A Note on Scattering in Deformed Space with Minimal Length

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Abstract: We consider the elastic scattering in deformed space with minimal length. We give the basic relation for the elastic scattering in deformed space. We also investigate the partial wave method in deformed space. It is shown that the relations for the scattering amplitude and cross-section formally coincides with ordinary ones

Key words: elastic scattering, minimal length, Coulomb potential

I. INTRODUCTION

In the recent years, a lot of works have been devoted to the quantum mechanical problems in a deformed space with minimal length. Such an interest was motivated by several independent lines of investigation as the string theory [1] and quantum gravity [2] where the existence of a finite lower bound for the possible resolution of length [3] was proposed. Kempf and his collaborators showed that finite resolution of length can be obtained from the deformed commutation relations [4-7]. One should note that deformed commutation relations were introduced earlier by Snyder [8].

The deformed algebra leading to the existence of a minimal length in a *D*-dimensional case takes the following form:

$$[X_i, P_j] = i\hbar\left(\delta_{ij}\left(1 + \beta P^2\right) + \beta' P_i P_j\right), \quad [P_i, P_j] = 0,$$

$$[X_i, X_j] = i\hbar\frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2}\left(P_i X_j - P_j X_i\right),$$

(1)

where β, β' are the deformation parameters. We assume that these quantities are nonnegative $\beta, \beta' \geq 0$. It is easy to show that minimal length equals $\hbar\sqrt{\beta + \beta'}$.

Deformed Heisenberg algebra (1) causes new complications in solving quantum mechanical problems. It is well known that algebra (1) does not have the position represen-

tation [5]. There are just a few known problems for which the energy spectra have been found exactly [9-14].

The several works were devoted to the hydrogen atom in the space with deformed Heisenberg algebra (1). In work [15] the partial case of deformation was considered when $2\beta = \beta'$, the spectrum was found perturbatively. In paper [16] a general case $2\beta \neq \beta'$ was considered, using the perturbation theory the spectrum of the hydrogen atom was found but when $l \neq 0$. For the *s*-states the corrections to the energy levels were found numerically. In work [17] a modified perturbation theory that allowed to obtain the analytical correction to the *s*-levels was proposed. In paper [18] the corrections to the energy for arbitrary *s*-levels of the hydrogen atom were calculated.

The scattering problem in deformed space was investigated in [19]. The elastic scattering was considered. Here we carry on the investigation of the scattering problem. We develop the partial wave method and compare it with previous results.

II. SCATTERING AMPLITUDE AND DIFFERENTIAL CROSS-SECTION

To investigate the scattering problem it is essential to introduce the representation of operators satisfying the algebra (1). It is well known that such an algebra has the momentum representation, but we use the approximate

representation fulfilling the algebra in the first order over the deformation parameters [17, 19]:

$$\begin{cases} X_i = x_i + \dfrac{2\beta - \beta'}{4}\left(x_i p^2 + p^2 x_i\right), \\ P_i = p_i + \dfrac{\beta'}{2} p_i p^2, \end{cases} \tag{2}$$

where $p^2 = \Sigma_{j=1}^{3} p_j^2$ and the operators x_i, p_j satisfy the canonical commutation relation. The position representation $x_i = x_i$, $p_j = i\hbar \partial / \partial x_j$ can be taken for the ordinary Heisenberg algebra. We note that in the special case $2\beta = \beta'$, the position operators commute in linear approximation over the deformation parameters, i.e. $[X_i, X_j] = 0$.

The Schrödinger equation with arbitrary potential $U(\mathbf{R})$ in canonical variables takes the following form:

$$\left(\frac{p^2}{2m} + \frac{\beta' p^4}{2m} + U(\mathbf{r}, \mathbf{p})\right)\Psi = E\Psi. \tag{3}$$

We suppose that $U(\mathbf{r}, \mathbf{p}) \to 0$ when $r \to \infty$ and motion of a scattered particle at large distances form the scattering center is free. The kinetic energy of a free particle reads:

$$E = \frac{\hbar^2 k^2}{2m}\left(1 + \beta' \hbar^2 k^2\right), \tag{4}$$

where $\mathbf{k}$ is the wave vector of an incident particle and $\mathbf{P} = \hbar\mathbf{k}(1 + \beta' \hbar^2 k^2 / 2)$ is the momentum of the particle. We investigate the elastic scattering so we have that after the scattering $k = k'$, where $\mathbf{k}'$ is the wave vector of a scattered particle.

Then we can write a formal solution of equation (3) in the form:

$$\Psi(\mathbf{r}) = \psi_k(\mathbf{r}) + \int G(\mathbf{r}, \mathbf{r}') \frac{2m}{\hbar^2} U(\mathbf{r}', \mathbf{p}')\Psi(\mathbf{r}')d\mathbf{r}', \tag{5}$$

where $G(\mathbf{r}, \mathbf{r}')$ is the Green's function.

As it was shown in paper [19], the asymptotic Green's function takes the following form:

$$G\left(|\mathbf{r} - \mathbf{r}'|\right) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|(1 + 2\beta'\hbar^2 k^2)} e^{ik|\mathbf{r} - \mathbf{r}'|}. \tag{6}$$

Using this Green's function (6) and after some simplifications we rewrite equation (5) in the form:

$$\Psi(\mathbf{r}) = \psi_k(\mathbf{r}) - \frac{m}{2\pi\hbar^2(1 + 2\beta'\hbar^2 k^2)} \times$$
$$\times \frac{e^{ikr}}{r}\int e^{-i\mathbf{k}'\mathbf{r}'} U(\mathbf{r}', \mathbf{p}')\Psi(\mathbf{r}')d\mathbf{r}', \tag{7}$$

where $\mathbf{k}' = k\mathbf{n}$.

Equation (7) can be represented in the form:

$$\Psi(\mathbf{r}) = e^{i\mathbf{k}\mathbf{r}} + \frac{e^{ikr}}{r} f, \tag{8}$$

where

$$f = -\frac{m}{2\pi\hbar^2(1 + 2\beta'\hbar^2 k^2)}\int e^{-i\mathbf{k}'\mathbf{r}'} U(\mathbf{r}', \mathbf{p}')\Psi(\mathbf{r}')d\mathbf{r}' \tag{9}$$

is the scattering amplitude.

As we see the first term in equation (8) corresponds to the wave function of an incident particle and the second term corresponds to the wave function of a scattered particle.

It is well known that the central problem of the scattering theory is the calculation of the differential cross-section. It was shown in [19] that differential cross-section in deformed space takes the same form as in the ordinary quantum mechanics:

$$\frac{d\sigma}{d\Omega} = |f|^2. \tag{10}$$

The last equation allows to calculate the differential cross-sections for an arbitrary potential of scattering. But as we can see from expression (9), the scattering amplitude and, as a consequence, differential cross-section can not be calculated exactly. It is necessary to use the approximate calculation. In the Born approximation we consider the scattering potential as a small perturbation and solve equation (7) by the method of successive approximation. In the first approximation we substitute the plane wave $\psi_k(\mathbf{r}) = \exp(i\mathbf{k}\mathbf{r})$ in relation (9).

As shown in [19], the differential cross-section for the Coulomb potential $U = e^2 / R$, where $R = \sqrt{\Sigma_i X_i^2}$ in deformed space takes the form:

$$\frac{d\sigma}{d\Omega} = \frac{m^2 e^4}{4\hbar^4 k^4 \sin^4\dfrac{\vartheta}{2}} + \frac{me^2}{\hbar^2 k^2 \sin^2\dfrac{\vartheta}{2}}\left(\frac{me^2}{2}(2\beta - \beta')\times\right.$$

$$\times\left[\ln\left(\hbar^2(2\beta - \beta')k^2 \sin^2\frac{\vartheta}{2}\right) + 2\gamma - 1 - \frac{1}{2\sin^2\dfrac{\vartheta}{2}}\right] + \tag{11}$$

$$\left.-\beta'\frac{me^2}{\sin^2\dfrac{\vartheta}{2}}\right).$$

The first term in (11) is the ordinary differential cross-section and other terms are caused by deformation of com-

mutation relations. As we see, the corrections to the differential cross-section nonanalytically depend on the deformation parameters, but when the deformation parameters tend to zero the corrections also tend to zero and we return to the ordinary differential cross-section. We also note that as in ordinary quantum mechanics the scattering amplitude for the particle in the Coulomb potential is ill defined in the Born approximation, so it is necessary to calculate the scattering amplitude for the Yukawa potential $U(R) = -e^2 \exp(\lambda R)/R$ and then one should tend the parameter λ to zero.

III. PARTIAL WAVE METHOD

When the potential of scattering is spherically symmetric, the angular momentum of scattering particle is the integral of motion, so it should note that particles with different orbital quantum numbers are scattered independently. Then as in ordinary quantum mechanics we can represent the scattering cross-section as a sum of partial cross-sections for the fixed values of the orbital quantum numbers l. So we assume that we have a spherically symmetric scattering potential, and consider the scattering process on this potential. The arbitrary solution of the Shrödinger equation can be represented in the form:

$$\Psi(\mathbf{r}) = \sum_{l=0}^{+\infty} A_l R_{kl}(r) P_l(\cos\vartheta), \qquad (12)$$

where $R_{nl}(r)$ are the radial wave functions and $P_l(\cos\vartheta)$ are the Legendre polynomials.

Let us first consider free particles: the Shrödinger equation for a free particle:

$$-\frac{\hbar^2 \nabla^2}{2m} \left(1 - \beta' \hbar^2 \nabla^2\right)\psi = \frac{\hbar^2 k^2}{2m}\left(1 + \beta' \hbar^2 k^2\right)\psi, \quad (13)$$

or in the equivalent form:

$$\left(\nabla^2 + k^2\right)\left(1 - \beta' \hbar^2\left(\nabla^2 - k^2\right)\right)\psi = 0. \qquad (14)$$

We represent the solution of the last equation as sum of two functions ψ_1 and ψ_2 which are the solutions of the following equations:

$$\left(\nabla^2 + k^2\right)\psi_1 = 0, \qquad (15)$$

$$\left(1 - \beta' \hbar^2\left(\nabla^2 - k^2\right)\right)\psi_2 = 0. \qquad (16)$$

The radial part of wave function ψ_1 takes the well known form:

$$R_{kl}(r) = \frac{1}{k^l}\sqrt{\frac{2}{\pi}}\, r^l \left(-\frac{1}{r}\frac{d}{dr}\right)^l \frac{\sin kr}{r}. \qquad (17)$$

Equation (16) gives us an unphysical solution and we reject it.

The asymptotic of solution (17) at large distances reads:

$$R_{kl}(r) = \sqrt{\frac{2}{\pi}}\, \frac{\sin\left(kr - l\frac{\pi}{2}\right)}{r}. \qquad (18)$$

Similarly to ordinary quantum mechanics we suppose that asymptotic of the wave function for the scattered particle is the same and differs from (18) by the phase shift only:

$$\overline{R}_{kl}(r) = \sqrt{\frac{2}{\pi}}\, \frac{\sin\left(kr - l\frac{\pi}{2} + \delta_l\right)}{r}, \quad r \to \infty. \qquad (19)$$

We use decomposition (12) and instead of function $R_{kl}(r)$ we take wave function (18) for the free particle and function (19) for the particle in the external field. Then we substitute the aforementioned decompositions in (8), so we obtain:

$$\psi_{\text{scatt}} = f\frac{e^{ikr}}{r} = \frac{1}{kr}\sum_{l=0}^{+\infty} i^l (2l+1) P_l(\cos\vartheta) \times$$
$$\times \left[C_l \sin\left(kr - l\frac{\pi}{2} + \delta_l\right) - \sin\left(kr - l\frac{\pi}{2}\right)\right]. \qquad (20)$$

After simple transformations we obtain:

$$f = \frac{i}{2k}\sum_{l=0}^{+\infty} (2l+1)\left(1 - e^{2i\delta_l}\right) P_l(\cos\vartheta). \qquad (21)$$

The last relation formally coincides with the expression for the scattering amplitude in the ordinary quantum mechanics, but we note that wave number k and phase shift δ_l depend on the deformation parameters.

Using relation (10) and after integration over all spatial angles we have:

$$\sigma = \frac{\pi}{k^2}\sum_{l=0}^{+\infty} (2l+1)\sin^2 \delta_l. \qquad (22)$$

Similarly to in ordinary quantum mechanics, the scattering cross-section relates to the imaginary part of the scattering amplitude in the forward direction, so we have:

$$\sigma = \frac{4\pi}{k}\operatorname{Im} f(0). \qquad (23)$$

The partial wave method also allows to obtain the phase shifts. Using the radial functions $\chi_{kl}(r) = rR_{kl}(r)$ and $\overline{\chi}_{kl}(r) = r\overline{R}_{kl}(r)$ for the free and scattered particles, respectively, we find:

$$
\chi_{kl}\frac{\partial \overline{\chi}_{kl}}{\partial r} - \overline{\chi}_{kl}\frac{\partial \chi_{kl}}{\partial r} - \beta'\hbar^2\left(\chi_{kl}\frac{\partial^3 \overline{\chi}_{kl}}{\partial r^3} - \overline{\chi}_{kl}\frac{\partial^3 \chi_{kl}}{\partial r^3} - \right.
$$
$$
\left. \frac{\partial \chi_{kl}}{\partial r}\frac{\partial^2 \overline{\chi}_{kl}}{\partial r^2} + \frac{\partial \overline{\chi}_{kl}}{\partial r}\frac{\partial^2 \chi_{kl}}{\partial r^2} \right) + 2\beta'\hbar^2\frac{l(l+1)}{r^2}\times \quad (24)
$$
$$
\left(\chi_{kl}\frac{\partial \overline{\chi}_{kl}}{\partial r} - \overline{\chi}_{kl}\frac{\partial \chi_{kl}}{\partial r} \right) = \int_0^r \chi_{kl}(r')\, U\, \overline{\chi}_{kl}(r')\, dr'.
$$

It is a very complicated problem to solve the last equation, but what we find most interesting is the phase shift at large distances from the scattering center, so we can use asymptotic wave functions (18) and (19). Taking into account the fact that the last term in the left hand side of relation (24) goes to zero at large distances from the scatterer we obtain:

$$
\sin\delta_l = \frac{-1}{k(1+2\beta'\hbar^2 k^2)}\times
$$
$$
\times \int_0^r \sin\left(kr' - l\frac{\pi}{2}\right) U(\mathbf{r}',\mathbf{p}') \sin\left(kr' - l\frac{\pi}{2} + \delta_l\right) dr'. \quad (25)
$$

So far as we consider the phase shifts at large distances from the scattering center we can tend to infinity the upper bound in integral (25). In the first approximation we can put in the right hand side of (25) asymptotic wave function of free particle $\chi_{kl}(r)$ instead of $\overline{\chi}_{kl}(r)$, so we have:

$$
\sin\delta_l = \frac{-1}{k(1+2\beta'\hbar^2 k^2)}\times
$$
$$
\times \int_0^{+\infty} \sin\left(kr' - l\frac{\pi}{2}\right) U(\mathbf{r}',\mathbf{p}') \sin\left(kr' - l\frac{\pi}{2}\right) dr'. \quad (26)
$$

Relation (26) corresponds to the first Born approximation in relation (9). To obtain a more accurate estimation it is necessary to use expressions (25), but in practice we obtain sufficiently good estimation in the Born approximation.

IV. CONCLUSIONS

In this paper, the elastic scattering in deformed space with minimal length was considered. We reviewed the main results obtained earlier. One should note that differential cross-section in deformed case equals to the square of module of the scattering amplitude, so we have the same expression as in the ordinary quantum mechanics. As shown, the correction to the differential cross-section for the Coulomb potential nonanalytically depends on deformation parameters but it tends to zero when the parameters go to zero. This dependence is caused by the shifted expansion for the distance operator. Then we also considered the partial wave method in deformed space. It was shown that in deformed case scattering on spherically symmetric potential is similar to the ordinary one. The particles that have different angular momentum (different orbital quantum numbers *l*) are scattered independently. So the expression for the scattering amplitude as a sum of partial amplitudes and the relation for cross-section formally coincide with the corresponding relations in ordinary quantum mechanics.

References

[1] D.J. Gross, P.F. Mende, *String theory beyond the Planck scale.* Nucl. Phys. B 303, 407-454 (1988).

[2] M. Maggiore, *A generalized uncertainty principle in quantum gravity.* Phys. Lett. B 304, 65-69 (1993).

[3] E. Witten, *Reflections on the Fate of Spacetime.* Phys. Today 49, 24 (1996).

[4] A. Kempf, *Uncertainty relation in quantum mechanics with quantum group symmetry.* J. Math. Phys. 35, 4483-4496 (1994).

[5] A. Kempf, G. Mangano, R.B. Mann, *Hilbert space representation of the minimal length uncertainty relation.* Phys. Rev. D 52, 1108-1118 (1995).

[6] A. Kempf, *On quantum field theory with nonzero minimal uncertainties in positions and momenta.* J. Math. Phys. 38, 1347-1372 (1997).

[7] A. Kempf, *Non-pointlike particles in harmonic oscillators.* J. Phys. A 30, 2093-2101 (1997).

[8] H.S. Snyder, *Quantized Space-Time.* Phys. Rev. 71, 38-41 (1947).

[9] C. Quesne, V.M. Tkachuk, *Harmonic oscillator with nonzero minimal uncertainties in both position and momentum in a SUSYQM framework.* J. Phys. A 36, 10373-10389 (2003).

[10] C. Quesne, V.M. Tkachuk, *More on a SUSYQM approach to the harmonic oscillator with nonzero minimal uncertainties in position and/or momentum.* J. Phys. A 37, 10095-10113 (2004).

[11] L.N. Chang, D. Minic, N. Okamura, T. Takeuchi, *Exact solution of the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations.* Phys. Rev. D 65, 125027 (8 pages) (2002).

[12] I. Dadić, L. Jonke, S. Meljanac, *Harmonic oscillator with minimal length uncertainty relations and ladder operators.* Phys. Rev. D 67, 087701 (4 pages) (2003).

[13] C. Quesne, V.M. Tkachuk, *Dirac oscillator with nonzero minimal uncertainty in position.* J. Phys. A 38, 1747-1765 (2005).

[14] T.V. Fityo, I.O. Vakarchuk, V.M. Tkachuk, *One-dimensional Coulomb-like problem in deformed space with minimal lengt.* J. Phys. A 39, 2143-2149 (2006).

[15] F. Brau, *Minimal length uncertainty relation and the hydrogen atom.* J. Phys. A 32, 7691-7696 (1999).

[16] S. Benczik, L.N. Chang, D. Minic, T. Takeuchi, *Hydrogen-atom spectrum under a minimal-length hypothesis.* Phys. Rev. A 72, 012104 (4 pages) (2005).

[17] M.M. Stetsko, V.M. Tkachuk, *Perturbation hydrogen-atom spectrum in deformed space with minimal length.* Phys. Rev. A 74, 012101 (5 pages) (2006).

[18] M. M. Stetsko, *Corrections to the ns levels of the hydrogen atom in deformed space with minimal length.* Phys. Rev. A 74, 062105 (5 pages) (2006); M.M. Stetsko, *Erratum: Corrections to the ns levels of the hydrogen atom in deformed space with minimal length.* [Phys. Rev. A 74, 062105 (2006)], Phys. Rev. A 78, 029907 (1 page) (2008).

[19] M.M. Stetsko, V.M. Tkachuk, *Scattering problem in deformed space with minimal length.* Phys. Rev. A 76, 012707, (7 pages) (2007).

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