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Analytical Representations of Divisors of Integers

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Abstract: Certain analytical expressions which "feel" the divisors of natural numbers are investigated. It is shown that these expressions encode to some extent the well-known algorithm of the sieve of Eratosthenes.
Key words: number theory, factorization, primality, one-variable calculus

I. NOTATION AND CONVENTIONS

Throughout this paper we shall adopt the following notation and conventions: n is a given natural number and k is a possible divisor of n. If k actually divides n then $j = n/k$. Let $f(x)$ denote any real analytic function defined in the neighborhood of the origin by a power series

$$f(x) = \sum_{j=0}^{\infty} c_j x^j \tag{1}$$

with all $c_j \neq 0$ ($i = 1, 2, 3...$). It will be shown that, with appropriate assumptions, the exponent of x in the expansion (1) around zero j labels half-lines or rays of divisors (see below).

II. MOTIVATION

The theory of divisors of integers is the cornerstone of the elementary number theory. It is convenient to introduce *the characteristic function for divisors*:

Definition. For any $n, k \in \mathbb{N}$

$$\bar{\alpha}_{nk} := \begin{cases} 1 & \text{if} \quad k \mid n \\ 0 & \text{if} \quad k \nmid n \end{cases} \tag{2}$$

Another quite obvious (and rather useless in numerical calculations) representation of (2) is:

$$\bar{\alpha}_{nk} = \frac{1}{\Gamma\left(1 - \operatorname{mod}(n,k)\right)} \tag{3}$$

where $\Gamma(s)$ denotes the Euler gamma function and $\operatorname{mod}(n,k)$ gives the remainder on division of n by k. In fact, (3) is more general than (2) since it may be calculated also for non-integer or even complex values of n and k, but this leads to some interpretation difficulties which we shall not discuss here.

Consider the following expression for some natural numbers n and k:

$$\alpha_{nk} = \left.\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{x^k}\right|_{x=0} \tag{4}$$

We will prove the following

Theorem. Apart from a trivial normalization factor, α_{nk} defined in formula (4) is equal to $\bar{\alpha}_{nk}$ defined in (4).

Proof. Expanding the exponential function in (4) in power series and performing term-by-term differentiation we get:

$$\alpha_{nk} = \left.\frac{\mathrm{d}^n}{\mathrm{d}x^n} \sum_{j=0}^{\infty} \frac{\left(x^k\right)^j}{j!}\right|_{x=0} = \left.\sum_{j=0}^{\infty} \frac{1}{j!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} x^{jk}\right|_{x=0} \tag{5}$$

Recall the general formulas for the n-th derivative of x^p with respect to x

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-n)}x^{p-n} = n!\binom{p}{n}x^{p-n} \tag{6}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^p = (-1)^n\frac{\Gamma(n-p)}{\Gamma(-p)}x^{p-n} \tag{7}$$

where the second formula stems from properties of the gamma function and is suitable for integer negative p (see e.g. [4]). Note that the order of derivative n does not have to be integer but for integer n both (6) and (7) reduce to the well-known elementary differentiation rule. Using (6) we get:

$$\begin{aligned}\alpha_{nk} = \sum_{j=0}^{\infty}\frac{1}{j!}\frac{\Gamma(jk+1)}{\Gamma(jk+1-n)}x^{jk-n}\bigg|_{x=0} = \\ = n!\sum_{j=0}^{\infty}\frac{1}{j!}\binom{jk}{n}x^{jk-n}\bigg|_{x=0}\end{aligned} \tag{8}$$

By simple inspection of (8) we see why this expression "feels" the divisors of the integer n. Indeed, when taking the limit $x \to 0$ the only non-zero term in the series appears when $jk = n$ for some integer j, and this occurs if and only if k divides n. All terms with $jk > n$ disappear in the limit $x \to 0$ whereas those with $jk < n$, although singular in $x = 0$, vanish since the binomial coefficient term is zero. Therefore, in the summation (8) at most only one term can survive in the limit process.■

The above reasoning might appear far too excessive. However, it guarantees that among divisors none has been omitted. It should also be stressed that it may be used as a starting point for various generalizations since n does not need to be integer.

It is easy to guess the normalizing factor:

$$\alpha_{nk} = \frac{1}{n!}\left(\frac{n}{k}\right)!\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{x^k}\bigg|_{x=0} \tag{9}$$

Using the same reasoning we can derive similar expression for α_{nk}:

$$\alpha_{nk} = \frac{(k!)^{n/k}}{n!}\left(\frac{n}{k}\right)!\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{\frac{x^k}{k!}}\bigg|_{x=0} \tag{10}$$

III. SIMPLE EXAMPLE

In a natural way coefficients α_{nk} may be regarded as a square matrix of arbitrarily large dimension where the running integer n labels rows and the potential divisor k labels columns. The entries of this matrix are either one or zero depending on whether k divides n or not. This matrix is always triangular, since of course no divisor can exceed a given number, and its determinant (for any dimension) is 1.

$$\begin{pmatrix}
\boldsymbol{n\diagdown k} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} & \dots \\
\mathbf{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{2} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{3} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{4} & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{5} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{6} & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{7} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\
\mathbf{8} & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\
\mathbf{9} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\
\mathbf{10} & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix} \tag{11}$$

(Matrix (11) is closely related to the Redheffer matrix, see e.g. [5], [6].) Introducing

$$\sigma_0(n) := \sum_{k=1}^{n}\alpha_{nk} \tag{12}$$

we see that $\sigma_0(n)$ just counts the number of all divisors of a given n including both unity and n itself.

It is known (see e.g. [1]) that the inverse of matrix (11) is:

$$\beta_{nk} = \begin{cases}\mu\left(\frac{n}{k}\right) & \text{if} \quad k \mid n \\ 0 \quad \text{if} \quad k \nmid n\end{cases} \tag{13}$$

where μ denotes the Möbius function:

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has squared prime factor} \\ +1 & \text{if } n \text{ is a square-free positive} \\ & \text{integer with an even number of prime factors} \\ -1 & \text{if } n \text{ is a square-free positive} \\ & \text{integer with an odd number of prime factors}\end{cases} \tag{14}$$

$$\begin{pmatrix}
\boldsymbol{n\diagdown k} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} & \dots \\
\mathbf{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{2} & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{3} & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{4} & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{5} & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{6} & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\
\mathbf{7} & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\
\mathbf{8} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\
\mathbf{9} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\
\mathbf{10} & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix} \tag{15}$$

Note that the numbers in (15) when summed in rows give zero except for the first row which stems from the following identity:

$$\sum_{d|n}\mu(d) = \delta_{n,1} \tag{16}$$

Matrices (11) and (15) are visualized in Fig. 1.

Somewhat similar but purely qualitative results have been published in [3].

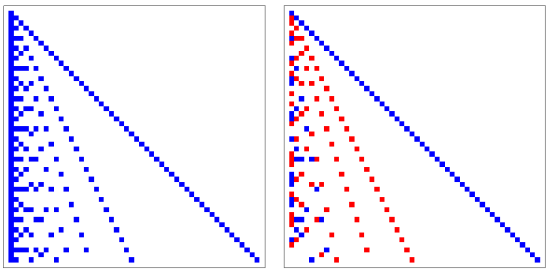

Fig. 1. Graphic distribution of divisors (11) for $n = 1, 2, ..., 50$ as a square matrix (left panel). Each blue square denotes $+1$. In the inverse matrix (15) blue square denotes $+1$ and red square denotes -1 (right panel)

IV. GENERAL CASE

The particular choice of the exponential function in (4) is not crucial to our reasoning. Indeed, instead of this function we can take *any* regular function $f(x)$ provided that it has all non-zero coefficients in its power series expansion

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 ... \qquad c_i \neq 0 \text{ for } i = 1, 2, 3, ... \tag{17}$$

Thus in general we have (up to the appropriate normalizing factor)

$$\alpha_{nk} = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big|_{x=0} \tag{18}$$

Denote the composition of f with monomial x^k (for each $k \in \mathbb{N}$)

$$g_k(x) \equiv f\left(x^k\right)$$

Obviously g_k is also analytic around zero and its power series at the origin has the form:

$$g_k(x) = \sum_{j=0}^{\infty} c_j x^{jk} = \sum_{j=0}^{\infty} \frac{g_k^{(0)}}{n!} x^n$$

Using the identity principle for powers series we immediately get that $g_k^{(0)} \neq 0$ if and only if $n = jk$ for some j, or equivalently, if and only if k is a divisor of n.

For example, taking

$$f(x) = \frac{x}{1-x} = x + x^2 + x^3 + ... \tag{19}$$

we get:

$$\alpha_{nk} = \sum_{j=1}^{\infty} \binom{jk}{n} x^{jk-n}\Big|_{x=0} \tag{20}$$

or simply

$$\alpha_{nk} = \sum_{j=1}^{\infty} \frac{x^{jk-n}}{(jk-n)!}\Big|_{x=0} \tag{21}$$

Taking

$$f(x) = \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ... \tag{22}$$

we get:

$$\alpha_{nk} = (-1)^{n/k} \frac{n}{k} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \binom{jk}{n} x^{jk-n}\Big|_{x=0} \tag{23}$$

The general explicit formula for α_{nk} using arbitrary function f satisfying (17) is:

$$\alpha_{nk} = \left(\frac{n}{k}\right)! \frac{1}{f^{(n/k)}(0)} \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} \binom{jk}{n} x^{jk-n}\Big|_{x=0} \tag{24}$$

where $f^{(j)}(0)$ denotes the j-th derivative of f with respect to x taken at $x = 0$. (If n/k in (24) is non-integer then the value of fractional derivative $f^{(n/k)}(0)$ is unimportant since in this case the sum vanishes.)

The table below contains normalizing factors for α_{nk}, for several different choices of function $f(x)$, obtained using (24).

$f(x) = e^x$	$\alpha_{nk} = \frac{n}{k}! \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big\|_{x=0}$
$f(x) = \ln(1-x)$	$\alpha_{nk} = -\frac{n}{k} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big\|_{x=0}$
$f(x) = \frac{x}{1-x}$	$\alpha_{nk} = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big\|_{x=0}$
$f(x) = \sqrt{1+x}$	$\alpha_{nk} = -\frac{(-2)^{n/k}}{\left(2\frac{n}{k}-3\right)!!} \frac{n}{k}! \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big\|_{x=0}$
$f(x) = \frac{1}{\sqrt{1+x}}$	$\alpha_{nk} = (-1)^{n/k} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{k}+\frac{1}{2}\right)} \frac{n}{k}! \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big\|_{x=0}$
$f(x) = (1+x)^{-3/2}$	$\alpha_{nk} = \frac{(-2)^{n/k}}{\left(2\frac{n}{k}+1\right)!!} \frac{n}{k}! \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big\|_{x=0}$
$f(x) = W(x)$	$\alpha_{nk} = (-1)^{n/k-1} \frac{\frac{n}{k}!}{\left(\frac{n}{k}\right)^{\frac{n}{k}-1}} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big\|_{x=0}$
$f(x) = \frac{1}{1-x-x^2}$	$\alpha_{nk} = \frac{1}{F_{\frac{n}{k}+1}} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big\|_{x=0}$

($W(x)$ is the Lambert W-function and F_n in the last row denotes the n-th Fibonacci number.)

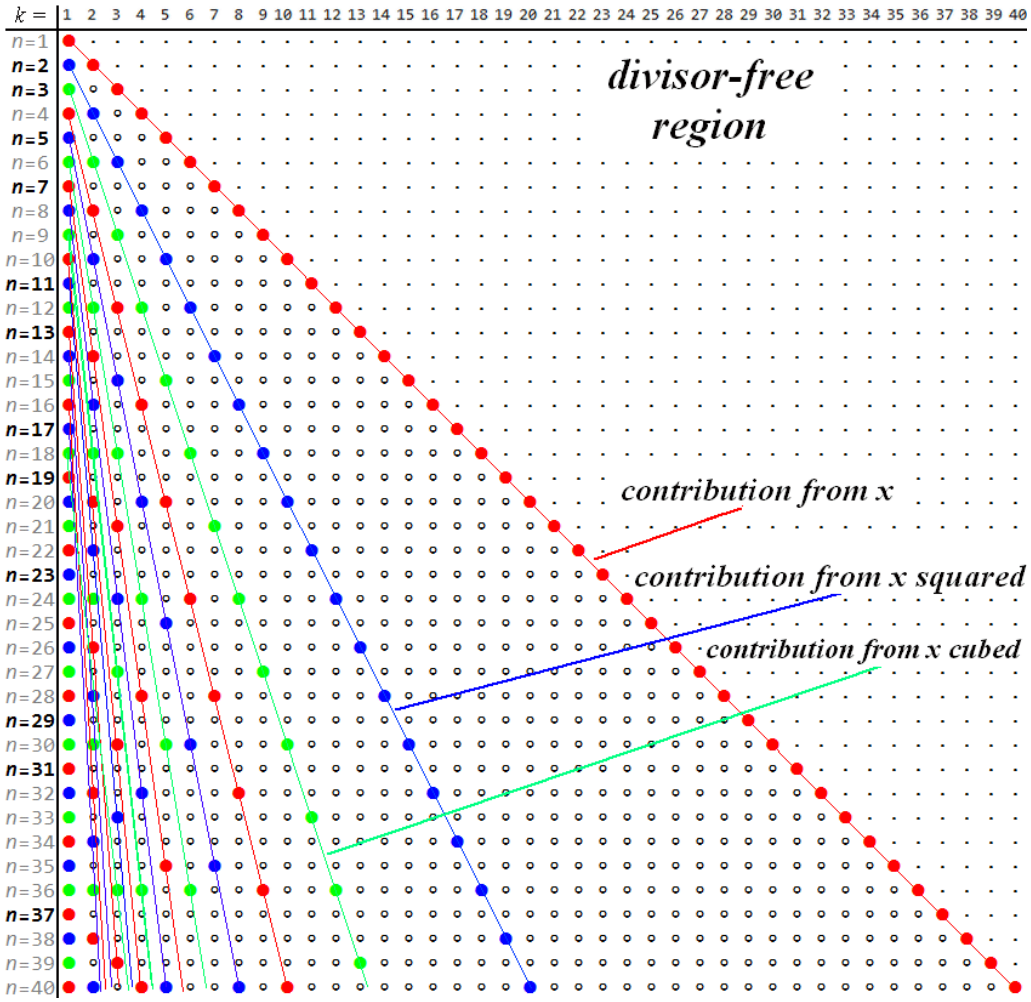

Fig. 2. Distribution of divisors of integers computed from α_{nk}. This figure illustrates how various terms in the sum (27) contribute to the whole pattern of divisors. Each term corresponds to a ray of divisors. Rows are labelled by consecutive integers n and columns are labelled by potential divisors k. Each colored disc means that given k actually divides n, otherwise there is a small black circle. To better visualize the whole pattern, discs are in 3 different colors and lines connecting them are drawn. Of course, above the diagonal ($k > n$) there cannot be any divisors

V. INTERPRETATION

Let us now explain in more detail how it all works. The thing is that all formulas for α_{nk} presented so far encode, at least to some extent, the ancient algorithm known as the sieve of Eratosthenes.

Indeed, consider as $f(x)$ the function $f(x) = x/(1-x)$ and let us temporarily restrict ourselves to the linear case: $f(x) \approx x$. According to the general formula (18) we have

$$\begin{aligned} \alpha_{nk} =& \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(x^k\right)\Big|_{x=0} = \\ =& \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n} x^k\Big|_{x=0} = \binom{k}{n} x^{k-n}\Big|_{x=0} = \delta_{k,n} \end{aligned} \tag{25}$$

and this produces a single line of ones on the diagonal $n = k$ in the divisor matrix (11) – cf. Fig. 2 above. This is equivalent to the trivial statement that all integers are divisible both by one and by themselves. Let us further consider more precise approximation $f(x) \approx x + x^2$. We get from (18) another sequence of ones on the line $n = 2k$. This is equivalent to selecting all even integers n and adding to the divisor matrix (11) their divisors $n/2$. Taking into account higher powers of x we select all numbers n which are multiplies of $3, 4, 5, \ldots$ and this adds to the matrix further lines of divisors: $n/3, n/4, n/5$, respectively.

Proceeding in the same way we finally arrive at the full expansion of $f(x)$:

$$f(x) = \frac{x}{1-x} = \sum_{j=1}^{\infty} x^j \tag{26}$$

which produces the entire sequence of lines $n = jk$ labelled by parameter $j = 1, 2, 3, \ldots$. In this way we have selected and

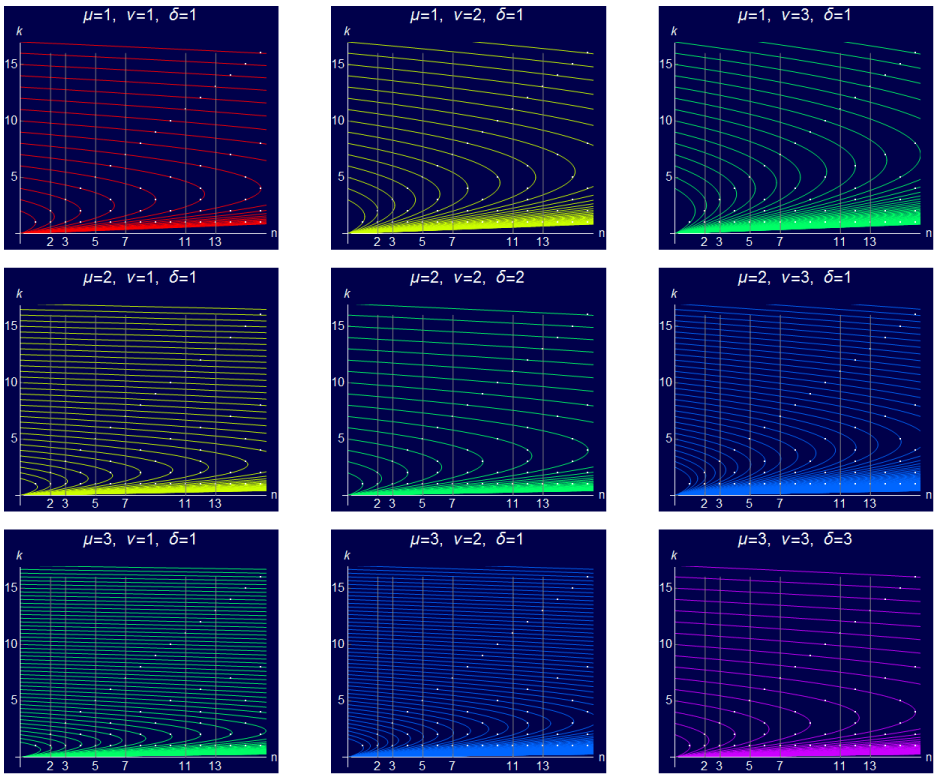

Fig. 3. Various families of parabolas (31) for $\mu = 1, 2, 3$ and $\nu = 1, 2, 3$. Step δ (32) described in the main text is also indicated

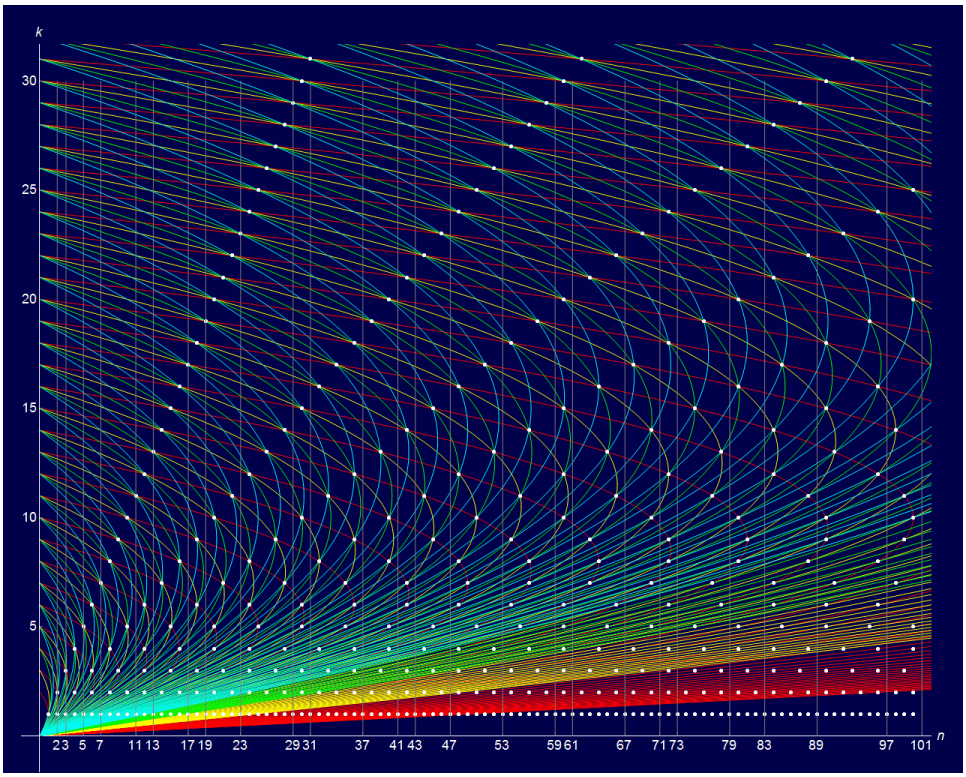

Fig. 4. Family of parabolas (31) for $\mu = 1$ and $\nu = 1$ (red), 2 (orange), 3 (green),and 4 (cyan) for $n < 100$. For clarity of the plot parameter i assumes only 50 consecutive values. Prime numbers among ns are indicated by vertical lines

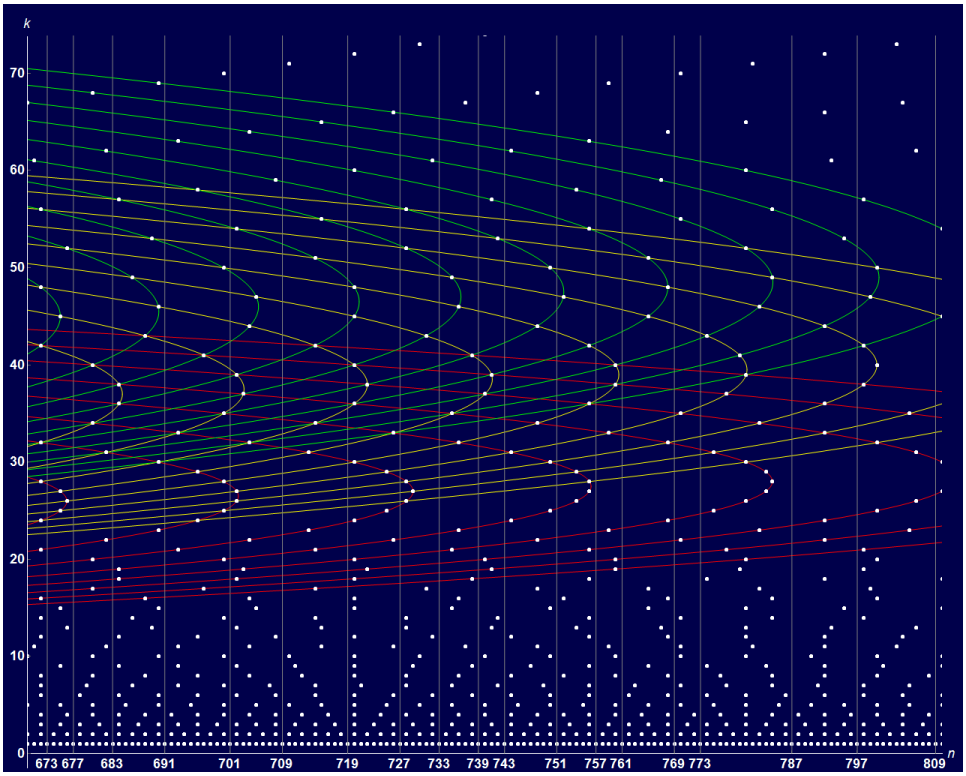

Fig. 5. Family of parabolas (31) for $\mu = 1$ and $\nu = 1$ (red), 2 (yellow) and 3 (green) around $n = 740$. For clarity of the plot parameter i assumes only 5 consecutive values. Prime numbers among ns are indicated by vertical lines

visualized *all divisors for all integers*. It is clear that there are certain well-defined numbers n (marked in bold in Fig. 2) which have exactly two divisors: unity and themselves, i.e. prime numbers: $2, 3, 5, 7, 11, 13, ...$ At the same time, we see the importance of condition $c_i \neq 0$ in (17) since even a single coefficient $c_i = 0$ would cause a skipping of certain divisors. In view of this the characteristic function for divisors may also be written in a very natural form as a sum over Kronecker deltas:

$$\alpha_{nk} = \sum_{j=1}^{n} \delta_{jk,n} \tag{27}$$

Note that combining (12), (18) and (26) gives:

$$\sigma_0(n) := \sum_{k=1}^{n} \alpha_{nk} = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \sum_{k=1}^{\infty} \frac{x^k}{1-x^k} \bigg|_{x=0} \tag{28}$$

Hence

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x^k} = \sum_{n=1}^{\infty} \sigma_0(n) x^n \tag{29}$$

which is consistent with the theory of Lambert series (see e.g. [2]) which is the generating function for the sequence $\sigma_0(n)$ where $\sigma_0(n)$ is the total number of divisors for a given integer n.

VI. CONCLUDING REMARKS

Let us add some elementary comments at the end of this note. As we have seen, all divisors k of integers n lie on rays passing through the origin of the coordinate system on the (n, k) plane and are labelled by an integer parameter $j = 1, 2, 3...$

$$n = jk \tag{30}$$

We have also seen that this simple condition has a natural interpretation since j may be identified with the exponent in x^j in the expansion (17). The key thing is that these rays must pass through certain points of an integer lattice and only then can a potential divisor be an actual divisor. For large n these rays typically get closer and closer to one another. Therefore, we qualitatively see why it is so difficult to factorize large integers.

Moreover, numerical experiments suggest that *all* divisors lie on countable families of parabolas passing through the origin (see Figures 3, 4 and 5 below). These parabolas are "quantized" in the sense that each family is characterized by two discrete parameters $\mu = 1, 2, 3...$ and $\nu = 1, 2, 3...$ and inside any family parabolas are labelled by another integer parameter i:

$$g_i^{(\mu\nu)}(k) = \frac{\mu}{\nu} k^2 - \frac{i}{\nu} k \tag{31}$$

Careful simulations using Mathematica revealed that parameter i assumes equidistant values with integer constant step:

$$\delta = \gcd(\mu, \nu) \tag{32}$$

starting from $i = \mu + \nu$ where $\gcd$ denotes greatest common divisor, i.e. $i = \mu + \nu,\ \mu + \nu + \delta,\ \mu + \nu + 2\delta, ...$

As far as I am aware the unexpected parabolas in the distribution of divisors have been independently noticed by Jeffrey Ventrella (see his popular book [7], page 33) but with no quantitative considerations.

Finally, it should be stressed that, unfortunately, expressions presented in this note do not tell us much about distribution of primes. They are even not very suitable for numerical calculations for large n, therefore they may be treated merely as a curiosity. Nevertheless, we have shown some unexpected relationship between the number theory and calculus.

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The results presented in this paper were inspired by experimenting with Wolfram Mathematica. Also all calculations were checked using this powerful software.

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