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Reflection of Plane Waves from Surface of a Generalized Thermo-viscoelastic Porous Solid Half-space with Impedance Boundary Conditions

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Abstract: A phenomenon of reflection of plane waves from a thermally insulated surface of a solid half-space is studied in the context of Lord-Shulman theory of generalized thermo-viscoelasticity with voids. The governing equations of generalized thermo-viscoelastic medium with voids are specialized in x-z plane. The plane wave solution of these equations shows the existence of three coupled longitudinal waves and a shear vertical wave in a generalized thermo-viscoelastic medium with voids. For incident plane wave (longitudinal or shear), three coupled longitudinal waves and a shear vertical wave reflect back in the medium. The mechanical boundary conditions on the free surface of solid half-space are considered as impedance boundary conditions, in which the shear force tractions are assumed to vary linearly with the tangential displacement components multiplied by the frequency. The impedance corresponds to the constant of proportionality. The appropriate potentials of incident and reflected waves in the half-space will satisfy the required impedance boundary conditions. A non-homogeneous system of four equations in the amplitude ratios of reflected waves is obtained. These amplitude ratios are functions of material parameters, impedance parameter, angle of incidence, thermal relaxation and speeds of plane waves. Using relevant material parameters for medium, the amplitude ratios are computed numerically and plotted against certain ranges of the impedance parameter and the angle of incidence.

Key words: generalized thermo-viscoelasticity, voids, thermal relaxation, plane waves, reflection, amplitude ratios

I. INTRODUCTION

Cowin and Nunziato [1] developed the theory of elastic material with voids. Iesan [2, 3] developed the theory of thermoelastic material with voids. Various dynamical problems and plane strain problems in the theory of elasticity and thermoelasticity with voids have appeared in literature. For example, Iesan [4], Ciarletta and Scalia [5], Chirita and Scalia [6], Chirita et al. [7], Iesan and Nappa [8], Chirita and D'Apice [9, 10] and Ciarletta et al. [11] have studied various outstanding dynamical problems in the theory of thermoelasticity with voids. Various problems on plane wave propagation in elasticity and thermoelasticity with voids were also studied by, for example, Puri and Cowin [12], Chandrasekharaiah [13, 14], Singh [15], Ciarletta and Straughan [16], Ciarletta, et al. [17] and Bucur et al. [18].

Iesan [19, 20] developed theories of thermoviscoelastic materials with voids by incorporating the memory effects. Some problems on waves and vibrations in thermoviscoelastic material with voids were studied by Sharma and Kumar [21], Svanadze [22], Tomar et al. [23], Chirita [24], Chirita and Danescu [25], D'Apice and Chirita [26] and Bucur [27]. Exploring various problems on wave propagation in thermoviscoelastic materials with voids is useful in civil engineering, seismology, nano-technology and bio-materials [28]. In the present paper we consider a generalized thermoviscoelastic solid half-space with voids, whose surface is subjected to impedance boundary conditions as in Godoy [29], where the tangential components of stress tensor depends linearly on tangential displacement components times frequency, respectively. A problem on reflection of plane (longitudinal or shear) wave in a generalized thermoviscoelastic medium with voids under these impedance boundary conditions is considered. The reflection coefficients (or amplitude ratios) of various reflected waves are analysed numerically to show the dependence on the angle of incidence, viscous, thermal and voids parameters and impedance parameters.

II. BASIC EQUATIONS

A fixed system of rectangular Cartesian axes $Ox_i(i = 1, 2, 3)$ is referred to the motion of the continuum. We assume that the continuum has achieved the given state at time t due to some prescribed motion. We restrict to the linear theory to study the behavior of porous solid in which the matrix is a thermoviscoelastic material and the interstices are void of material. Initially the body is assumed free from stresses. In a distributed body, the mass density at time t has the decomposition $\gamma\nu$ where γ is the density of the matrix material and ν is the volume fraction field. For isotropic and homogeneous case, the system of field equations for thermoviscoelastic material with voids given by Iesan [19] are organized in the context of Lord and Shulman [30] theory after neglecting body forces and heat sources as
(a) the equations of motion

$$\frac{\partial t_{rs}}{\partial x_r} = \frac{\partial^2 u_s}{\partial t^2}, \tag{1}$$

$$\frac{\partial H_r}{\partial x_r} + g = \rho K^* \frac{\partial^2 \phi}{\partial t^2}, \tag{2}$$

(b) the energy equation

$$\rho T_0 \frac{\partial \eta}{\partial t} = \frac{\partial Q_r}{\partial x_r}, \tag{3}$$

(c) the constitutive equations

$$\begin{aligned} t_{rs} =& \left(\lambda + \lambda^* \frac{\partial}{\partial t}\right) e_{pp}\delta_{rs} + 2\left(\mu + \mu^* \frac{\partial}{\partial t}\right) e_{rs} \\ &+ \left(b + b^* \frac{\partial}{\partial t}\right)\phi\delta_{rs} - \beta T \delta_{rs}, \end{aligned} \tag{4}$$

$$H_r = \left(\alpha + \alpha^* \frac{\partial}{\partial t}\right)\frac{\partial \phi}{\partial x_r} + \tau^* \frac{\partial T}{\partial x_r}, \tag{5}$$

$$g = -\left(b + \gamma^* \frac{\partial}{\partial t}\right) e_{pp} - \left(\xi + \xi^* \frac{\partial}{\partial t}\right)\phi + mT, \tag{6}$$

$$\rho\eta = \beta e_{pp} + aT + m\phi, \tag{7}$$

$$Q_r + \tau_0 \frac{\partial Q_r}{\partial t} = \kappa \frac{\partial T}{\partial x_r} + \zeta \frac{\partial^2 \phi}{\partial x_r \partial t}, \tag{8}$$

$$e_{rs} = \frac{1}{2}\left(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r}\right), \tag{9}$$

where subscripts p, r and s range from 1 to 3, t_{rs} are the components of the stress tensor, H_r are the components of the equilibrated stress vector, g is the intrinsic equilibrated body force, η is the entropy per unit mass, Q_r are the components of the heat flux vector, e_{rs} are the components of the strain tensor, ρ is the mass density of the medium, K^* is the equilibrated inertia, $u_r(x_1, x_2, x_3, t)$ are the components of the displacement vector, $\phi(x_1, x_2, x_3, t)$ is an increment in void fraction field from constant value ν_0 in reference configuration, $T(x_1, x_2, x_3, t)$ is an increment in temperature from the constant reference temperature T_0, δ_{rs} are the components of the Kronecker delta, λ and μ are well known Lame's constant parameters, b, α, ξ and ξ^* are the constant parameters corresponding to voids present in the medium, β, τ^*, m, κ, ζ and a are the constant thermal parameters and λ^*, μ^*, b^*, α^* and γ^* are the constant viscoelastic parameters, τ_0 is thermal relaxation time. In what follows, the following notations are used

$$C_e = aT_0,$$

$$\lambda_0 = \lambda + \lambda^* \frac{\partial}{\partial t},$$

$$\mu_0 = \mu + \mu^* \frac{\partial}{\partial t},$$

$$b_0 = b + b^* \frac{\partial}{\partial t},$$

$$\alpha_0 = \alpha + \alpha^* \frac{\partial}{\partial t},$$

$$\gamma_0 = b + \gamma^* \frac{\partial}{\partial t},$$

$$\xi_0 = \xi + \xi^* \frac{\partial}{\partial t}.$$

Using equations (4) to (9) in equations (1) to (3), we obtain the following field equations

$$\begin{aligned} &\mu_0 \frac{\partial^2 u_s}{\partial {x_r}^2} + (\lambda_0 + \mu_0)\frac{\partial^2 u_m}{\partial x_m \partial x_s} \\ &+ b_0 \frac{\partial \phi}{\partial x_s} - \beta \frac{\partial T}{\partial x_s} = \rho \frac{\partial^2 u_s}{\partial t^2}, \end{aligned} \tag{10}$$

$$\alpha_0 \frac{\partial^2 \phi}{\partial {x_r}^2} - \gamma_0 \frac{\partial u_r}{\partial x_r} - \xi_0 \phi + \tau^* \frac{\partial^2 T}{\partial {x_r}^2} + mT = \rho K^* \frac{\partial^2 \phi}{\partial t^2}, \tag{11}$$

$$\kappa \frac{\partial^2 T}{\partial {x_r}^2} + \zeta \frac{\partial^3 \phi}{\partial {x_r}^2 \partial t} - \beta T_0 \left(\frac{\partial^2 u_r}{\partial x_r \partial t} + \tau_0 \frac{\partial^3 u_r}{\partial x_r \partial t^2} \right) - m T_0 \left(\frac{\partial \phi}{\partial t} + \tau_0 \frac{\partial^2 \phi}{\partial t^2} \right) - C_e \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right) = 0, \tag{12}$$

We now restrict our attention to plane strain in the $(1;3)$ plane with displacement components $u_1 = u_1(x_1, x_3, t)$, $u_2 = 0$, $u_1 = u_3(x_1, x_3, t)$. In what follows x_1 and x_3 are denoted as x and z, respectively. Equations (10) to (12) are specialized in $x - z$ plane as

$$\mu_0 \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial z^2} \right) + (\lambda_0 + \mu_0) \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_3}{\partial x \partial z} \right) + b_0 \frac{\partial \phi}{\partial x} - \beta \frac{\partial T}{\partial x} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{13}$$

$$\mu_0 \left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial z^2} \right) + (\lambda_0 + \mu_0) \left(\frac{\partial^2 u_1}{\partial x \partial z} + \frac{\partial^2 u_3}{\partial z^2} \right) + b_0 \frac{\partial \phi}{\partial z} - \beta \frac{\partial T}{\partial z} = \rho \frac{\partial^2 u_3}{\partial t^2}, \tag{14}$$

$$\alpha_0 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - \gamma_0 \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z} \right) - \xi_0 \phi + \tau^* \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + mT = \rho K^* \frac{\partial^2 \phi}{\partial t^2}, \tag{15}$$

$$\kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + \zeta \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - \beta T_0 \left\{ \left(\frac{\partial^2 u_1}{\partial x \partial t} + \frac{\partial^2 u_3}{\partial z \partial t} \right) + \tau_0 \left(\frac{\partial^3 u_1}{\partial x \partial t^2} + \frac{\partial^3 u_3}{\partial z \partial t^2} \right) \right\} - m T_0 \left(\frac{\partial \phi}{\partial t} + \tau_0 \frac{\partial^2 \phi}{\partial t^2} \right) - C_e \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right) = 0. \tag{16}$$

Using the following Helmholtz representations of displacement components in terms of potentials q and ψ

$$u_1 = \frac{\partial q}{\partial x} - \frac{\partial \psi}{\partial z}, \;\; u_3 = \frac{\partial q}{\partial z} + \frac{\partial \psi}{\partial x} \tag{17}$$

the equations (13) to (16) result in the following equations

$$\mu_0 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = \rho \frac{\partial^2 \psi}{\partial t^2}, \tag{18}$$

$$(\lambda_0 + 2\mu_0) \left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial z^2} \right) + b_0 \phi - \beta T = \rho \frac{\partial^2 q}{\partial t^2}, \tag{19}$$

$$\alpha_0 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - \gamma_0 \left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial z^2} \right) - \xi_0 \phi + \tau^* \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + mT = \rho K^* \frac{\partial^2 \phi}{\partial t^2}, \tag{20}$$

$$\kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} \right) + \zeta \left(\frac{\partial^3 \phi}{\partial x^2 \partial t} + \frac{\partial^3 \phi}{\partial z^2 \partial t} \right) - \beta T_0 \left\{ \left(\frac{\partial^3 q}{\partial x^2 \partial t} + \frac{\partial^3 q}{\partial z^2 \partial t} \right) + \tau_0 \left(\frac{\partial^4 q}{\partial x^2 \partial t^2} + \frac{\partial^4 q}{\partial z^2 \partial t^2} \right) \right\} - m T_0 \left(\frac{\partial \phi}{\partial t} + \tau_0 \frac{\partial^2 \phi}{\partial t^2} \right) - C_e \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right) = 0. \tag{21}$$

We seek the plane wave solutions of equations (18) to (21) in the following form

$$\{q, \phi, T, \psi\} = \{A, B, C, D\} \times \exp[ik(x \sin\theta + z \cos\theta - Vt)], \tag{22}$$

where A, B, C and D are arbitrary constants. k is the wavenumber, V is the complex phase speed and θ is the angle of propagation. With the help of (22), the non-trivial plane wave solution of equation (18) leads to

$$V^2 = \frac{\mu_0}{\rho}. \tag{23}$$

which is the speed of shear vertical (SV) wave.

With the help of (22) the plane wave solutions of equations (19) to (21) lead to following cubic velocity equation

$$\begin{aligned} &\left(\frac{1 - \bar{\xi_0}}{\rho} - \epsilon_2 \bar{m} \right) \Gamma^3 - [\bar{\kappa}(1 - \bar{\xi_0}) + \frac{c_2}{\rho} + \epsilon_3 c_3 \\ &+ \epsilon_3 \bar{m} + \frac{c_1 (1 - \bar{\xi_0})}{\rho} - \epsilon_2 c_1 \bar{m} + \frac{b_0 \bar{\gamma}_0}{\rho} + \epsilon_1 b_0 \bar{m} \\ &+ \beta \gamma_0 \epsilon_2 + \beta \epsilon_1 (1 - \bar{\xi_0})] \Gamma^2 + \big[\bar{\kappa} c_2 - \epsilon_3 c_3 + \bar{\kappa} c_1 (1 - \bar{\xi_0}) \\ &+ \frac{c_1 c_2}{\rho} - c_1 c_3 \epsilon_3 + c_1 \bar{m} \epsilon_3 + b_0 \bar{\gamma}_0 \bar{\kappa} - b_0 c_3 \epsilon_1 - \beta \epsilon_3 \bar{\gamma}_0 \\ &+ \beta c_2 \epsilon_1 \big] \Gamma - (c_1 c_2 \bar{\kappa} - c_1 c_3 \epsilon_3) = 0, \end{aligned} \tag{24}$$

where

$$\Gamma = \rho v^2,$$

$$c_1 = \lambda_0 + 2\mu_0,$$

$$c_2 = \frac{\alpha_0}{K^*},$$

$$c_3 = -\frac{\tau^*}{K^*},$$

$$\epsilon_1 = \frac{\beta T_0}{\rho C_e},$$

$$\epsilon_2 = \frac{mT_0}{\rho C_e},$$

$$\epsilon_3 = \frac{\zeta^*\omega^2}{C_e},$$

$$\bar{\kappa} = \frac{\kappa}{C_e(\tau_0 + \frac{i}{\omega})},$$

$$\bar{\gamma}_0 = \frac{\gamma_0}{\rho K^*\omega^2},$$

$$\bar{m} = \frac{m}{\rho K^*\omega^2},$$

$$\bar{\xi}_0 = \frac{\xi_0}{\rho K^*\omega^2},$$

$$\zeta^* = \frac{i\zeta}{\omega(\tau_0 + \frac{i}{\omega})}.$$

The real parts of the roots of cubic velocity equation (24) correspond to the speeds of three coupled ($P1$, $P2$ and $P3$) waves, which are longitudinal in nature (D'Apice and Chirita [26]).

III. REFLECTION FROM A PLANE SURFACE

We consider a half-space $z < 0$ containing a generalized thermoviscoelastic material with voids. The plane surface of the half-space is taken along the x-axis. The negative z-axis is taken as normal into the thermoviscoelastic half-space $z < 0$ as shown in Fig. 1. Following Godoy et al. [29], we assume that the surface of half-space is subjected to impedance boundary conditions, where the tangential tractions are proportional to tangential displacement components time frequency, respectively. Therefore, in the present problem, the impedance boundary conditions at $z = 0$ are expressed as

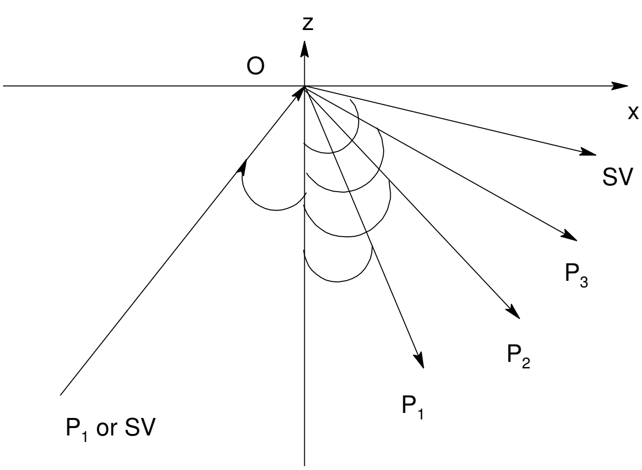

Fig. 1. Reflection of plane waves at a stress-free surface of a poro-thermo-viscoelastic solid half-space

$$t_{33} = 0,\ \ t_{31} + \omega Z u_1 = 0,\ \ H_3 = 0,\ \ Q_3 = 0, \quad (25)$$

where

$$t_{33} = \lambda_0 \frac{\partial u_1}{\partial x} + (\lambda_0 + 2\mu_0)\frac{\partial u_3}{\partial z} + b_0\phi - \beta T,$$

$$t_{31} = \mu_0\left(\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\right),$$

$$(1 + \tau_0\frac{\partial}{\partial t})Q_3 = \kappa\frac{\partial T}{\partial z} + \zeta\frac{\partial^2\phi}{\partial z\partial t}$$

$$H_3 = \alpha_0\frac{\partial\phi}{\partial z} + \tau^*\frac{\partial T}{\partial z},$$

and ω is the frequency of wave and Z is the impedance parameter of dimension stress/velocity, which is assumed strictly real. For $Z = 0$, the impedance boundary conditions reduce to traction-free boundary conditions and $|Z| \to +\infty$ corresponds to vanishing of the tangential component of the displacement vector. An incident P_1 or SV wave moves in half-plane $z < 0$ and strikes the free surface $z = 0$, then four reflected waves, namely, P_1, P_2, P_3 and SV are generated in the half-plane $z < 0$. The appropriate potentials for incident and reflected waves in the half-space are:

$$q = A_0\ exp\{ik_1(x\sin\theta_0 + z\cos\theta_0 - v_1t)\} + \sum_{j=1}^{3} A_j\ exp\{ik_j(x\sin\theta_j - z\cos\theta_j - v_jt)\}, \quad (26)$$

$$\phi = p_1A_0\ exp\{ik_1(x\sin\theta_0 + z\cos\theta_0 - v_1t)\} + \sum_{j=1}^{3} p_jA_j\ exp\{ik_j(x\sin\theta_j - z\cos\theta_j - v_jt)\}, \quad (27)$$

$$T = q_1A_0\ exp\{ik_1(x\sin\theta_0 + z\cos\theta_0 - v_1t)\} + \sum_{j=1}^{3} q_jA_j\ exp\{ik_j(x\sin\theta_j - z\cos\theta_j - v_jt)\}, \quad (28)$$

$$\psi = B_0\ exp\{ik_4(x\sin\theta_0 + z\cos\theta_0 - v_4t)\} + B_1\ exp\{ik_4(x\sin\theta_4 - z\cos\theta_4 - v_4t)\}. \quad (29)$$

where $v_i = Re(V_i),\ (i = 1, 2, ., 4)$ and the expression for $\frac{p_j}{k_j^2}$ and $\frac{q_j}{k_j^2},\ (j = 1, 2, 3)$ are given as

$$\frac{p_j}{k_j^2} = \frac{(\tau^* - \frac{m}{k_j^2})(\lambda_0 + 2\mu_0 - \rho v_j^2) + \frac{\beta\gamma_0}{k_j^2}}{b_0(\tau^* - \frac{m}{k_j^2}) + \beta(\alpha_0 + \frac{\xi_0}{k_j^2} - \rho K^* v_j^2)},$$

$$\frac{q_j}{k_j^2}=\frac{-(\alpha_0+\frac{\xi_0}{k_j^2}-\varrho K^* v_j^2)(\lambda_0+2\mu_0-\rho v_j^2)+\frac{b_0\gamma_0}{k_j^2}}{b_0(\tau^*-\frac{m}{k_j^2})+\beta(\alpha_0+\frac{\xi_0}{k_j^2}-\rho K^* v_j^2)}.$$

The potentials given in equations (26) to (29) satisfy boundary conditions (25) if the following relations (Snell's law for present problem) hold

$$\frac{\sin\theta_0}{v_1 \ or \ v_4}=\frac{\sin\theta_1}{v_1}=\frac{\sin\theta_2}{v_2}=\frac{\sin\theta_3}{v_3}=\frac{\sin\theta_4}{v_4} \tag{30}$$

$$k_1v_1=k_2v_2=k_3v_3=k_4v_4 \tag{31}$$

and
(a) incident P wave

$$\sum_{j=1}^{4}a_{ij}Z_j=b_i,\ \ (i=1,2,.,4) \tag{32}$$

where
$Z_j=\dfrac{A_j}{A_0},\ (j=1,2,3)$ and $Z_4=\dfrac{B_1}{A_0}$ are reflection coefficients of reflected P_1,P_2,P_3 and SV waves, and
$b_1=-1,\ \ b_2=-1,\ \ b_3=1,\ \ b_4=1,$

$$a_{1j}=\frac{(\frac{v_1}{v_j})^2[(\lambda-iw\lambda^*)+2(\mu-iw\mu^*)(1-(\frac{v_j}{v_1})^2\sin^2\theta_0)-(b-iwb^*)\frac{p_j}{k_j^2}+\beta\frac{q_j}{k_j^2}]}{(\lambda-iw\lambda^*)+2(\mu-iw\mu^*)\cos^2\theta_0-(b-iwb^*)\frac{p_1}{k_1^2}+\beta\frac{q_1}{k_1^2}},$$

$$a_{14}=-\frac{2(\mu-iw\mu^*)\frac{v_1}{v_4}\sin\theta_0\sqrt{1-(\frac{v_4}{v_1})^2\sin^2\theta_0}}{(\lambda-iw\lambda^*)+2(\mu-iw\mu^*)\cos^2\theta_0-(b-iwb^*)\frac{p_1}{k_1^2}+\beta\frac{q_1}{k_1^2}},$$

$$a_{2j}=(\frac{v_1}{v_j})\left[\frac{2\sqrt{1-(\frac{v_j}{v_1})^2\sin^2\theta_0}+iZ_j}{-2\cos\theta_0+iZ_1}\right],\ \ Z_j=\frac{v_jZ}{\mu-i\omega\mu^*},\ \ (j=1,2,3),$$

$$a_{24}=(\frac{v_1}{v_4})^2\left[\frac{1-2(\frac{v_4}{v_1})^2\sin^2\theta_0+iZ_4\sqrt{1-(\frac{v_4}{v_1})^2\sin^2\theta_0}}{\sin\theta_0(-2\cos\theta_0+iZ_1)}\right],\ \ Z_4=\frac{v_4Z}{\mu-i\omega\mu^*},$$

$$a_{3j}=\frac{(\frac{v_1}{v_j})(wp_j\zeta+iq_j\kappa)\sqrt{1-(\frac{v_j}{v_1})^2\sin^2\theta_0}}{(wp_1\zeta+iq_1\kappa)\cos\theta_0},\ \ (j=1,2,3),$$

$$a_{34}=0,$$

$$a_{4j}=\frac{(\frac{v_1}{v_j})[(\alpha-iw\alpha^*)p_j+\tau^*q_j]\sqrt{1-(\frac{v_j}{v_1})^2\sin^2\theta_0}}{[(\alpha-iw\alpha^*)p_1+\tau^*q_1]\cos\theta_0},$$

$$(j=1,2,3),$$

$$a_{44}=0,$$

(b) incident SV wave:

$$\sum_{j=1}^{4}c_{ij}Y_j=d_i,\ \ (i=1,2,.,4) \tag{33}$$

where
$Y_j=\dfrac{A_j}{B_0}\ (j=1,2,3)$ and $Y_4=\dfrac{B_1}{B_0}$ are reflection coefficients of reflected P_1,P_2,P_3 and SV waves, and,
$d_1=-1,\ d_2=-1,\ d_3=0,\ d_4=0,$

$$c_{1j} = \frac{(\frac{v_4}{v_j})^2[(\lambda - iw\lambda^*) + 2(\mu - iw\mu^*)[1 - (\frac{v_j}{v_4})^2 \sin^2\theta_0] - (b - iwb^*)\frac{p_j}{k_j^2} + \beta\frac{q_j}{k_j^2}]}{(\mu - iw\mu^*)\sin 2\theta_0}, \;\; c_{14} = -1,$$

$$c_{2j} = \sin\theta_0(\frac{v_4}{v_j})\left[\frac{2\sqrt{1 - (\frac{v_j}{v_4})^2\sin^2\theta_0} + iZ_j}{1 - 2\sin^2\theta_0 - iZ_4\cos\theta_0}\right], \;\; (j = 1, 2, 3),$$

$$c_{24} = \frac{1 - 2\sin^2\theta_0 + iZ_4\cos\theta_0}{1 - 2\sin^2\theta_0 - iZ_4\cos\theta_0},$$

$$c_{3j} = (\frac{v_4}{v_j})(w\zeta p_j + i\kappa q_j)\sqrt{1 - (\frac{v_j}{v_4})^2\sin^2\theta_0}, \;\; (j = 1, 2, 3), \;\; c_{34} = 0,$$

$$c_{4j} = (\frac{v_4}{v_j})[(\alpha - iw\alpha^*)p_j + \tau^* q_j]\sqrt{1 - (\frac{v_j}{v_4})^2\sin^2\theta_0}, \;\; (j = 1, 2, 3), \;\; c_{44} = 0.$$

IV. NUMERICAL RESULTS AND DISCUSSION

Various experimental studies including Hobbs et al. [31], Wang et al. [32] and Gondcharton et al. [33]) have shown the presence of voids in copper material. Therefore, it is relevant to consider an example of copper material for the purpose of numerical illustration of theoretical results. To illustrate the dependence of amplitude ratios of various reflected waves on the angle of incidence, impedance parameter and other material parameters, the following relevant physical constants of copper material are taken as in Chirita and Danescu [25] and Bucur [27], that is,

$$\lambda = 7.76 \times 10^{11} \text{ dyn/cm}^2,$$

$$\mu = 3.86 \times 10^{11} \text{ dyn/cm}^2,$$

$$\rho = 8.954\ gm/\text{cm}^3,$$

$$c = 3.4303 \times 10^4 \text{ dyn/cm}^2\ {}^o\text{C},$$

$$b = 2 \times 10^3 \text{ dyn/cm}^2,$$

$$\alpha = 1.688 \text{ dyn},$$

$$\beta = 0.4 \times 10^{-1} \text{ dyn/cm}^2\ {}^o\text{C},$$

$$\xi = 1.475 \text{ dyn/cm}^2,$$

$$m = 0.2 \times 10^7 \text{ dyn/cm}^2\ {}^o\text{C},$$

$$\kappa = 0.386 \times\ 10^8 \text{ dyn/s}\ {}^o\text{C},$$

$$T_0 = 293\ K,$$

$$K^* = 1.75 \times 10^{-11} \text{ cm}^2,$$

$$\lambda^* = 0.1 \text{ dyn s/cm}^2,$$

$$\mu^* = 0.2 \text{ dyn s/cm}^2,$$

$$b^* = 0.1 \times 10^{-3} \text{ dyn s/cm}^2,$$

$$\xi^* = 0.3 \text{ dyn s/cm}^2,$$

$$\alpha^* = 0.1 \text{ dyn s},$$

$$\gamma^* = 0.5 \times 10^{-7} \text{ dyn s/cm}^2,$$

$$\tau^* = 0.3 \times 10^{-7} \text{ dyn/}\ {}^o\text{C},$$

$$\zeta = 1.5 \times 10^{-11} \text{ dyn}.$$

For the above values of material parameters, the non-homogeneous systems (32) and (33) of linear equations in amplitude ratios of reflected waves are solved by using a Fortran program of Gauss elimination method. For incident $P1$ wave, the amplitude ratios Z_1, Z_2, Z_3 and Z_4 of reflected P_1, P_2, P_3 and SV waves are plotted against the range $0^o \leq$

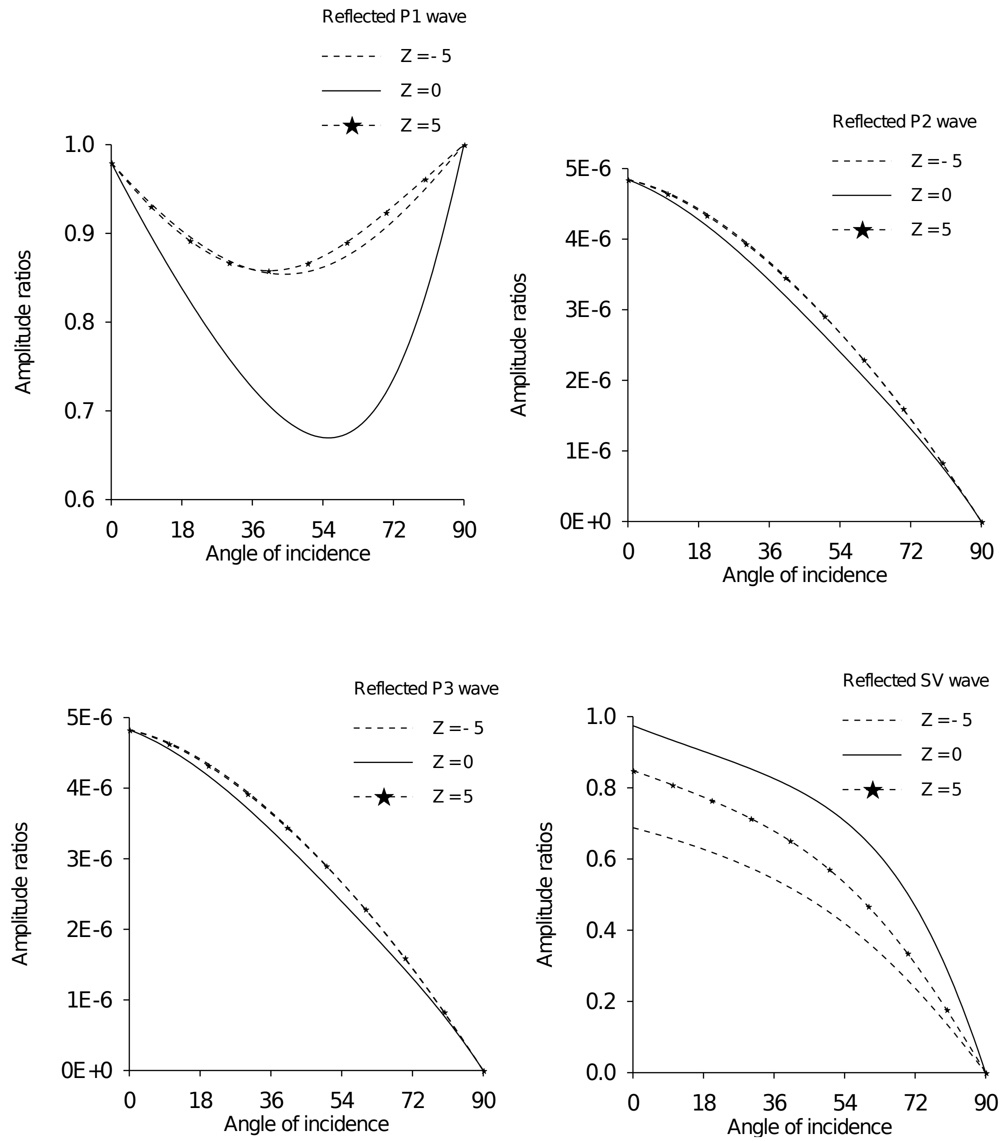

Fig. 2. Variations of the amplitude ratios of reflected $P1, P2, P3$ and SV waves against the angle of incidence θ_0 of incident $P1$ wave when $Z = -5, 0$ and 5

$\theta_0 \leq 90^o$ of angle of incidence in Fig. 2 by solid lines, when impedance parameter $Z = 0$. The amplitude ratios Z_1 of reflected $P1$ wave is 0.98 at $\theta_0 = 0^o$ (normal incidence). It decreases to value 0.6695 at $\theta_0 = 55^o$ and then increases to value one at $\theta_0 = 90^o$ (grazing incidence). The amplitude ratios Z_2 and Z_3 of reflected $P2$ and $P3$ waves are very smaller in comparison to that of $P1$ wave. The maximum values of the amplitude ratios Z_2 and Z_3 of reflected $P2$ and $P3$ waves are 0.4841e-05 and 0.4825e-05 at normal incidence. These reduce to zero at grazing incidence. The amplitude ratio Z_4 of reflected SV is 0.9742 at normal incidence and it also reduces to zero at grazing incidence. Similar variations for impedance parameters $Z = -5$ and $Z = 5$ are also shown in Fig. 2 by a dashed line and a dashed line with stars as center symbols, respectively. The comparison of these dashed lines with a solid line shows the effect of the impedance parameter at each angle of incidence of $P1$ wave.

For incident $P1$ wave, the amplitude ratios Z_1, Z_2, Z_3 and Z_4 of reflected P_1, P_2, P_3 and SV waves are plotted against the range $-20 \leq Z \leq 20$ of impedance parameter in Fig. 3 by dashed line, dashed line with squares and solid line with stars for $\theta_0 = 30^o, 60^o$ and 90^o, respectively. The comparison of these three variations shows the effect of three different angle of incidences in a particular range of impedance parameter. It is observed that there is no impact of impedance at grazing incidence.

For incident SV wave, the amplitude ratios Y_1, Y_2, Y_3 and Y_4 of reflected P_1, P_2, P_3 and SV waves are plotted against the range $1^o \leq \theta_0 \leq 45^o$ of the angle of incidence in Fig. 4 by solid lines, when impedance parameter

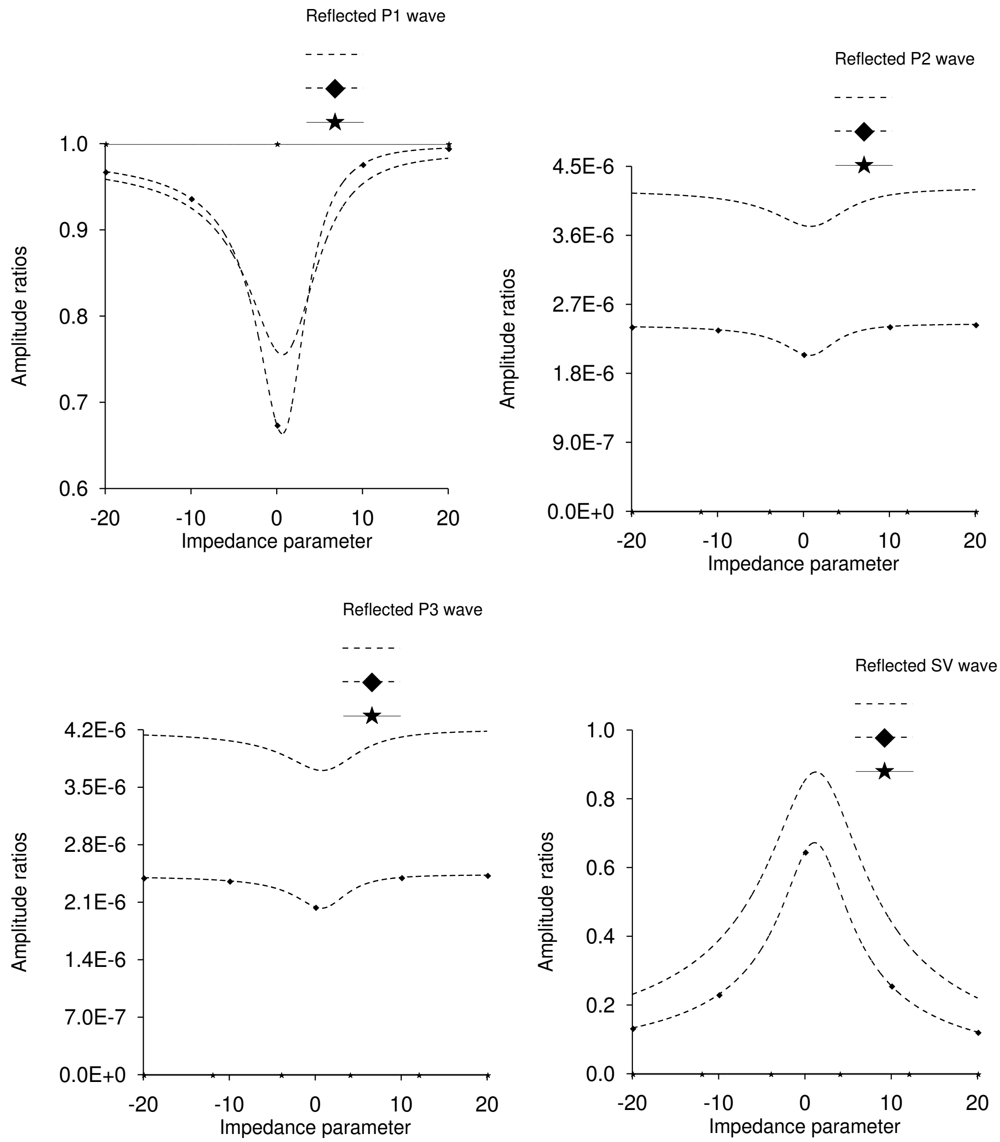

Fig. 3. Variations of the amplitude ratios of reflected $P1, P2, P3$ and SV waves against the impedance parameter Z for incident $P1$ wave when $\theta_0 = 30^o, 60^o$ and 90^o

$Z = 0$. Beyond $\theta_0 > 45^o$, a phase change occurs. The amplitude ratios Y_1 of reflected $P1$ wave is zero at $\theta_0 = 1^o$ (near normal incidence). It increases to its maximum value 0.5472 at $\theta_0 = 34^o$ and then decreases sharply to its minimum value zero at $\theta_0 = 90^o$ (grazing incidence). In this case also, the amplitude ratios Y_2 and Y_3 of reflected $P2$ and $P3$ waves are observed very smaller in comparison to that of $P1$ wave. The maximum values of the amplitude ratios Y_2 and Y_3 of reflected $P2$ and $P3$ waves are $0.7350e-04$ and $0.7355e-04$ at $\theta_0 = 25^o$. These amplitude ratios of reflected $P2$ and $P3$ waves reduce to zero at $1^o and 45^o$. The amplitude ratio Y_4 of reflected SV is one at $\theta_0 = 1^o$ and it reduces to 0.5416 at $\theta_0 = 39^o$ and increases sharply to one at 45^o. Similar variations for impedance parameters $Z = -5$ and $Z = 5$ are also shown in Fig. 4 by a dashed line and a dashed line with stars as center symbols, respectively. The comparison of these dashed lines with solid line shows the effect of the impedance parameter at each angle of incidence of SV wave.

For incident SV wave, the amplitude ratios Y_1, Y_2, Y_3 and Y_4 of reflected P_1, P_2, P_3 and SV waves are plotted against the range $-20 \leq Z \leq 20$ of the impedance parameter in Fig. 5 by a dashed line, a dashed line with stars and solid line with squares for $\theta_0 = 15^o, 30^o$ and 45^o, respectively. The comparison of these three variations shows the effect of three different angle of incidence in a particular range of the impedance parameter. It is observed that there is no impact of impedance at $\theta_0 = 45^o$.

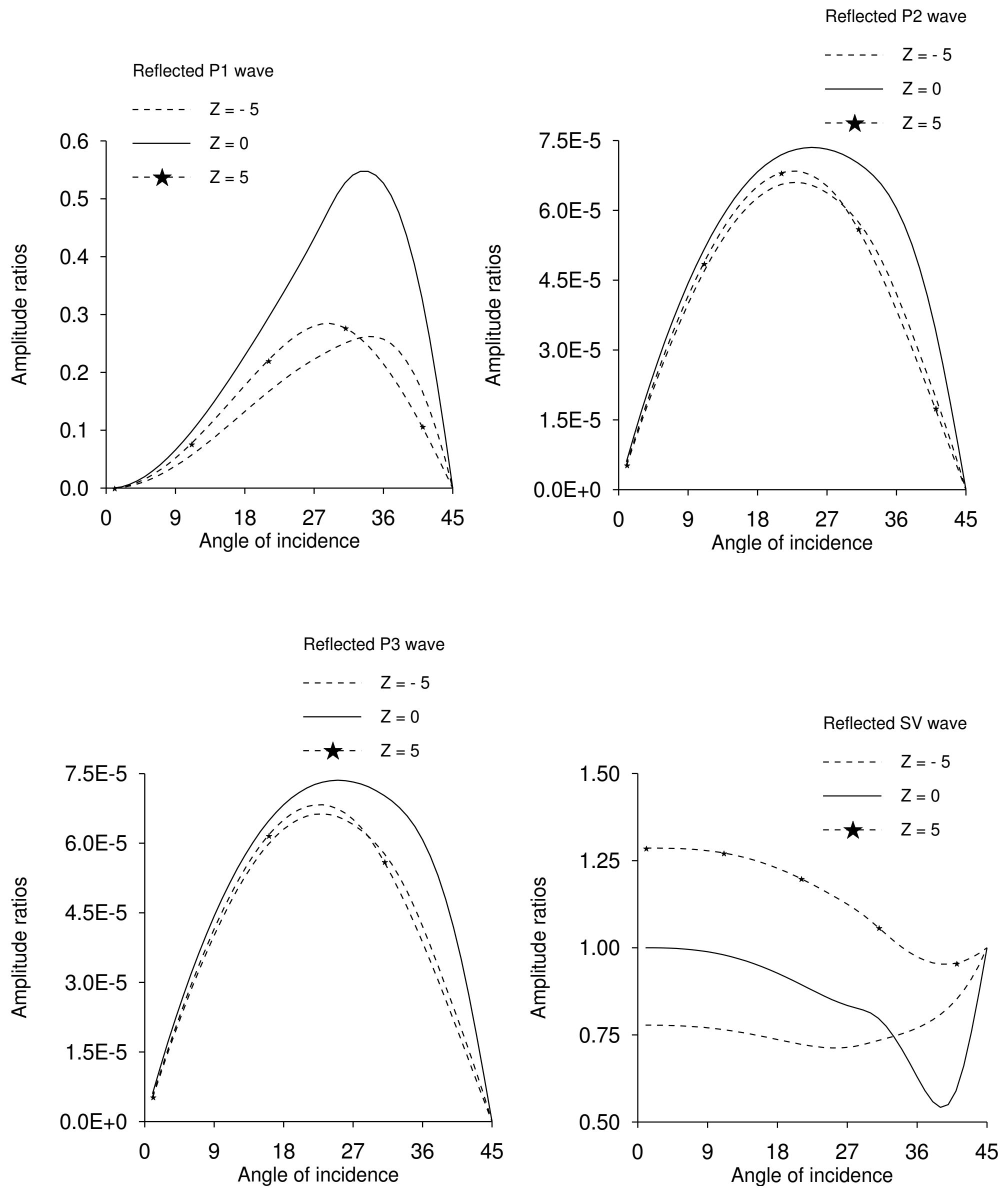

Fig. 4. Variations of the amplitude ratios of reflected $P1, P2, P3$ and SV waves against the angle of incidence θ_0 of incident SV wave when $Z = -5, 0$ and 5

V. CONCLUSIONS

Plane waves in a thermoviscoelastic medium with voids is studied in the context of the Lord and Shulman theory of generalized thermoelasticity. The solution of specialized governing equations of medium shows the existence of three coupled longitudinal waves ($P1, P2$ and $P3$) and a shear vertical (SV) wave. The relations between the amplitude ratios of various reflected waves are obtained for incidence of both $P1$ and SV waves. For a particular material representing the medium, the amplitude ratios of the reflected waves are computed and plotted against the angle of incidence and impedance parameter. The numerical discussion of these plots provide some vital observations:

1. Introduction of the impedance parameter in the tangential stress component changes significantly the amplitude ratios of all reflected waves for incidence of both $P1$ and SV waves.
2. For incident $P1$ wave, the impedance parameter significantly changes the amplitude ratios of all reflected waves at each angle of incidence except grazing incidences. From Fig. 2 it is also observed that the presence of impedance parameter changes significantly the amplitude ratios of reflected SV wave at normal incidence and the amplitude ratios of reflected $P1, P2$ and $P3$ waves remain unaffected at normal incidence.

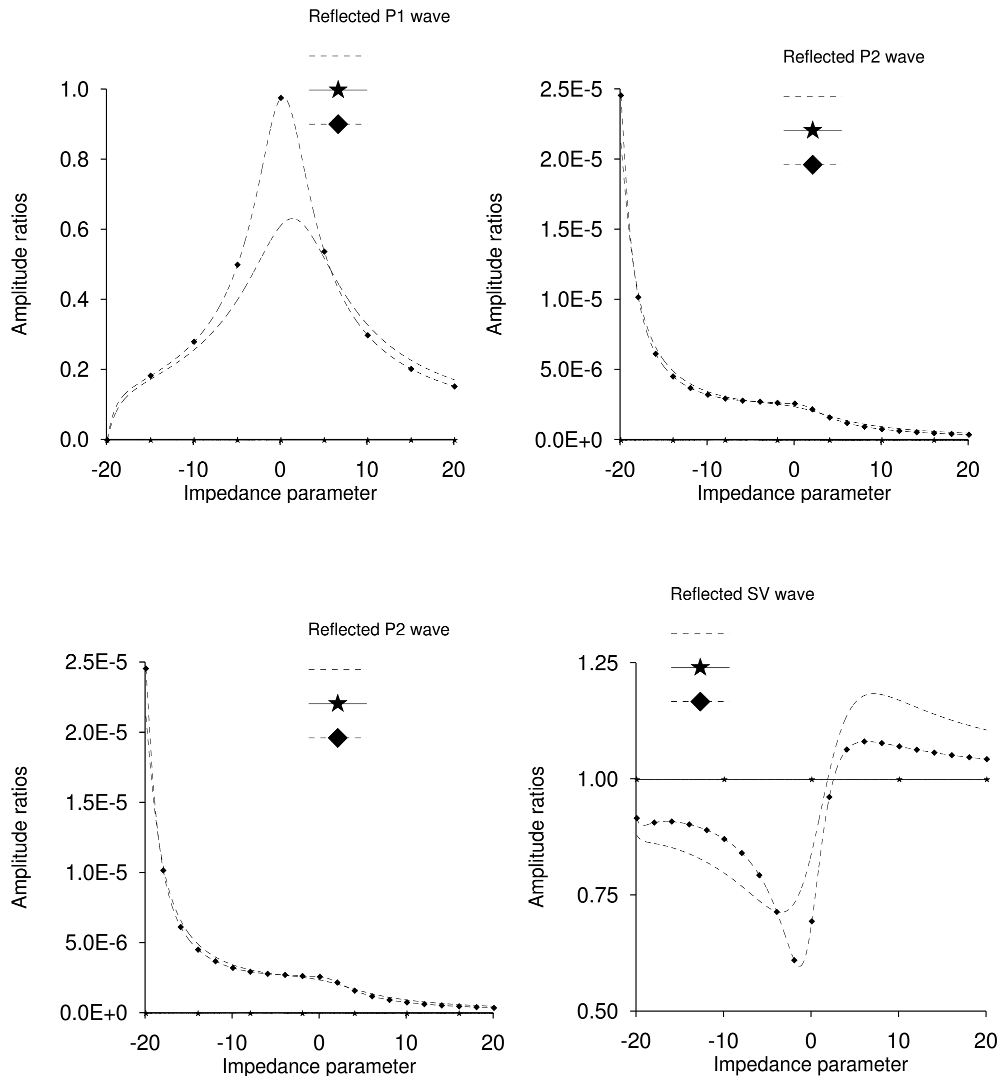

Fig. 5. Variations of the amplitude ratios of reflected $P1, P2, P3$ and SV waves against the impedance parameter Z for incident SV wave when $\theta_0 = 15^o, 30^o$ and 45^o

3. For incident SV wave, the impedance parameter significantly changes the amplitude ratios of all reflected waves at each angle of incidence except at $\theta_0 = 45^o$. Again from Fig. 4, it is also observed that the presence of impedance parameter changes significantly the amplitude ratios of reflected SV wave at normal incidence and the amplitude ratios of reflected $P1, P2$ and $P3$ waves remain unaffected at normal incidence.

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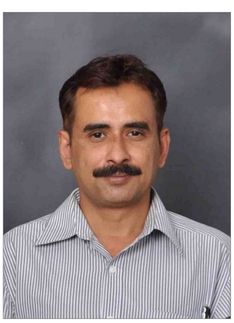

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