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Formulation and Solution of Space-Time Fractional KdV-Burgers Equation

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Abstract: The space-time fractional KdV-Burgers equation has been derived using the semi-inverse method and Agrawal's variational method. The modified Riemann-Liouville definition is used for the fractional differential operators. The derived fractional equation is solved using the fractional sub-equation method.

Key words: Fractional Euler-Lagrange equation, space-time fractional KdV-Burgers equation, modified Riemann-Liouville fractional definition, fractional sub-equation method

I. INTRODUCTION

All forces in nature are nearly non-conservative: dissipative and/or dispersive forces. Classical mechanics, using integer differential equations, treated conservative forces while the non-integer differential equations can be used to describe the non-conservative forces. Fractional calculus is a field of mathematics that grows out of the traditional definitions of calculus. Fractional calculus has gained importance during the last decades mainly due to its applications in various fields of science and engineering. Some of the areas of present day applications of fractional calculus include fluid flow, rheology, dynamical process in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, viscoelasticity, electro-chemistry of corrosion, chemical physics, optics, and signal processing, and so on [1-7].

There are different kinds of fractional integration and differentiation operators. The most famous one is the Riemann-Liouville definition [8-11], which has been used in various fields of science and engineering successfully, but this definition leads to the result that constant function differentiation is not zero. Caputo put definitions which give zero value for fractional differentiation of constant function, but these definitions require that the function should be smooth and differentiable [8-11]. Recently, Jumarie derived definitions for the fractional integral and derivative called modified Riemann-Liouville [12-15], which are suitable for continuous and nondifferentiable functions and give differentiation of a constant function equal to zero. The modified Riemann-Liouville fractional definitions are used effectively in many different problems [16-20].

It was shown that non-integer derivatives in the Lagrangian describe non-conservative forces. Riewe [21, 22] derived a method using a fractional Lagrangian that leads to a fractional Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion. Hamilton's equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. Further study of the fractional Euler-Lagrange can be found in the works of Agrawal [23-25]. He presented generalized Euler-Lagrange equations for unconstrained and constrained fractional variational problems. Baleanu and coworkers [26, 27] used the fractional Euler-Lagrange equation to model fractional Lagrangian and Hamiltonian formulations. El-Wakil et al derived the time fractional forms of some mathematical-physics equations [28] using Agrawal's variational method [23-25] and used them to describe the electrostatic potential in some plasma systems [29].

Several methods have been used to solve fractional differential equations such as: Laplace transformation method, Fourier transformation method, iteration method, and operational method [8-11, 30]. However, most of these methods are suitable for special types of fractional differential equations, namely the linear with constant coefficients. However, some papers deal with the existence and multiplicity of solution of the nonlinear fractional differential equation using techniques of nonlinear analysis such as: Adomian decomposition method [31], homotopy perturbation method [32] and variational iteration method [33].

It is common knowledge that many physical problems (such as non-linear shallow-water waves and wave motion in plasmas) can be described by the KdV-type equations. The KdV-type equations also have applications in quantum field theory, plasma physics and solid-state physics. For example, the kink soliton can be used to calculate energy, momentum flow and topological charge in the quantum field. In order to study the problems of the flow of liquids containing gas bubbles, the fluid flow in elastic tubes, etc., the control equation can be reduced to the so-called KdV-Burgers equation. This equation is equal to the KdV equation if a viscous dissipation term is added. The KdV-Burgers equation can be thought of as a generalization of the KdV and Burgers equations. This equation combines nonlinearity, linear dissipation and dispersion terms. This is a well known nonlinear model of viscous elastic medium and is found in many physical phenomena. The Burgers equation is a special case of the KdV-Burgers equation has been found to describe various kinds of phenomena such as a mathematical model of turbulence and the approximate theory of flow through a shock wave traveling in viscous fluid.

In this paper, the space-time fractional KdV-Burgers equation is derived using Agrawal's technique [23-25] and the modified Riemann-Liouville derivative [12-15], and solved by the improved fractional sub-equation method [34, 35].

The modified Riemann-Liouville fractional derivative $D_x^\alpha f(x)$ is defined in the form [12-15]

$$D_x^\alpha[f(x)-f(a)] = \frac{1}{\Gamma(-\alpha)}\int_a^x d\xi\, f(\xi)/(x-\xi)^{\alpha+1}, \quad \alpha<0, \tag{1a}$$

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \times\frac{d}{dx}\left\{\int_a^x \frac{d\xi[f(\xi)-f(a)]}{(x-\xi)^\alpha}\right\},\ 0<\alpha<1, \tag{1b}$$

$$D_x^\alpha f(x) = \frac{d^n}{dx^n}\left\{D_x^{\alpha-n}f(x)\right\}, \quad n\le\alpha<n+1,\ n\ge 1. \tag{1c}$$

Some properties of the modified Riemann-Liouville (mRL) fractional derivative were summarized in [12-15], useful formulae include

$$D_x^\alpha C=0,\ \alpha>0,\ C \text{ is a constant}, \tag{2a}$$

$$D_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha},\ \gamma>0, \tag{2b}$$

$$D_x^\alpha[u(x)\,v(x)] = D_x^\alpha[u(x)]v(x) + u(x)\,D_x^\alpha[v(x)], \tag{3}$$

$$D_x^\alpha f(u(x)) = D_x^\alpha u(x)\left(\frac{df}{du}\right), \tag{4a}$$

$$= D_u^\alpha f(u)\left(\frac{du}{dx}\right)^\alpha, \tag{4b}$$

$$\begin{aligned}&\int_a^b (dx)^\alpha u(x) D_x^\alpha v(x) = \alpha![u(x)v(x)\Big|_a^b\\ &-\int_a^b (dx)^\alpha v(x)D_x^\alpha u(x),\\ &\qquad u(x),v(x)\in[a,\,b],\quad 0<\alpha<1.\end{aligned} \tag{5}$$

$v(x)$ is non-differentiable, $u(x)$ is non-differentiable in (3) and (4a) and differentiable in (4b), and $f(u(x))$ is differentiable in (4a) and non-differentiable in (4b).

II. SPACE-TIME-FRACTIONAL KDV-BURGERS EQUATION FORMULATION

The space-time-fractional KdV-Burgers (STFKdV-Burgers) equation in (1+1)-dimension can be formulated as follows: The regular KdV-Burgers equation has the form

$$\begin{aligned}&\frac{\partial}{\partial t}u(x,t) + Au(x,t)\frac{\partial}{\partial x}u(x,t)\\ &+B\frac{\partial^2}{\partial x^2}u(x,t) + C\frac{\partial^3}{\partial x^3}u(x,t)=0.\end{aligned} \tag{6}$$

Using the potential function $U(x,t)$ where $u(x,t) = U_x(x,t)$ gives the potential equation of the regular KdV-Burgers equation in the form

$$\begin{aligned}&U_{xt}(x,t) + AU_x(x,t)U_{xx}(x,t)\\ &+Bu_{xx}(x,t) + CU_{xxxx}(x,t)=0,\end{aligned} \tag{7}$$

where the subscripts denote partial differentiation of the function with respect to the parameter. The Euler-Lagrange equation of the regular KdV-Burgers equation can be derived using the semi-inverse method [36, 37] as follows:
The functional of the potential equation can be represented by

$$\begin{aligned}J(U) = &\int_R dx\int_T dt\, U(x,t)\Big\{c_1U_{xt}(x,t) + c_2AU_x(x,t)\\ &\times U_{xx}(x,t) + c_3Bu_{xx}(x,t) + c_4CU_{xxxx}(x,t)\Big\},\end{aligned} \tag{8}$$

where c_1, c_2, c_3 and c_4 are constant Lagrangian multipliers. Here R refers to the boundaries of the space domain, while T denotes to the initial and final values of the time. Integrating (8) by parts where $U_x|_R$=$U_t|_T$=$U_{xx}|_R$=0, $u_{xx}(x,t)$

is considered as a fixed function and applying the variation of this functional with respect to $U(x,t)$ lead to

$$\delta J(U)=\int_R dx\int_T dt\Big\{-c_1\Big[U_t(x,t)\delta U_x(x,t)+U_x(x,t)\delta U_t(x,t)\Big]-\frac{3}{2}c_2A[U_x(x,t)]^2\delta U_x(x,t)+c_3Bu_{xx}(x,t)\delta U(x,t)+2c_4C\left[U_{xx}(x,t)\delta U_{xx}(x,t)\right]\Big\}.$$

Integrating by parts using $U_x|_R=U_t|_T=U_{xx}|_R=0$ and optimizing this variation, $\delta J(U)=0$, give

$$2c_1U_{xt}(x,t)+3c_2AU_x(x,t)U_{xx}(x,t)+c_3Bu_{xx}(x,t)+2c_4CU_{xxxx}(x,t)=0.$$

Comparing the above equation with (7) gives constant Lagrangian multipliers as

$$c_1=\frac{1}{2},\quad c_2=\frac{1}{3},\quad c_3=1,\quad c_4=\frac{1}{2}.$$

The functional relation yields directly the Lagrangian of the potential equation as

$$L(U,U_t,\,U_x,\,U_{xx})=\frac{1}{2}U_t(x,t)U_x(x,t)+\frac{1}{6}A\left[U_x(x,t)\right]^3-BU(x,t)u_{xx}(x,t)-\frac{1}{2}C\left[U_{xx}(x,t)\right]^2. \tag{9}$$

Similarly, the Lagrangian of the space-time-fractional version of the KdV-Burgers equation could be written in the form

$$\begin{aligned}&F(U,D_t^\alpha U,\,D_x^\beta U,\,D_x^{\beta\beta}U)=\\&=\frac{1}{2}D_t^\alpha U(x,t)D_x^\beta U(x,t)+\frac{1}{6}A[D_x^\beta U(x,t)]^3\\&-BU(x,t)D_x^{\beta\beta}u(x,t)-\frac{1}{2}C[D_x^{\beta\beta}U(x,t)]^2,\end{aligned} \tag{10}$$

where $D_z^{\gamma\gamma}f(z)=D_z^\gamma[D_z^\gamma f(z)]$ while the fractional derivative $D_z^\gamma f(z)$ is the mRL fractional derivative [12-15] defined by (1). The functional of the STFKdV-Burgers equation takes the form

$$J_F(U)=\int_R(dx)^\beta\int_T(dt)^\alpha F\left(U,D_t^\alpha U,\,D_x^\beta U,\,D_x^{\beta\beta}U\right). \tag{11}$$

The variation of this functional with respect to $U(x,t)$ leads to

$$\delta J_F(U)=\int_R(dx)^\beta\int_T(dt)^\alpha\Big\{\left(\frac{\partial F}{\partial U}\right)\delta U+\left(\frac{\partial F}{\partial D_t^\alpha U}\right)\times\delta D_t^\alpha U+\left(\frac{\partial F}{\partial D_x^\beta U}\right)\delta D_x^\beta U+\left(\frac{\partial F}{\partial D_x^{\beta\beta}U}\right)\delta D_x^{\beta\beta}U\Big\}. \tag{12}$$

Integrating this equation by parts using the definition (5) and optimizing this relation, $\delta J_F(U)=0$, the Euler-Lagrange equation for the STFKdV-Burgers equation has the form

$$\begin{aligned}&\left(\frac{\partial F}{\partial U}\right)-D_t^\alpha\left(\frac{\partial F}{\partial D_t^\alpha U}\right)\\&-D_x^\beta\left(\frac{\partial F}{\partial D_x^\beta U}\right)+D_x^{\beta\beta}\left(\frac{\partial F}{\partial D_x^{\beta\beta}U}\right)=0,\end{aligned} \tag{13}$$

with the constraints that $\delta U|_R$=$\delta U|_T$=0.
Substituting the Lagrange of STFKdV-Burgers (10) into this Euler-Lagrange formula gives

$$\begin{aligned}&-BD_x^{\beta\beta}u(x,t)-D_t^\alpha[D_x^\beta U(x,t)]\\&-\frac{1}{2}AD_x^\beta[D_x^\beta U(x,t)]^2-CD_x^{\beta\beta}[D_x^{\beta\beta}U(x,t)]=0.\end{aligned} \tag{14}$$

Substituting $u(x,t)=D_x^\beta U(x,t)$ and using formula (3) lead to

$$\begin{aligned}&D_t^\alpha u(x,t)+Au(x,t)D_x^\beta u(x,t)\\&+BD_x^{\beta\beta}u(x,t)+CD_x^{\beta\beta\beta}u(x,t)=0,\end{aligned} \tag{15}$$

which is the space-time-fractional Koreweg-de Vries-Burgers equation.

III. SPACE-TIME-FRACTIONAL KDV-BURGERS EQUATION SOLUTION

In this section, the STFKdV-Burgers equation will be solved using a fractional sub-equation method [34, 35].

Considering the traveling wave transformations $u(x,t)=\Phi(\xi)$, $\xi=x+vt$, (15) can be reduced to the following nonlinear fractional ordinary differential equation (FODE) using relation (4) for the case of $\beta=\alpha$:

$$\begin{aligned}&v^\alpha\,D_\xi^\alpha\Phi(\xi)+A\Phi(\xi)D_\xi^\alpha\Phi(\xi)\\&+BD_\xi^{\alpha\alpha}\Phi(\xi)+CD_\xi^{\alpha\alpha\alpha}\Phi(\xi)=0.\end{aligned} \tag{16}$$

The fractional sub-equation method [34, 35] assumes solution of this equation as

$$\Phi(\xi)=\sum_{k=0}^{n}a_k\varphi^k(\xi), \tag{17}$$

where $\varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha\varphi(\xi)=\sum_{j=0}^{m}b_j\varphi^j(\xi), \tag{18}$$

where a_k, $k=0$, ..., n are constant coefficients to be determined later and b_j, $j=0$, ..., m are arbitrary coefficients.

Balancing the highest order derivative term and nonlinear term in (16) the value of n can be determined, which has in this problem the value $n=2$.
We suppose that (16) has the following formal solution:

$$\Phi(\xi)=a_0+a_1\varphi(\xi)+a_2[\varphi(\xi)]^2, \tag{19}$$

where a_0, a_1 and a_2 are constant coefficients to be determined and $\varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha \varphi(\xi) = b_0 + b_2\varphi^2(\xi), \tag{20}$$

where b_0 and b_2 are arbitrary coefficients.
Using the generalized Exp-function method via Mittag-Leffler functions, Zhang et al [38] first obtained the following solution of the fractional Riccati equation (20)

$$\varphi_1(\xi) = -\sqrt{-b_0}\tanh_\alpha(\sqrt{-b_0}\xi), \quad b_0 < 0, b_2 = 1, \tag{21a}$$

$$\varphi_2(\xi) = -\sqrt{-b_0}\coth_\alpha(\sqrt{-b_0}\xi), \quad b_0 < 0, b_2 = 1, \tag{21b}$$

$$\varphi_3(\xi) = \sqrt{b_0}\tan_\alpha(\sqrt{b_0}\xi), \quad b_0 > 0, b_2 = 1, \tag{21c}$$

$$\varphi_4(\xi) = -\sqrt{b_0}\cot_\alpha(\sqrt{b_0}\xi), \quad b_0 > 0, b_2 = 1, \tag{21d}$$

$$\varphi_5(\xi) = -\Gamma(1+\alpha)/(\xi^\alpha + \omega), \quad b_0 = 0, b_2 = 1, \quad \omega = \text{constant}, \tag{21e}$$

with generalized hyperbolic and trigonometric functions

$$\tanh_\alpha(x) = \sinh_\alpha(x)/\cosh_\alpha(x), \quad \coth_\alpha(x) = \cosh_\alpha(x)/\sinh_\alpha(x), \tag{22a}$$

$$\sinh_\alpha(x) = [E_\alpha(x) - E_\alpha(-x)]/2, \quad \cosh_\alpha(x) = [E_\alpha(x) + E_\alpha(-x)]/2, \tag{22b}$$

$$\tan_\alpha(x) = \sin_\alpha(x)/\cos_\alpha(x), \quad \cot_\alpha(x) = \cos_\alpha(x)/\sin_\alpha(x), \tag{22c}$$

$$\sin_\alpha(x) = [E_\alpha(ix) - E_\alpha(-ix)]/(2i), \quad \cos_\alpha(x) = [E_\alpha(ix) + E_\alpha(-ix)]/2, \tag{22d}$$

where $i = \sqrt{-1}$ and $E_\alpha(x)$ is the Mittag-Leffler function defined by

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)}. \tag{22e}$$

Substituting (17) along with (18) and setting the coefficient of $\varphi^k(\xi)$ equal to zero lead to a set of algebraic equations in terms of coefficients a_0, a_1, a_2, b_0 and b_2. Solving the algebraic set of equations by Maple gives the following cases:

Case 1: In this case, A and B are arbitrary while $C = 0$. This case describes the Burgers equation. The coefficients of (19), using the Maple package, have the forms:

$$b_0 = \text{arbitrary}, \quad b_2 = 1, \quad a_0 = -v^\alpha/A, \quad a_1 = -2B/A, \quad a_2 = 0, \tag{23}$$

where $A \neq 0$, the solutions of the STFKdV-Burgers equation using the fractional Riccate equation solutions (21) are as follows:

$$\Phi_{11}(\xi) = -v^\alpha/A + (2B/A)\sqrt{-b_0}\tanh_\alpha(\sqrt{-b_0}\xi), b_0 < 0, \tag{24a}$$

$$\Phi_{12}(\xi) = -v^\alpha/A + (2B/A)\sqrt{-b_0}\coth_\alpha(\sqrt{-b_0}\xi), \ b_0 < 0, \tag{24b}$$

$$\Phi_{13}(\xi) = -v^\alpha/A - (2B/A)\sqrt{b_0}\tan_\alpha(\sqrt{b_0}\xi), \quad b_0 > 0, \tag{24c}$$

$$\Phi_{14}(\xi) = -v^\alpha/A + (2B/A)\sqrt{b_0}\cot_\alpha(\sqrt{b_0}\xi), \quad b_0 > 0, \tag{24d}$$

$$\Phi_{15}(\xi) = -v^\alpha/A + (2B/A)\Gamma(1+\alpha)/(\xi^\alpha + \omega), \quad b_0 = 0, \quad \omega = \text{arbitrary constant}. \tag{24e}$$

Case 2: The second case has A and C arbitrary while $B = 0$. This describes the KdV equation. The set of coefficients of the solution of (19) using the Maple package is

$$b_0 = \text{arbitrary}, \ \mathrm{b_2 = 1}, \ \mathrm{a_0 = -(8b_0C + v^\alpha)/A}, \quad a_1 = 0, \quad a_2 = -12C/A, \tag{25}$$

where $A \neq 0$, these coefficients lead to the second set of solutions of the STFKdV-Burgers equation in the following forms

$$\Phi_{21}(\xi) = -(8b_0C + v^\alpha)/A + (12C/A)b_0\left[\tanh_\alpha(\sqrt{-b_0}\xi)\right]^2, \quad b_0 < 0, \tag{26a}$$

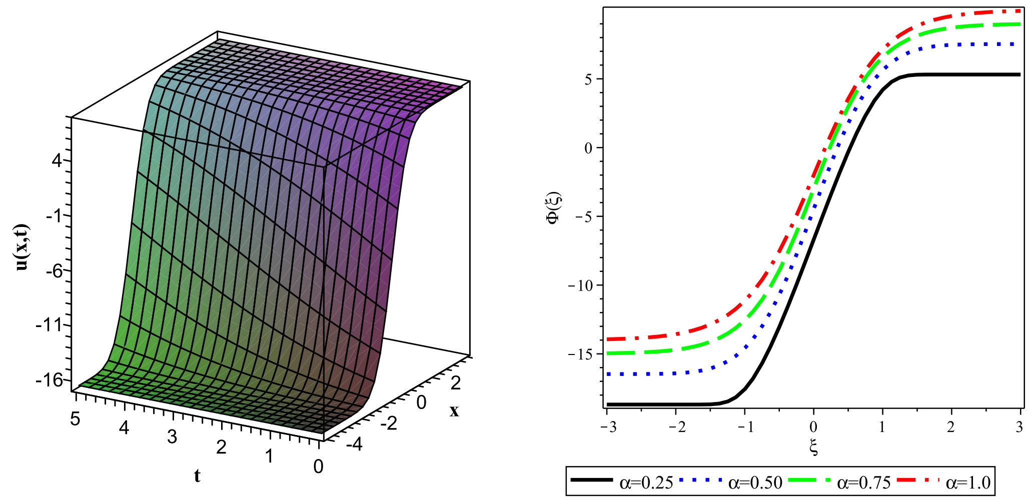
$$\Phi_{22}(\xi) = -(8b_0C + v^\alpha)/A + (12C/A)b_0\left[\coth_\alpha(\sqrt{-b_0}\xi)\right]^2, \quad b_0 < 0, \tag{26b}$$

$$\Phi_{23}(\xi) = -(8b_0C + v^\alpha)/A - (12C/A)b_0\left[\tan_\alpha(\sqrt{b_0}\xi)\right]^2, \quad b_0 > 0, \tag{26c}$$

$$\Phi_{24}(\xi) = -(8b_0C + v^\alpha)/A - (12C/A)b_0\left[\cot_\alpha(\sqrt{-b_0}\xi)\right]^2, \quad b_0 > 0, \tag{26d}$$

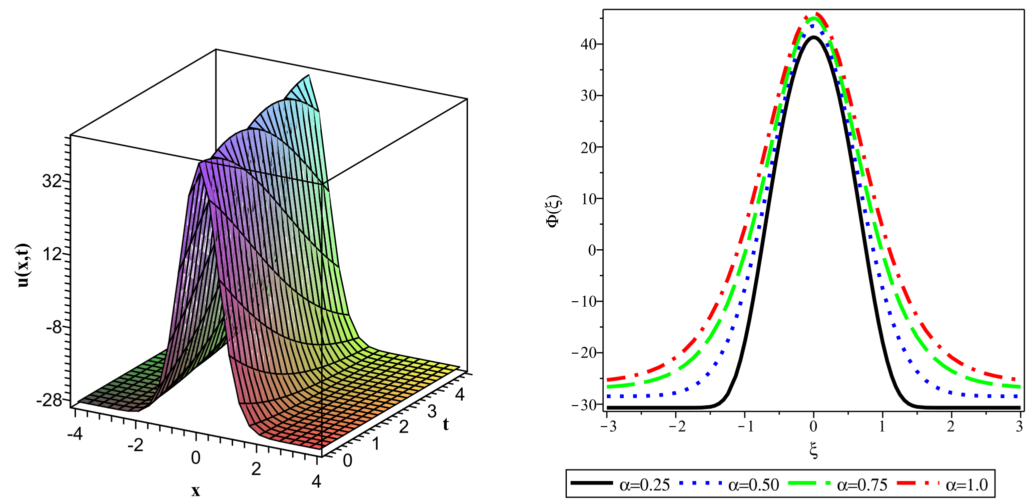
$$\Phi_{25}(\xi) = -(8b_0C + v^\alpha)/A - (12C/A)[\Gamma(1+\alpha)/(\xi^\alpha + \omega)]^2, \quad b_0 = 0, \quad \omega = \text{constant}. \tag{26e}$$

Case 3: The third case has A, B and C arbitrary, which describe the KdV-Burgers equation. The set of coefficients of



(a) 3-dimensions, $\alpha = 0.5$ (b) 2-dimensions

Fig. 1. The solution $\Phi_{11}(\xi)$, $\xi = x + vt$ for $A = 0.1$, $B = 0.6$, $C = 0$, $b_0 = -1$, $v = 0.2$



(a) 3-dimensions, $\alpha = 0.5$ (b) 2-dimensions

Fig. 2. The solution $\Phi_{21}(\xi)$, $\xi = x + vt$ for $A = 0.1$, $B = 0$, $C = 0.6$, $b_0 = -1$, $v = 0.2$

the solution of STFKdV-Burgers equation is given by

$$b_0 = -\frac{B^2}{100C^2}, \quad b_2 = 1, \quad a_0 = \frac{3B^2 - 25v^\alpha C}{25AC}, \quad a_1 = -\frac{12B}{5A}, \quad a_2 = -\frac{12C}{A}, \tag{27}$$

where $AC \neq 0$. As the coefficient b_0 is negative, therefore the solutions corresponding to this set of coefficients are two solutions only and represented by the following forms:

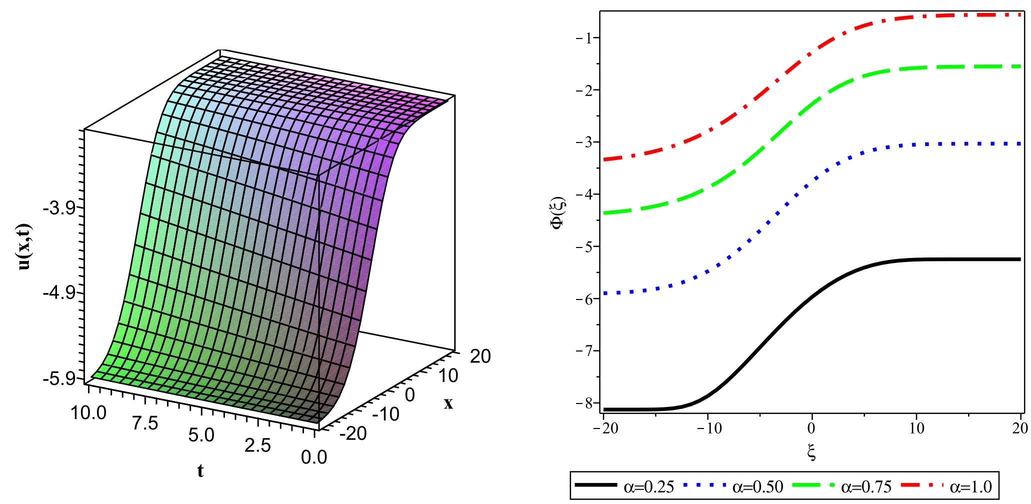
$$\Phi_{31}(\xi) = \frac{3B^2 - 25v^\alpha C}{25AC} + \frac{6B^2}{25AC}\tanh_\alpha\left(\frac{B}{10C}\xi\right) - \frac{3B^2}{25AC}\left[\tanh_\alpha\left(\frac{B}{10C}\xi\right)\right]^2. \tag{28a}$$

$$\Phi_{32}(\xi) = \frac{3B^2 - 25v^\alpha C}{25AC} + \frac{6B^2}{25AC}\coth_\alpha\left(\frac{B}{10C}\xi\right) - \frac{3B^2}{25AC}\left[\coth_\alpha\left(\frac{B}{10C}\xi\right)\right]^2. \tag{28b}$$

IV. RESULTS AND DISCUSSION

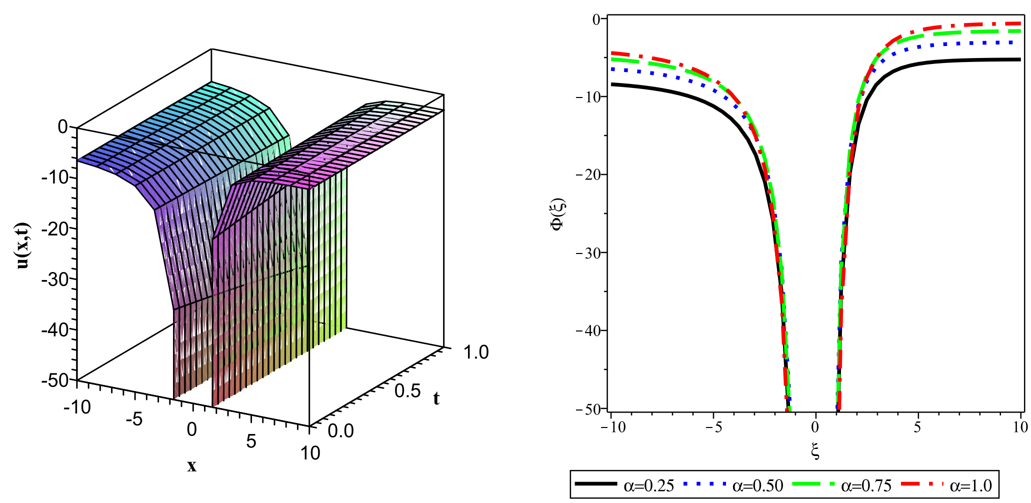
The real world physical processes are modeled by nonlinear partial differential equations. First order nonlinear partial differential equations model nonlinear waves and arise in gas dynamics, water waves, elastodynamics, chemical reactions, transport of pollutants, flood waves in rivers, chromatography, traffic flow, and a wide range of biological and ecological systems. Second order partial differential equations govern nonlinear diffusion processes, including thermodynamics, chemical reactions, dispersion of pollutants, and population dynamics, the simplest and best understood is Burger's equation. Third order partial differential equations arise in the study of dispersive wave motion as the KdV equation, including water waves, plasma waves, waves in elastic media. A generalization of these two equations (KdV and Burgers) known as the KdV-B equation is very useful approximation to describe phenomena in fluid mechanics and plasma physics.

The real world physical processes can be better modeled by fractional differential equations rather by integer-order differential equations. The space-time fractional KdV-Burgers' equations are derived using Agrawal's technique [23-25] and



(a) 3-dimensions, $\alpha = 0.5$ (b) 2-dimensions

Fig. 3. The solution $\Phi_{31}(\xi)$, $\xi = x + vt$ for $A = 0.1$, $B = 0.6$, $C = 0.6$, $b_0 = -\frac{B^2}{100C^2}$, $v = 0.2$



(a) 3-dimensions, $\alpha = 0.5$ (b) 2-dimensions

Fig. 4. The solution $\Phi_{32}(\xi)$, $\xi = x + vt$ for $A = 0.1$, $B = 0.6$, $C = 0.6$, $b_0 = -\frac{B^2}{100C^2}$, $v = 0.2$

the modified Riemann-Liouville derivative which were defined by Jumarie [12-15], and solved at different three cases. The first case takes the dissipation coefficient equal to zero, which tends to Burgers' equation, the second case takes the dispersion coefficient equal to zero, which tends to KdV equation, and the third case takes the dispersion and dissipation coefficient arbitrary that are not equal to zero, using the fractional sub-equation method [34, 35].

Fig. (1a) shows the solution $\Phi_{11}(\xi)$ in 3-dimensions where a shock wave formulated as the dispersion term in the KdV-Burgers equation equals to zero, and Fig (1b) shows the relation of $\Phi_{11}(\xi)$ and the position at different values of the fractional parameter α, where as shown as α increase the amplitude of the shock wave increase. If the dissipation coefficient in the KdV-Burgers equation equals to zero this equation tends to KdV equation, where a soliton solution will be obtained as shown in Fig (2a) the relation between$\Phi_{21}(\xi)$, position, and time. Fig. (2b) shows the relation of $\Phi_{21}(\xi)$ and the position at different values of the fractional parameter α, where a bell shape formulated which increased in the amplitude and width as the fractional parameter increased. Figures (3) and (4) show that at dispersion and dissipation coefficients are not equal to zero, where Fig (3) shows the relation between $\Phi_{31}(\xi)$ and the position in 3-dimensions and 2-dimensions at different values of fractional parameter, respectively. Fig. (4) shows the relation between $\Phi_{32}(\xi)$ and the position in 3-dimensions and 2-dimensions at different values of fractional parameter, respectively when the balance between nonlinear and dispersion effect is strong and can result in the formation of explosive waves which appear in different fields as tsunami.

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