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Are the Stieltjes constants irrational? Some computer experiments

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Abstract: Khnichin's theorem is a surprising and still relatively little known result. It can be used as a specific criterion for determining whether or not any given number is irrational. In this paper we apply this theorem as well as the Gauss-Kuzmin theorem to several thousand high precision (up to more than 53 000 significant digits) initial Stieltjes constants γ_n, $n = 0, 1, 2, \ldots, 5000$ in order to confirm that, as is commonly believed, they are irrational numbers (and even transcendental). We also study the normality of these important constants.

Key words: Continued fractions, Riemann zeta function, Khinchin's theorem, experimental mathematics, normality

I. INTRODUCTION

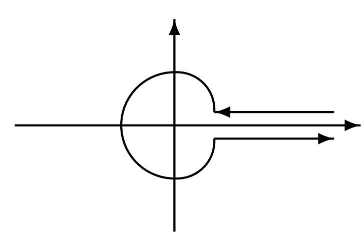
The famous zeta function $\zeta(s)$ discovered by L. Euler in 1737 and published in 1744 [1] as a function of real variable was investigated by G.F.B. Riemann in the complex domain in his famous memoir submitted in 1859 to the Prussian Academy [2]. It is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \Re(s) > 1. \tag{1}$$

It is divergent in the most interesting area of the complex plane, i.e., in the so called critical strip $0 \leq \Re(s) \leq 1$ where all complex zeros of zeta lie. However, as was shown by Riemann, the definition (1) does contain information about the zeta function on the entire complex plane but the process of analytic continuation must be used in order to reveal global behavior of this function. In fact, Riemann in his paper analytically continued Eq. (1) to the whole complex plane except $s = 1$ by means of the following contour integral:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{P}} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}\ , \tag{2}$$

where the integration is performed along the following path $\mathcal{P}$:



Till now dozens of integrals and series representing the $\zeta(s)$ function have been known, for collection of such formulas see, for example, the entry *Riemann Zeta Function* in [3] and references cited therein and [4].

Another representation of this function is given by a power series where certain constants γ_n appear. These constants are essentially coefficients of the Laurent series expansion of the zeta function around its only simple pole at $s=1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n \, (s-1)^n \,. \tag{3}$$

The primary definition of these fundamental constants was found by Th.J. Stieltjes and presented in a letter to Ch. Hermite dated June 23, 1885 [5, letters no. 71–74]

$$\gamma_n = \lim_{m\to\infty} \left[\left(\sum_{k=1}^{m} \frac{(\ln k)^n}{k} \right) - \frac{(\ln m)^{n+1}}{n+1} \right] . \tag{4}$$

It should be noted that when $n=0$ and $k=1$ the numerator in the first summand in Eq. (4) is 0^0 which is an undetermined expression. However, if we accept the convention $0^0 = 1$ then in one formula (4) we can also encode the usual formula for γ_0.

Effective numerical computing of the constants γ_n is quite a challenge because the formulas (4) converge extremely slowly. Even when $n = 0$, which corresponds to the well-known Euler-Mascheroni constant γ_0, in order to obtain just 10 accurate digits one has to sum up exactly 12 366 terms whereas in order to obtain 10 000 digits (which is indeed required in some applications) one would have to sum up an unrealistically large number of terms: nearly $5 \cdot 10^{4342}$, which is of course far beyond capabilities of the present day computers. However, various fast algorithms were found to efficiently compute the specific value of the zeroth Stieltjes constant γ_0, i.e. the fundamental Mascheroni-Euler constant, see e.g. [6, 7]. For $n > 0$ the situation is still worse. Therefore we have to seek for other faster algorithms. In 1992, J.B. Keiper [8] published an effective algorithm based on numerical quadrature of certain integral representation of the zeta function and alternating series summation using Bernoulli numbers. Keiper's algorithm was later implemented in a widely used program *Mathematica*. An efficient but rather complicated method based on Newton-Cotes quadrature was proposed by R. Kreminski in 2003 [9]. Quite recently, F. Johansson has presented a particularly efficient method [10].

The Appendix at the end of the present paper describes yet another method of computing Stieltjes constants which is perhaps not as efficient as Johansson's approach, yet it is by far simpler and may be easily and quickly used in practical calculations for obtaining γ_n up to $n \sim 10\,000$ with

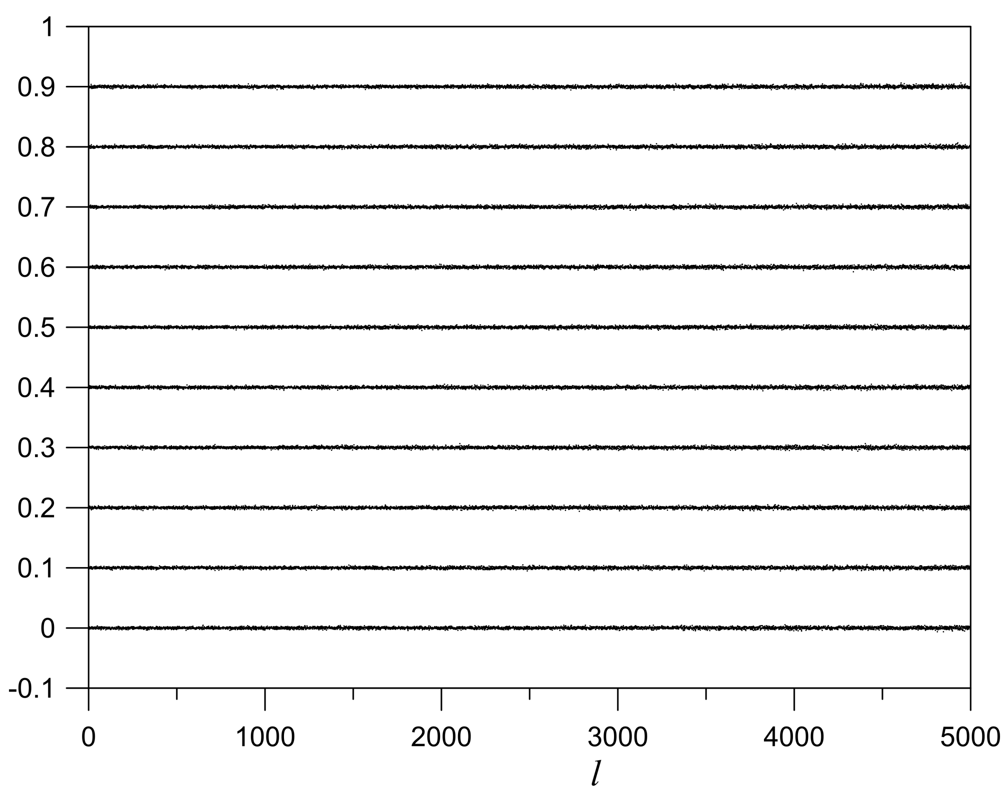


Fig. 1. The plot of the differences between 0.1 and actual frequencies of digits $0, 1, \ldots, 9$ for all 5001 Stieltjes constants. The data for digit a is plotted at y value $a \times 0.1$ for clarity. By zooming in the above figure one can easily discern tiny chaotic oscillations

accuracy $\sim 50\,000$ significant digits. As far as we know, the method presented in the Appendix is currently the most efficient method for very high-precision numerical calculation of the Stieltjes constants.

We proceed as follows. First, we use the algorithm presented in the Appendix to calculate 5001 γ_n with accuracies ranging from about 53 000 significant digits (γ_0) to about 24 000 digits (γ_{5001}). Having these numbers we intend to provide an argument in favor of their irrationality. Then we consider the question of their normality, as real expansions in the base equal 10. Finally, in Sec. III, we develop γ_n's into continuous fractions and next use the remarkable theorems due to Khinchin, Lévy and Gauss-Kuzmin. The obtained results support the common opinion that γ_n are indeed irrational.

II. NORMALITY

Let us recall that a number r is normal in base b if each finite string of k consecutive digits appears in this expansion with asymptotic frequency b^{-k}. In the usual decimal base we have that each digit $0, 1, 2, \ldots, 9$ appears in the expansion of the number r with limiting frequency 0.1, each 2-digits string $00, 01, \ldots, 99$ appears with density 0.01. Having the first 5001 Stieltjes constants with accuracies as described earlier we checked that each digit $0, 1, 2, \ldots, 9$ appears almost exactly with frequency 0.1. It is difficult to represent this 5001×10 data points in one plot. In Fig. 1 we employed the following artifice: the frequency $h_n(0)$ of appearance of digit 0 in the Stieltjes constant γ_n is plotted at x-axis value n with the y value $0.1 - h_n(0)$, i.e. the distance from the expected value 0.1, which in this case of $a = 0$ should be around 0.1. In general, the frequency $h_n(a)$ of appearance of digit a in the Stieltjes constant γ_n is plotted with the y value $a \times 0.1 + (0.1 - h_n(a))$. We also calculated density of 100 strings of two digits $00, 01, \ldots, 99$ for all 5001 Stieltjes constants γ_n. Now the result consisted of half a million points, which is impossible to represent on the plot. Instead, in Tab. 1 we present for each pattern of digits ab the maximal difference between calculated frequency of appearance and the expected value of 0.01 and the number n of the Stieltjes constant γ_n for which this discrepancy appeared. The difference between the actual computed value of the frequency of two digits patterns and the expected value 0.01 was typically of a few percent.

III. CONTINUED FRACTIONS EXPANSIONS

Continued fractions often reveal various profound and unexpected properties of irrational numbers that are normally hidden in their traditional decimal (or other basis) notation, see e.g. [11].

In this Section we exploit three facts about the continued fractions: the existence of the Khinchin constant, Khinchin-Lévy constant and the Gauss-Kuzmin distribution, see e.g. [12, chapter III, §15], [13, §1.8, §2.17], to support the irrationality of Stieltjes constants γ_n. The paper [14] presents the regular continued fraction for the Euler's-Mascheroni constant γ_0. Let

$$r = [a_0(r); a_1(r), a_2(r), a_3(r), \ldots] = = a_0(r) + \cfrac{1}{a_1(r) + \cfrac{1}{a_2(r) + \cfrac{1}{a_3(r) + \ddots}}}, \tag{5}$$

be the continued fraction expansion of the real number r, where $a_0(r)$ is an integer and all denominators $a_k(r)$ ("partial quotients") with $k \geq 1$ are positive integers. Let us remark that rational numbers have a finite number of coefficients a_k. Khinchin has proved [12], see also [15], that limits of geometrical means of $a_k(r)$ are the same for almost all real r:

$$\lim_{l\to\infty} \big(a_1(r) \ldots a_l(r)\big)^{\frac{1}{l}} = \prod_{m=1}^{\infty} \left\{1 + \frac{1}{m(m+2)}\right\}^{\log_2 m} \equiv \equiv K_0 = 2.685452001\ldots \tag{6}$$

The Lebesgue measure of (all) the exceptions is zero and include *rational numbers*, quadratic irrationals and some irrational numbers too, like for example the Euler constant $e = 2.7182818285\ldots$ for which the limit (6) is infinity.

The constant K_0 is called the Khinchin constant, see e.g. [13, §1.8]. If the quantities

$$K(r; l) = \big(a_1(r)a_2(r)\ldots a_l(r)\big)^{\frac{1}{l}}, \tag{7}$$

for a given number r are close to K_0 we can regard it as an indication that r is irrational.

We developed the fractional parts of Stieltjes constants (in Sec. II, investigating the normality, we used the whole number, e.g. $\gamma_{61} = 111670.9578149410793387893\ldots$ and we use in this section only digits after the decimal dot) using built in PARI/GP [16] the function $\texttt{contfrac}(r, \{nmax\})$ which creates the row vector $\mathbf{a}(r)$ whose components are the denominators $a_k(r)$ of the continued fraction expansion of r, i.e. $\mathbf{a} = [a_0(r); a_1(r), \ldots, a_l(r)]$ means that

$$r \approx a_0(r) + \cfrac{1}{a_1(r) + \cfrac{1}{a_2(r) + \cfrac{1}{\ddots \, \cfrac{1}{a_l(r)}}}}. \tag{8}$$

The parameter $nmax$ limits the number of terms $a_{nmax}(r)$; if it is omitted the expansion stops with a declared precision

Tab. 1. In columns A, C, E and G the two digits patterns are given, in columns B, D, F and H the maximal differences between 0.01 and the frequency that a given pattern ab, $a, b = 0, 1, \ldots, 9$ appears among the digits of the γ_n, $n = 0, 1, 2, \ldots, 5001$

A	B $(\times 10^{-3})$	C	D $(\times 10^{-3})$	E	F $(\times 10^{-3})$	G	H $(\times 10^{-3})$
00	2.4914	25	1.7898	50	2.0046	75	1.9586
01	2.0114	26	2.3187	51	2.1064	76	2.0058
02	2.0771	27	2.0847	52	2.2251	77	2.1520
03	2.3235	28	2.5891	53	2.2773	78	2.1413
04	1.8466	29	2.1732	54	1.9028	79	2.2307
05	1.9006	30	1.9310	55	2.2080	80	1.8309
06	1.8525	31	2.0466	56	2.4565	81	2.1083
07	2.4075	32	2.0625	57	1.8966	82	1.8493
08	2.4080	33	2.1236	58	1.9259	83	2.1614
09	2.0118	34	1.9970	59	2.0112	84	2.3112
10	2.1949	35	2.2988	60	1.9846	85	2.6315
11	2.3476	36	2.1588	61	1.9017	86	1.9200
12	1.8161	37	2.2839	62	1.9813	87	2.1604
13	1.9746	38	1.9860	63	2.3341	88	2.4448
14	2.3346	39	2.1897	64	2.2752	89	2.3153
15	2.1317	40	2.1021	65	1.9558	90	1.8766
16	1.8801	41	2.2182	66	2.3915	91	2.2997
17	1.8627	42	2.1976	67	2.3017	92	2.1946
18	2.0085	43	1.9233	68	2.1579	93	1.8714
19	2.3663	44	2.5452	69	1.8103	94	1.8551
20	1.8711	45	1.9193	70	2.0240	95	2.7646
21	2.0741	46	1.9071	71	1.9349	96	1.9379
22	2.2366	47	2.1403	72	1.9635	97	2.0152
23	2.2588	48	1.9612	73	1.9174	98	1.9536
24	2.3669	49	1.9473	74	1.9815	99	2.0863

of representation of real number r at the last significant partial quotient: the values of the convergents $P_k(r)/Q_k(r)$

$$\frac{P_k(r)}{Q_k(r)} = a_0(r) + \cfrac{1}{a_1(r) + \cfrac{1}{a_2(r) + \cfrac{1}{a_3(r) + \ddots + \cfrac{1}{a_k}}}}\ , \quad (9)$$

approximate the value of r with accuracy at least $1/Q_k^2$ [12, Theorem 9, p.9]:

$$\left| r - \frac{P_k(r)}{Q_k(r)} \right| < \frac{1}{Q_k^2(r)}\ , \quad (10)$$

hence when $1/Q_k^2$ is smaller than the accuracy of the number r the process stops.

We checked that the PARI precision set to `\p 120000` digits is sufficient in the sense that scripts with larger precision generated exactly the same results: the rows $\mathbf{a}(\gamma_n)$ obtained with accuracy 140 000 digits were the same for all n as those obtained for accuracy 120 000 and the continued fractions with accuracy set to 100 000 digits had different denominators $a_k(\gamma_n)$. The number of partial quotients a_k varied from over 110 000 for initial Stieltjes constants to 48 027 for γ_{5001}, i.e. the value of $l(n)$ was roughly twice the number of digits in the expansion of γ_n. However, there have been cases of extremely large values of partial quotients. The largest was $a_{13034}(\gamma_{2366}) = 17\,399\,017\,050$ for γ_{2366}, marked by the red arrow at the top in Fig. 2.

With the precision set to 120 000 digits we have expanded each γ_n, $n = 0, 2, \ldots 5000$ into its the continued fractions ($\doteq$ means "approximately equal")

$$\gamma_n \doteq [a_0(n); a_1(n), a_2(n), a_3(n), \ldots, a_{l(n)}(n)] \equiv \mathbf{a}(n), \tag{11}$$

without specifying the parameter $nmax$, thus the length of the vector $\mathbf{a}(n)$ depended on γ_n and it turns out that the number $l(n)$ of denominators was contained between 53 000 for Stieltjes constants with index around 5000 and 110 000 for gammas with smallest index n. The value of the product $a_1a_2\ldots a_{l(n)}$ was typically of the order 10^{47000} for beginning Stieltjes constants to 10^{23000} for the last γ_n's. It means that if these Stieltjes constants are rational numbers P/Q then Q are larger than those big numbers, for justification see e.g. [12, Theorems 16 & 17]. Next for each n we have calculated the geometrical means:

$$K_n(l(n)) = \left(\prod_{k=1}^{l(n)} a_k(n)\right)^{1/l(n)}. \tag{12}$$

The results are presented in Fig. 3. Values of $K_n(l(n))$ are scattered around the red line representing K_0. To gain some insight into the rate of convergence of $K_n(l(n))$ we have plotted in Fig. 4 the number of sign changes $S_K(n)$ of $K_n(m) - K_0$ for each n when $m = 100, 101, \ldots l(n)$, i.e.

$$\begin{gathered} S_K(n) = \text{number of such } m \text{ that} \\ (K_n(m+1) - K_0)(K_n(m) - K_0) < 0. \end{gathered} \tag{13}$$

The largest $S_K(n)$ was 961 and it occurred for the γ_{1175} and for 124 gammas there were no sign changes at all. It is well known that the convergence to Khinchin's constant is very slow. In Fig. 4 for each γ_n we present the closest to the Khnichin constant K_0 value of the "running" geometrical means

$$K_n(m) = \left(\prod_{k=1}^{m} a_k(n)\right)^{1/m}, \quad m = 100, 101, \ldots, l(n). \tag{14}$$

Let the rational P_k/Q_k be the n-th partial convergent of the continued fraction:

$$\frac{P_k}{Q_k} = [a_0; a_1, a_2, a_3, \ldots, a_k]. \tag{15}$$

For almost all real numbers r the denominators of the finite continued fraction approximations fulfill [12, chapter III, §15]:

$$\begin{aligned} \lim_{k\to\infty} \left(Q_k(r)\right)^{1/k} &= e^{\pi^2/12\ln 2} \equiv \\ &\equiv L_0 = 3.275822918721811\ldots, \end{aligned} \tag{16}$$

where L_0 is called the Khinchin-Lévy's constant [13, §1.8]. Again the set of exceptions to the above limit is of the Lebesgue measure zero and it includes rational numbers, quadratic irrational, etc.

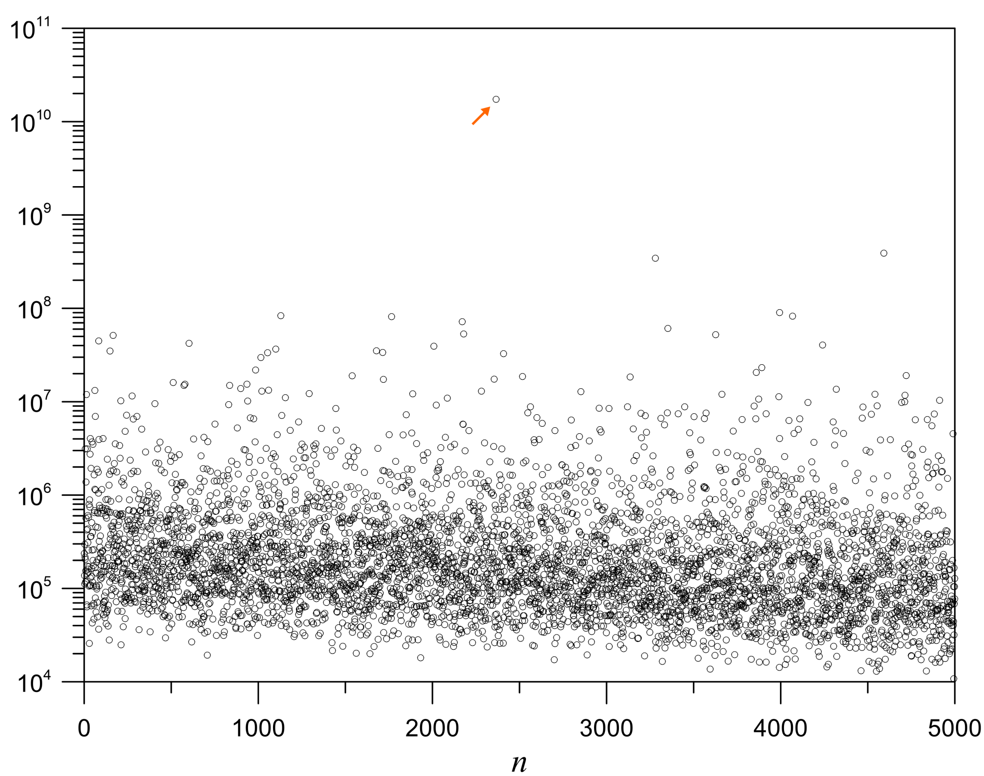


Fig. 2. The plot of maximal $a_k(n)$ for $n = 0, 1, 2, 3, \ldots, 5000$. The red arrow indicates $a_{13034}(\gamma_{2366}) = 17\,399\,017\,050$

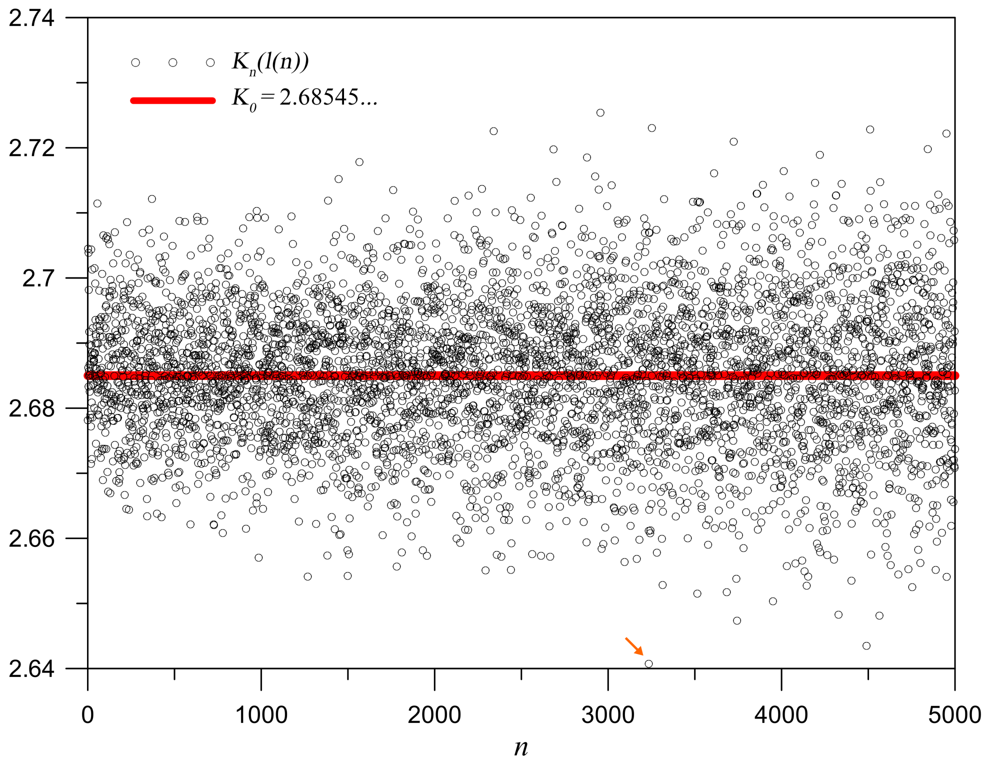


Fig. 3. The plot of $K_l(n(l))$ for $n = 0, 1, 2, 3, \ldots, 5000$. There are 384 points closer to K_0 than 0.001 and 30 points closer to K_0 than 0.0001. The largest value of $|K_0 - K_n(l(n))|$ is 4.47×10^{-2} and it occurred for the Stieltjes constant number $n = 3235$ (marked with the red arrow), the smallest value of $|K_0 - K_n(l(n))|$ is 1.02×10^{-5} and it occurred for γ_{1563}

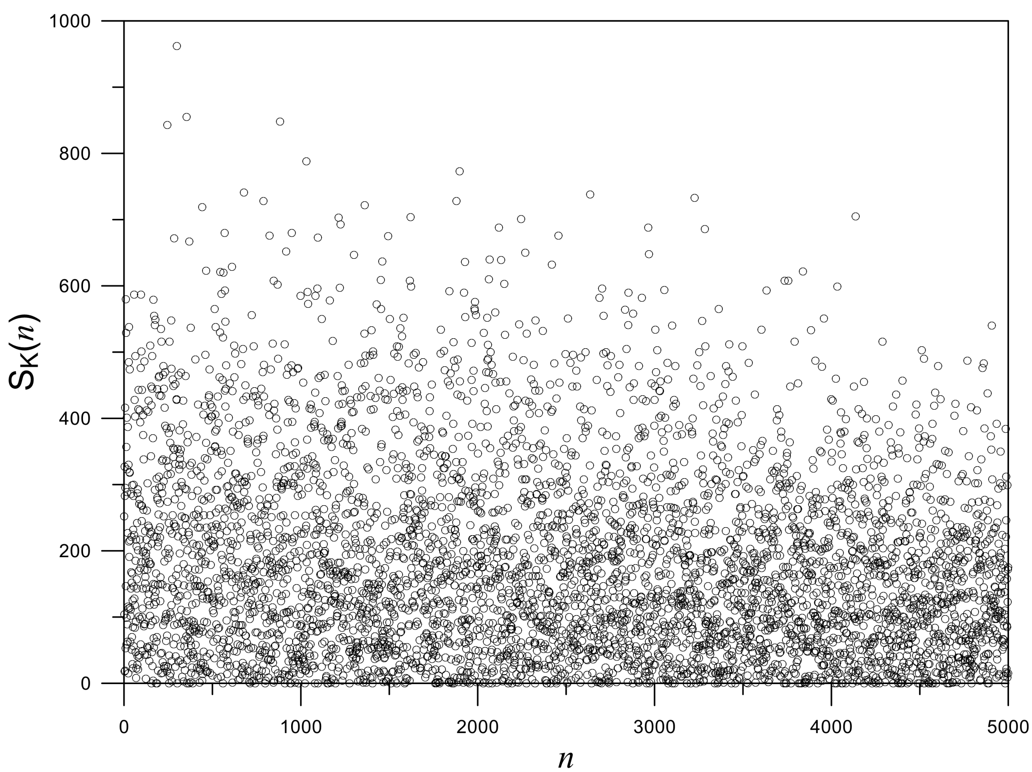


Fig. 4. The number of sign changes $S_K(n)$ for each n, i.e. the number of such m that $(K_n(m+1) - K_0)(K_n(m) - K_0) < 0$ (the initial transient values of m were skipped – sign changes were detected for $m = 100, 101, \ldots l(n)$)

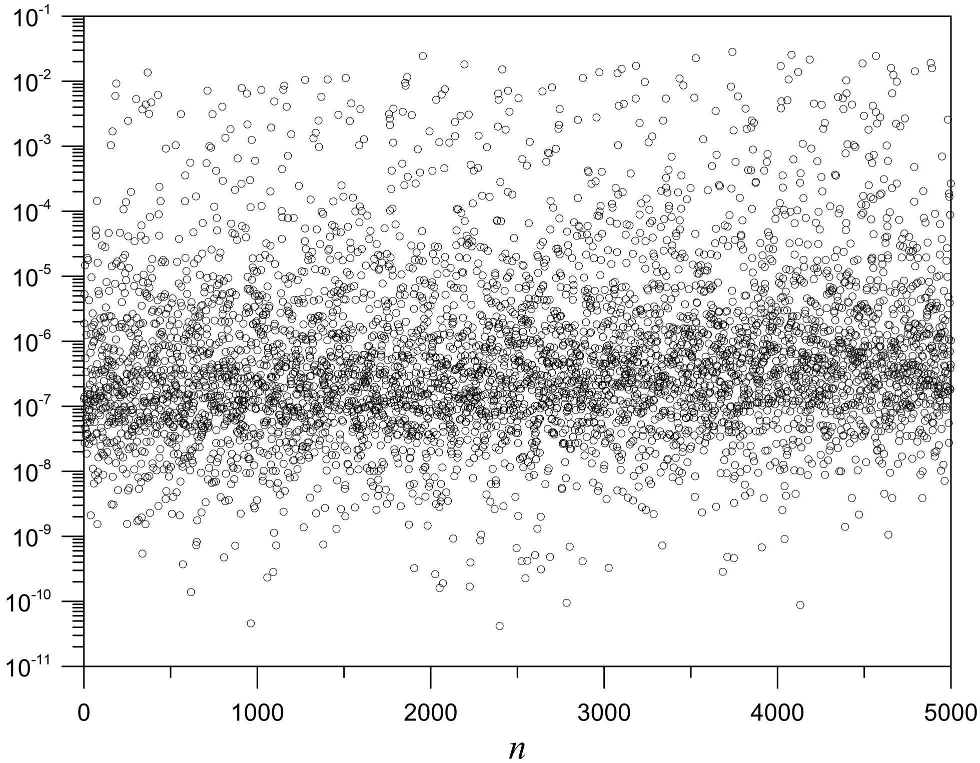


Fig. 5. The plot of the closest to the Khnichin constant K_0 values of the "running" geometrical means $K_n(m)$

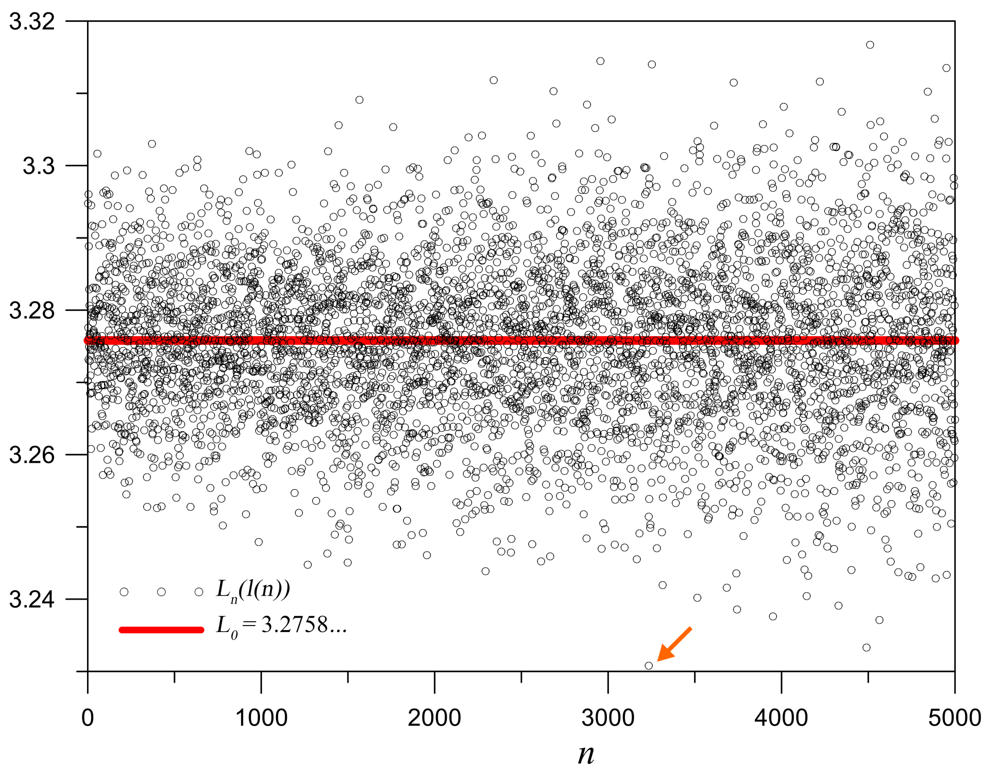


Fig. 6. The plot of $L_n(l(n))$ for $n = 0, 1, 2, \ldots, 5000$. There are 352 points closer to L_0 than 0.001 and 38 closer to L_0 than 0.0001. The largest value of $|L_0 - L_n(l(n))|$ is 4.503×10^{-2} and it occurred for the Stieltjes constant number $l = 3235$ (marked with the red arrow), the smallest value of $|L_0 - L_n(l(n))|$ is 2.336×10^{-6} and it occurred for the Stieltjes constant number $n = 3226$

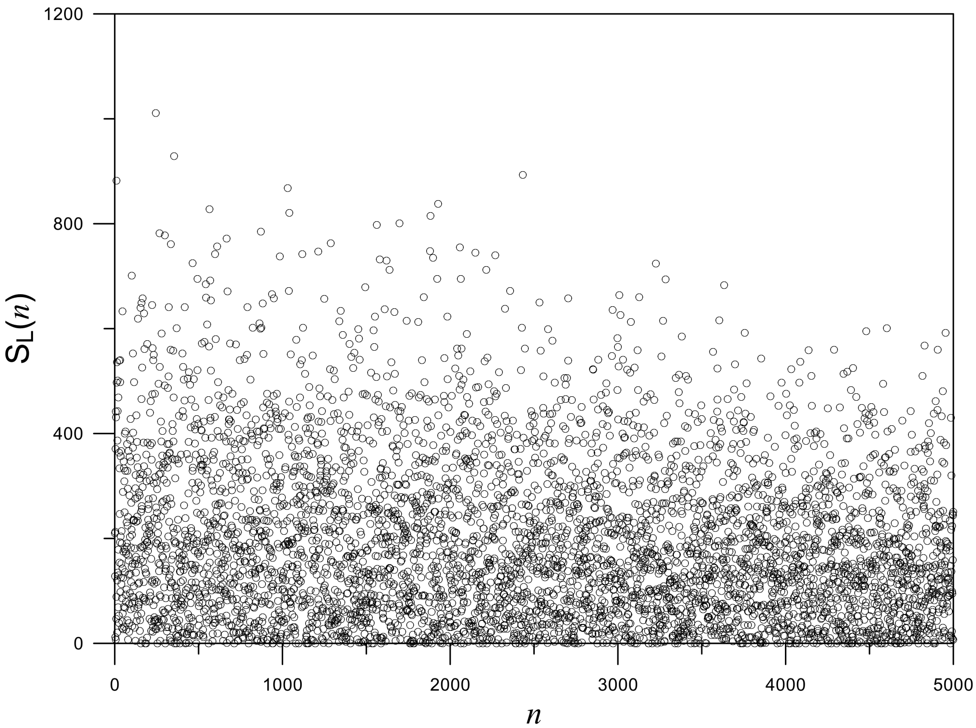


Fig. 7. The number of sign changes $S_L(n)$ for each n, i.e. the number of such m that $(L_n(m+1) - L_0)(L_n(m) - L_0) < 0$ (the initial transient values of m were skipped – sign changes were detected for $m = 100, 101, \ldots l(n)$)

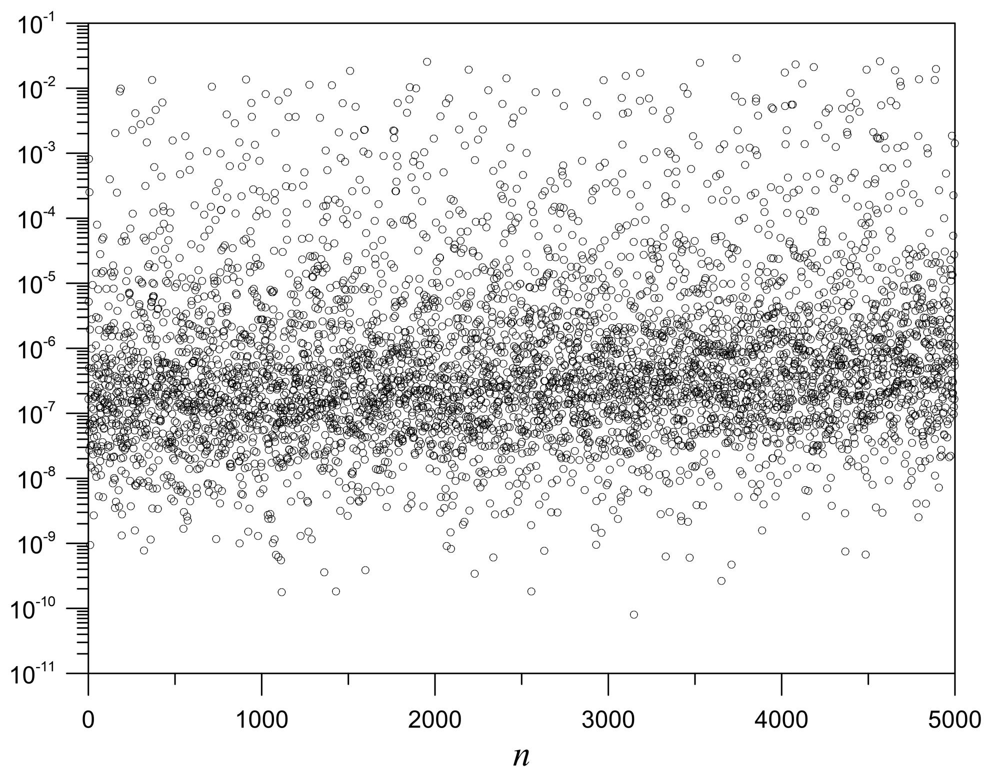


Fig. 8. The plot of the closest to the Khinchin-Lévy constant L_0 values of the "running" values of $\sqrt[m]{Q_n(m)}, n = 0, 1, 2, \ldots, 5000$

Let the rational $P_{l(n)}(\gamma_n)/Q_{l(n)}(\gamma_n)$ be the l-th partial convergent of the continued fractions (11) of γ_n:

$$\frac{P_{l(n)}(\gamma_n)}{Q_{l(n)}(\gamma_n)} = \mathbf{a}(n) \doteq \gamma_n. \tag{17}$$

For each Stieltjes constant γ_n we calculated the partial convergents $P_{l(n)}(\gamma_n)/Q_{l(n)}(\gamma_n)$ using the recurrence:

$$\begin{gathered} P_0 = a_1,\ \ Q_0 = 1,\ \ P_1 = 1 + a_1 a_2,\ \ Q_1 = a_1, \\ P_k = a_k P_{k-1} + P_{k-2},\ \ Q_{\ k} = a_k Q_{k-1} + Q_{k-2},\ \ k \geq 2. \end{gathered} \tag{18}$$

Next from these denominators $Q_{l(n)}(\gamma_n)$ we have calculated the quantities $L_n(l(n))$:

$$L_n(l(n)) = \left(Q_{l(n)}\right)^{1/l(n)}, \quad n = 0, 1, 2, \ldots, 5000. \tag{19}$$

The obtained values of $L_n(l(n))$ are presented in Fig. 6. These values scatter around the red line representing the Khinchin-Lévy's constant L_0 and are contained in the interval $(L_0 - 0.053, L_0 + 0.053)$. Again this plot is somehow misleading because there are Stieltjes constants $\gamma(n)$ for which there appear sign changes of $L_0 - L_n(m)$, $m = 1, 2, \ldots, l(n)$. As in the case of $K_n(m)$ Fig. 7 presents the number of sign changes of the difference $L_n(m) - L_0$ of the denominator of the m-th convergent P_m/Q_m

$$\begin{gathered} S_L(n) = \text{number of such } m \text{ that} \\ (L_n(m+1) - L_0)(L_n(m) - L_0) < 0. \end{gathered} \tag{20}$$

The maximal number of sign changes was 922 and appeared for the Stieltjes constant γ_{771} and there were 117 gammas without sign changes.

Finally, we looked into the distribution of the values of partial quotients $a_l(n)$. The Gauss-Kuzmin theorem [12, chapter III, §15] asserts that the density $d(k)$ of the denominators a_m, $m = 1, 2, \ldots l$, with the value k is given by

$$\lim_{l\to\infty} \frac{\sharp\{m : a_m = k\}}{l} = \log_2\left(\frac{1+\frac{1}{k}}{1+\frac{1}{1+k}}\right), \tag{21}$$

for almost all real numbers. In Fig. 9 the results are presented for the Stieltjes constants.

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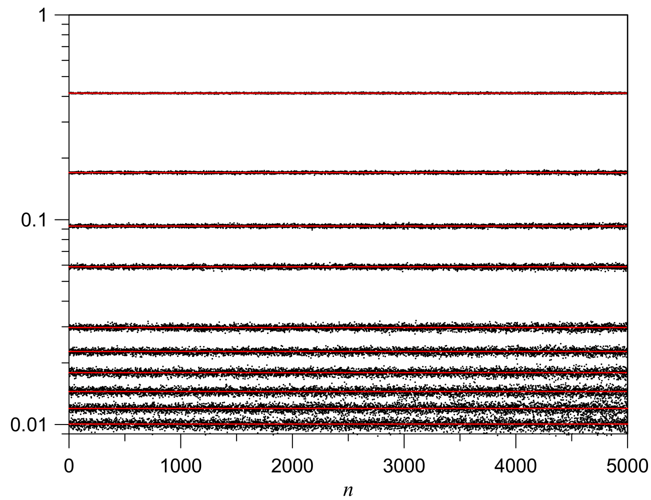

Fig. 9. The plot of the density of partial quotients a_l equal to $k = 1, 2, \ldots, 10$ from top to bottom for first 5001 Stieltjes constants. In red are the values of Eq. (21) plotted. The y axis is logarithmic to move the plots apart

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Appendix: Obtaining high precision numerical values of Stieltjes constants

In 1997, it was shown by one of the authors of the present note [17] (K.M.) that the Riemann zeta function may be expressed as

$$\zeta(s)=\frac{1}{s-1}\left[A_0+\left(1-\frac{s}{2}\right)A_1+\left(1-\frac{s}{2}\right)\left(2-\frac{s}{2}\right)\frac{A_2}{2!}+\ldots\right]= \quad \text{(A1)}$$

$$=\frac{1}{s-1}\sum_{k=0}^{\infty}\frac{A_k}{k!}\prod_{i=1}^{k}\left(i-\frac{s}{2}\right)= \quad \text{(A2)}$$

$$=\frac{1}{s-1}\sum_{k=0}^{\infty}\frac{\Gamma\left(k+1-\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\frac{A_k}{k!}, \quad s\in\mathbb{C}\backslash\{1\}, \quad \text{(A3)}$$

where

$$A_k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}(2j+1)\zeta(2j+2)= \quad \text{(A4)}$$

$$= \frac{1}{2}\sum_{j=0}^{k}\binom{k}{j}(2j+1)\frac{(2\pi)^{2j+2}B_{2j+2}}{(2j+2)!}\ . \quad \text{(A5)}$$

Here B_n denotes the n^{th} Bernoulli numbers. However, the particular choice of nodes in $s=2,4,6,\ldots$, albeit the most natural, is by no means the only one. One only requires that the prescribed points be strictly equally spaced. For the purpose of present calculations we choose the following sequence of points:

$$1+\varepsilon, 1+2\varepsilon, 1+3\varepsilon, \ldots,$$

where ε is a certain real, not necessarily small number.

More precisely, define certain entire function φ as:

$$\varphi(s) := (s-1)\zeta(s), \qquad s\neq 1,$$

together with $\varphi(1)=1$ which stems from the appropriate limit. Then, instead of Eq. (A1), we have

$$\varphi(s)=\sum_{k=0}^{\infty}\frac{\Gamma\left(k-\frac{s-1}{\varepsilon}\right)}{\Gamma\left(-\frac{s-1}{\varepsilon}\right)}\frac{\alpha_k}{k!}\ ,$$

with

$$\alpha_k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\varphi(1+j\varepsilon). \quad \text{(A6)}$$

Note that coefficients α_k depend on ε but for simplicity we shall temporarily drop this dependence in notation.

As mentioned in the Introduction, the Stieltjes constants are essentially coefficients of the Laurent series expansion of the zeta function around its only simple pole at $s=1$:

$$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n\,(s-1)^n\,. \quad \text{(A7)}$$

Now directly from Eq. (A7) we have:

$$\gamma_n = \frac{(-1)^n}{n+1}\frac{d^{n+1}}{ds^{n+1}}\varphi(s)\bigg|_{s-1}\,.$$

Then, after some elementary calculations, we get the following useful result:

$$\gamma_n = \frac{(-1)^n n!}{\varepsilon^{n+1}}\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!}\alpha_k S(k,n+1), \quad \text{(A8)}$$

where $S(k,i)$ are signed Stirling numbers of the first kind. Note that in the literature there are different conventions concerning denotation and indices of Stirling numbers which can be confusing. Here we shall adopt the following convention involving the Pochhammer symbol:

$$(x)_k\equiv\frac{\Gamma(k+x)}{\Gamma(x)}=\prod_{i=0}^{k-1}(x+i)=(-1)^k\sum_{i=0}^{k}(-1)^i S(k,i)x^i.$$

Denoting

$$\beta_{nk}\equiv(-1)^{n+k}\frac{n!}{k!}\frac{S(k,n+1)}{\varepsilon^{n+1}}\ ,$$

we can rewrite Eq. (A8) as formally an infinite matrix product

$$\gamma_n = \sum_{k=n+1}^{\infty}\beta_{nk}\,\alpha_k\ . \quad \text{(A9)}$$

The summation over k starts from $n+1$ since $\beta_{nk}\equiv 0$ for $k\leq n$. Accuracy of α_1 is equal to accuracy of precomputed values of $\varphi(s)$ in equidistant nodes. When k grows the accuracy of consecutive α_k quickly tends to zero. Thus there always exists a certain cut-off value of $k = k_0$. Therefore the summation in Eq. (A9) may be performed to this value:

$$\gamma_n = \sum_{k=n+1}^{k_0} \beta_{nk}\, \alpha_k \ . \tag{A10}$$

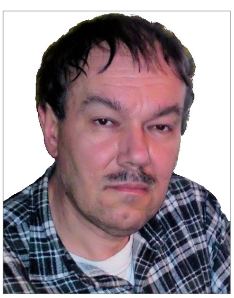
Numerical experiments confirm that adding more terms does not affect the value of the sum in Eq. (A10). As pointed earlier, ε need not be small, while choosing smaller ε greatly accelerates convergence of the series. However, it also turns out that smaller ε implies smaller k_0. What is really important is that all significant digits of γ_n obtained from the finite sum in Eq. (A10) are correct.

Obviously, γ_n eventually does not depend on ε although α_k as well as the rate of convergence of Eq. (A8) does. In fact, series in Eq. (A8) converges for any value of $\varepsilon > 0$ but the rate of convergence becomes terribly small for $\varepsilon \gg 1$. On the other hand, the smaller ε, the faster the rate of convergence. However, since α_k also depends on ε, choosing smaller value for ε requires higher accuracy of precalculated values of $\varphi(s)$, which in turn may be very time-consuming. Hence, an appropriate compromise in choosing ε is needed.

Formula (A8) is particularly suited for numerical calculations. As already pointed above, one has to choose parameter ε in order to optimally perform the calculations. Typically the algorithm has three simple steps:

1. Tabulating $\varphi(1 + j\varepsilon), j = 0, 1, 2, \ldots$ This requires an appropriate choice of parameter ε (see below) and is most time-consuming. The program which seems most convenient for this purpose is a small but extremely efficient program PARI/GP which has implemented the particularly optimal zeta procedure. The first of the authors used the Cyfronet ZEUS computer in Cracow, where calculating a single value of $\varphi(s)$ with 51 000 significant digits requires about 13 minutes. Since this procedure may easily be parallelized, in order to compute 10 000 values of φ 20 independent routines were performed (each calculating 500 values of φ) which took nearly one week.
2. Calculating α_k using Eq. (A6) and the precomputed values.
3. Calculating Stieltjes constants using Eq. (A8).

Contrary to the above step 1 which requires a powerful computer, steps 2 and 3 can be quickly performed on a typical PC. Several properties concerning accuracies may be obtained experimentally. It should be stressed out that given α_k calculating single γ_n with accuracy of about 50 000 digits requires several minutes on a very modest PC machine.



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