\ldots, \tag{A5}$$

accurate to 6 decimal places. We compute

$$\prod_{n=1}^{m}\left(1+\frac{1}{2j_n}\right)^{\frac{1}{m}} = 4\log\sqrt{2\pi} \approx \approx 3.67575413\underline{2}70457994521\ldots, \tag{A6}$$

accurate to 9 decimal places and also, when taking the limit $s = 10^3$, the Euler-Mascheroni constant:

$$\gamma = \left[4(s+\frac{1}{2})\prod_{n=1}^{m}\left(1-\frac{s}{j_n}\right)^{-\frac{1}{m}} - (1+\frac{1}{s})\right]s^2 \approx$$
$$\approx 0.57\underline{7}6530477\ldots,$$

accurate to 3 decimal places.

Appendix B

There is a recurrence relation for the generalized zeta series $Z_\nu(s)$ over the zeros of the Bessel function of the first kind found in Sneddon [16, p. 149], which satisfies

$$\sum_{r=0}^{m}\frac{m!\Gamma(m+\nu+1)}{(m-r)!\Gamma(m+\nu-r+1)}(-4)^r Z_\nu(2r+2) = \frac{1}{4(\nu+m+1)}\,, \tag{A7}$$

from which, with additional simplifications, we re-write this formula slightly differently as to keep it compact by defining the summand term

$$K(r,m,\nu) = (-4)^r\frac{m!\Gamma(m+\nu+1)}{(m-r)!\Gamma(m-r+\nu+1)}\,, \tag{A8}$$

so that

$$Z_\nu(2m+2) = \frac{1}{K(m,m,\nu)}\left(\frac{1}{4(m+\nu+1)} - \sum_{r=1}^{m}K(r-1,m,\nu)Z_\nu(2r)\right), \tag{A9}$$

and further expanding yields

$$\begin{aligned} Z_\nu(2m+2) = \frac{1}{4}\sum_{r=1}^{m}\frac{Z_\nu(2r)}{(-4)^{m-r}(m-r+1)!(\nu+1)_{m-r+1}} + \\ +(-1)^m\left(\frac{1}{4}\right)^{m+1}\frac{1}{m!(m+\nu+1)(\nu+1)_m}\,, \end{aligned} \tag{A10}$$

also found in [16, Eq. (39)]. The gamma terms of the type

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)(x+2)\cdots(x+k-1) = \sum_{n=0}^{k}(-1)^n s(n,k)x^n, \tag{A11}$$

is the rising factorial (which can be written in terms of the Pochhammer symbol) result in a finite k^{th} degree polynomial function with integer coefficients, i.e., the Stirling numbers of the first kind $s(n,k)$. Also for the case $m = 0$ the summation term in (A10) assumed to be 0 to bootstrap the recurrence. These formulas generate $Z_\nu(2m)$ as shown in Eq. (70) and Tab. 1.

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