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Two Arguments that the Nontrivial Zeros of the Riemann Zeta Function are Irrational. Part II

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Abstract: We extend the results of our previous computer experiment performed on the first 2600 nontrivial zeros γ_l of the Riemann zeta function calculated with 1000 digits accuracy to the set of 40 000 first zeros given with 40 000 decimal digits accuracy. We calculated the geometric means of the denominators of continued fractions expansions of these zeros and for all cases we get values very close to the Khinchin's constant, which suggests that γ_l are irrational. Next we have calculated the n-th square roots of the denominators Q_n of the convergents of the continued fractions obtaining values very close to the Khinchin-Lévy constant, again supporting the common opinion that γ_l are irrational.

Key words: zeros of the Riemann's zeta function, continued fractions, Khinchin and Levy constant, irrationality and normality of numbers

I. INTRODUCTION

The famous Riemann's $\zeta(s)$ function [1, 2] has trivial zeros at even negative integers: $-2, -4, -6, \ldots$ and infinity of nontrivial complex zeros $\rho_l = \beta_l + i\gamma_l$, $\rho_{-l} = = \beta_l - i\gamma_l = \overline{\beta_l + i\gamma_l} = \overline{\rho_l}$ $(l = 1, 2, 3, \ldots)$ in the critical strip: $\beta_l \in (0, 1)$, $\gamma_l \in \mathbb{R}$. The Riemann Hypothesis (RH) asserts that $\beta_l = \frac{1}{2}$ for all l – i.e. all zeros lie on the critical line $\Re(s) = \frac{1}{2}$. It is commonly believed that all nontrivial zeros ρ_l of zeta are simple, which means that $\zeta'(\rho_l) \neq 0$. This property is still unproven but is indeed highly desirable since there are many useful formulas in the number theory where this derivative is in the denominator.

There is little hope to obtain analytical formulas for the imaginary parts γ_l of the nontrivial zeros of $\zeta(s)$. However in the paper [3] LeClair presented remarkable formula:

$$\gamma_l \approx 2\pi \frac{l - \frac{11}{8}}{W\left(\frac{l - \frac{11}{8}}{e}\right)}\,, \tag{1}$$

where $W(x)$ is the Lambert function: $x = W(x)e^{W(x)}$, see e.g. [4, p. 111, §4.13]. Values obtained from (1) are indeed very close to actual imaginary parts of zeros of zeta γ_l – the larger the l, the better the approximation, but are distributed far more uniformly than the zeros. In [3] LeClair remarks that for $l = 10^p$ equation (1) gives correct roughly p digits.

In the previous paper [5] we exploited two facts about the continued fractions: the existence of the Khinchin constant and Khinchin-Lévy constant, see e.g. [6, §1.8], to support the irrationality of γ_l. Let

$$r = [a_0(r); a_1(r), a_2(r), a_3(r), \ldots] = \\ = a_0(r) + \cfrac{1}{a_1(r) + \cfrac{1}{a_2(r) + \cfrac{1}{a_3(r) + \ddots}}}\,, \tag{2}$$

be the continued fraction expansion of the real number r, where $a_0(r)$ is an integer and all $a_k(r)$ with $k \geq 1$ are positive integers. Khinchin proved certain deep and amazing (and still not widely known) theorem [7], see also [8], that limits of geometric means of $a_n(r)$ are the same for almost all real r:

$$\lim_{n\to\infty} \big(a_1(r)\dots a_n(r)\big)^{\frac{1}{n}} = \prod_{m=1}^{\infty}\left\{1+\frac{1}{m(m+2)}\right\}^{\log_2 m} \equiv \\ \equiv K_0 \approx 2.685452001\dots \tag{3}$$

The Lebesgue measure of all the exceptions is zero. These exceptions include: rational numbers, quadratic irrationals as well as some particular irrational numbers like, for example, the Euler constant $e = 2.7182818285\dots$ for which the limit (3) is infinity.

The constant K_0 is called the Khinchin constant, see e.g. [6, §1.8]. Therefore, if the quantities

$$K(r;n) = \big(a_1(r)a_2(r)\dots a_n(r)\big)^{\frac{1}{n}}, \tag{4}$$

for a given number r are close to K_0 we can regard it as an indication that r is irrational. And this is the key idea of both the present work and the previous one [5].

Let the rational P_n/Q_n be the n-th partial convergent of the continued fraction:

$$\frac{P_n}{Q_n} = [a_0; a_1, a_2, a_3, \dots, a_n], \tag{5}$$

i.e. convergent is obtained by taking a finite number of initial segments of a continued fraction. The values of the convergents $P_k(r)/Q_k(r)$ approximate the value of r with accuracy of at least $1/Q_kQ_{k+1}$ [7, Theorem 9, p.9]:

$$\left|r - \frac{P_k}{Q_k}\right| < \frac{1}{Q_kQ_{k+1}} < \frac{1}{Q_k^2 a_{k+1}} \le \frac{1}{Q_k^2}. \tag{6}$$

For almost all real numbers r the denominators of the finite continued fraction approximations fulfill:

$$\lim_{n\to\infty} \big(Q_n(r)\big)^{1/n} = e^{\pi^2/12\ln 2} \equiv L_0 = \\ = 3.275822918721811\dots, \tag{7}$$

where L_0 is called the Khinchin-Lévy's constant [6, §1.8]. Again the set of exceptions to the above limit is of the Lebesgue measure zero and it includes rational numbers, quadratic irrational, etc.

II. COMPUTER EXPERIMENTS WITH ZETA ZEROS

Some time ago we found that G. Beliakov and Y. Matiyasevich [9] calculated first 40 000 nontrivial zeros of $\zeta(s)$ with 40 000 digits accuracy and made them publicly available at [10]. The method used during this high-precision numerical calculations is described in [11]. Thus we were able to repeat computer experiments presented in [5] on a much larger set of zeros of $\zeta(s)$ given with a much higher number of digits.

In the computer experiments we used the PARI [12] which has a built in function $\texttt{contfrac}(r, \{nmax\})$. This function creates the row vector $\mathbf{a}(r)$ whose components are the denominators $a_n(r)$ of the continued fraction expansion of r, i.e. $\mathbf{a} = [a_0(r); a_1(r), \dots, a_n(r)]$ means that

$$r \approx a_0(r) + \cfrac{1}{a_1(r) + \cfrac{1}{a_2(r) + \cfrac{1}{\ddots \cfrac{1}{a_n(r)}}}}. \tag{8}$$

The parameter $nmax$ limits the number of terms $a_{nmax}(r)$; if it is omitted the expansion stops with a declared precision of representation of real numbers at the last significant partial quotient. With the precision set to 90 000 digits we expanded each γ_l, $l = 1, 2, \dots 40\,000$ with 40 000 accurate decimal digits value into its continued fractions

$$\gamma_l \doteq [a_0(l); a_1(l), a_2(l), a_3(l), \dots, a_{n(l)}(l)] \equiv \mathbf{a}(l), \tag{9}$$

(here $\doteq$ denotes approximate equality) without specifying the parameter $nmax$, thus the length $n(l)$ of the vector $\mathbf{a}(l)$ depended on γ_l and it turns out that the number of denominators was contained between 77 000 and 78 000. The value of the product $a_1a_2\dots a_{n(l)}$ was typically of the order $10^{33\,000}$–$10^{33\,500}$. Next for each l we have calculated the geometric means:

$$K_l(n(l)) = \left(\prod_{k=1}^{n(l)} a_k(l)\right)^{1/n(l)}. \tag{10}$$

The results are presented in Fig. 1. Values of $K_l(n(l))$ are scattered around the red line representing K_0 and are contained in the interval $(K_0 - 0.06, K_0 + 0.06)$. For the set of zeros reported in the paper [5] values of $K_l(n(l))$ were contained in the interval $(K_0 - 0.3, K_0 + 0.3)$. Fig. 1 is in some sense misleading, because there are cases when the difference $K_l(m) - K_0$ changes the sign for earlier m, thus at these points $K_l(m)$ are exactly equal to K_0. We are not able to repeat calculations presented in Fig. 2 in [5] showing the number of sign changes of $K_l(m) - K_0$: it would take a few years of CPU time as there are over 15 times more zeros with 40 times more digits, thus assuming linear complexity of the problem it will take over 600 more time than in the previous experiment, which took 2 CPU days. In Fig. 2 we present plots of $K_l(m)$ as a function of m for a few zeros γ_l with sign changes of $K_l(m) - K_0$. We also encountered zeros γ_l without sign change of $K_l(m) - K_0$, some of them are

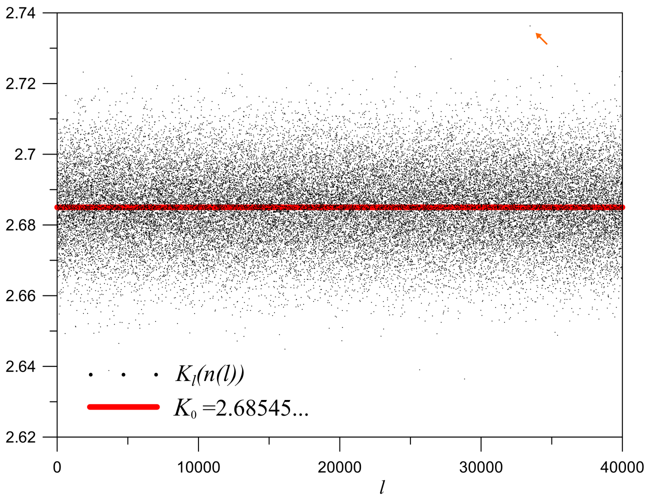

Fig. 1. The plot of $K_l(n(l))$ for $l = 1, 2, 3, \ldots, 40\,000$. There are 2976 points closer to K_0 than 0.001 and 292 points closer to K_0 than 0.0001. The largest value of $|K_0 - K_l(n(l))|$ is 5.08×10^{-2} and it occurred for the zero number $l = 33\,473$ (marked with the red arrow), the smallest value of $|K_0 - K_l(n(l))|$ is 9.2×10^{-8} and it occurred for the zero number $l = 17\,408$

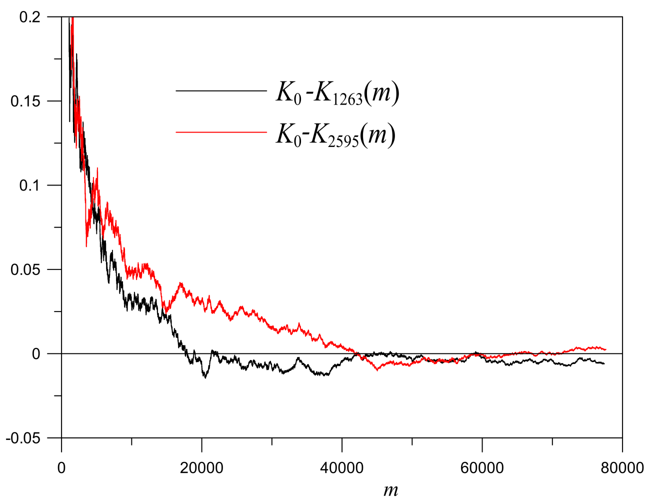

Fig. 2. The plot of the difference $K_0 - K_l(m)$ for $l = 1263$ and $l = 2595$. There are 267 sign changes for γ_{1263} and 218 sign changes for γ_{2595}

plotted in Fig. 3. The plots of the difference $K_0 - K_l(m)$ without sign changes seem to follow the power-like dependence $|K_l(m) - K_0| \sim m^{-\alpha_l}$, where the parameters α_l very weakly depend on the zero number l and are close to 0.9. Thus there are different ways of reaching the limit $m \to \infty$ of the sequence $K_l(m)$ depending on the zero index l.

We obtained from G. Beliakov the zero $\gamma_{14\,299}$ with 70 000 digits accuracy. In Fig. 4 we present the plot of $K_{14\,299}(m) - K_0$ vs m. It took 52 hours on 3.9 GHz CPU to get data for this plot. There are 322 sign changes of $K_0 - K_{14\,299}(m)$ present on Fig. 4.

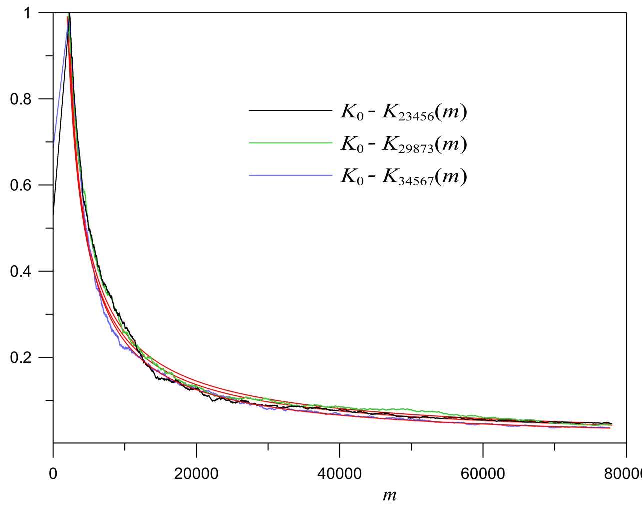

Fig. 3. The plots of the difference $K_0 - K_l(m)$ for $l = 23\,456,\ 29\,873,\ 34\,567$. In red are the fits to the power-like dependence plotted for these three zeros. These fits are represented by $m^{-\alpha}$ with parameter α almost the same for all three zeros and contained in the interval $(0.85, 0.92)$

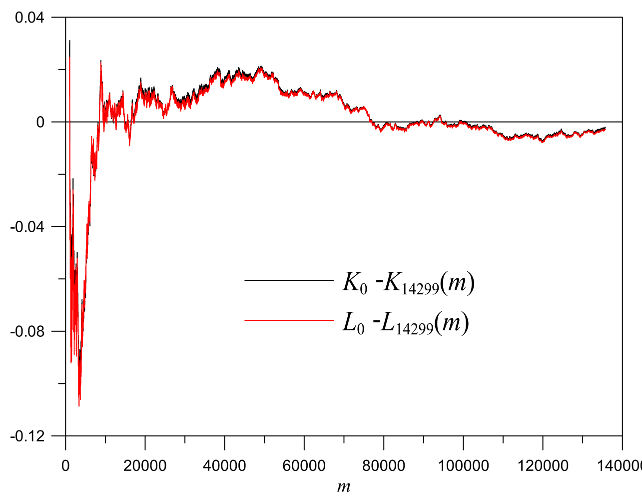

Fig. 4. The plots of the differences $K_0 - K_{14\,299}(m)$ (black) and $L_0 - l_{14\,299}(m)$ (red). The number of denominators $a(n)$ was 135 721

Let the rational $P_{n(l)}(\gamma_l)/Q_{n(l)}(\gamma_l)$ be the n-th partial convergent of the continued fractions (9):

$$\frac{P_{n(l)}(\gamma_l)}{Q_{n(l)}(\gamma_l)} = \mathbf{a}(l) \doteq \gamma_l \,. \tag{11}$$

For each zero γ_l using PARI function `contfracpnqn(a)` we calculated the partial convergents $P_{n(l)}(\gamma_l)/Q_{n(l)}(\gamma_l)$. Next from these denominators $Q_{n(l)}(\gamma_l)$ we have calculated the quantities $L_l(n(l))$:

$$L_l(n(l)) = \left(Q_{n(l)}\right)^{1/n(l)}, \quad l = 1, 2, \ldots, 40\,000. \tag{12}$$

The obtained values of $L_l(n(l))$ are presented in Fig. 5. These values scatter around the red line representing the Khinchin-Lévy's constant L_0 and are contained in the interval $(L_0-0.05, L_0+0.05)$, while in the previous paper [5] this interval was $(L_0-0.36, L_0+0.36)$. Again this plot is somehow misleading because there are zeros $\gamma(l)$ for which there appear sign changes of $L_0 - L_l(m)$. In Fig. 4 we present the plot of $L_0 - L_{14\,299}(m)$ for the zero number 14 299 which is known with 70 000 digits accuracy. It took 82 hours on 3.9 GHz CPU to get data for this plot. This plot is practically identical with the plot of $K_0 - K_{14\,299}(m)$. There are 229 sign changes of $L_0 - L_{14\,299}(m)$ present in Fig. 4. These sign changes of $K_0 - K_{14\,299}(m)$ and of $L_0 - L_{14\,299}(m)$ appears almost at the same arguments m, see Fig. 6.

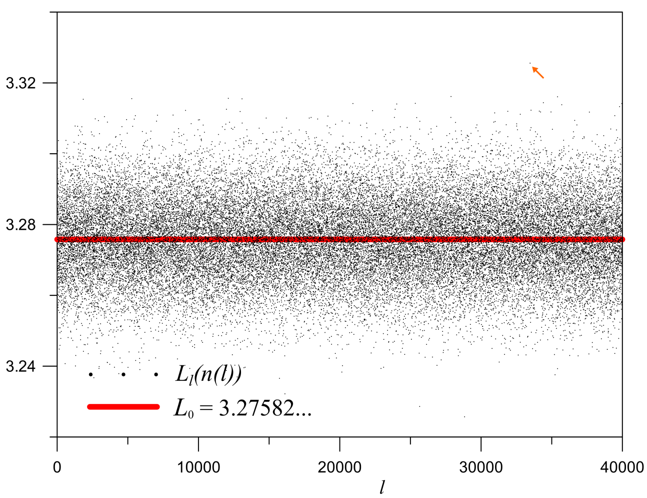

Fig. 5. The plot of $L_l(n(l))$ for $l = 1, 2, 3, \ldots, 40\,000$. There are 2906 points closer to L_0 than 0.001 and 275 zeros closer to L_0 than 0.0001 The largest value of $|L_0 - L_l(n(l))|$ is 5.01×10^{-2} and it occurred for the zero number $l = 28\,831$ (marked with the red arrow), the smallest value of $|L_0 - L_l(n(l))|$ is 1.72×10^{-6} and it occurred for the zero number $l = 14\,768$

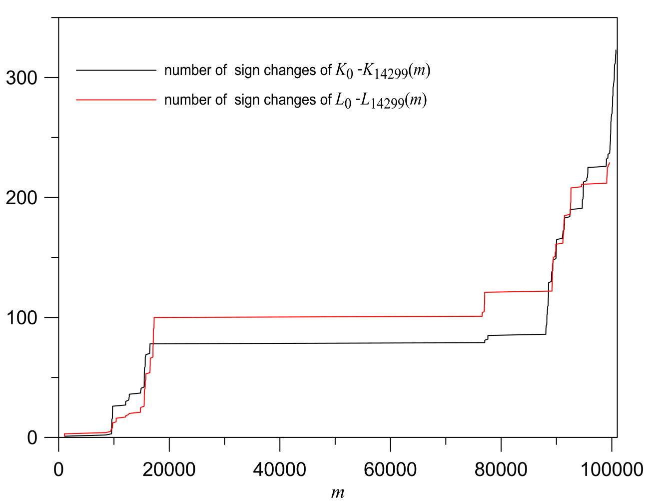

Fig. 6. The plot of the number of sign changes of difference $K_0 - K_{14\,299}(m)$ and $L_0 - L_{14\,299}(m)$ as the function of m

III. NORMALITY

In [13, Th.2] P.D.T.A. Elliott assuming RH proved that the sequence $\alpha\gamma_l$, $(l = 1, 2, \ldots)$ is uniformly distributed modulo 1 for every real nonzero α. Further results about the distribution of $\alpha\gamma_l$ were later obtained in [14] and [15].

Let us recall that a number x is normal in base b if each finite string of k consecutive digits appears in this expansion with asymptotic frequency b^{-k}. In the usual decimal base we have that each digit $0, 1, 2, \ldots, 9$ appears in the expansion of the number x with limiting frequency 0.1, each 2-digits string $00, 01, \ldots, 99$ appears with density 0.01. Having the first 40 000 nontrivial zeros of the zeta function with 40 000 digits accuracy we checked that each digit $0, 1, 2, \ldots, 9$ appears almost exactly with frequency 0.1. It is difficult to represent this 400 000 data points in one plot. In Fig. 7 we employed the following artifice: the frequency $h_l(0)$ of appearance of digit 0 in the zero γ_l is plotted at x-axis value l with the y value $0.1 - h_l(0)$, i.e. the distance from the expected value 0.1. In general, the frequency $h_l(n)$ of appearance of digit n in the zero γ_l is plotted with the y value $n \times 0.1 + 0.1 - h_l(0)$. We also calculated density of 100 strings of two digits $00, 01, \ldots, 99$ for all 40 000 zeros γ_l. Now the result consisted of four million points, which is impossible to represent on the plot. Instead, in Tabs. 1 and 2 we present for each pattern of digits ab the maximal difference between the calculated frequency of appearance and the expected value of 0.01 and the number l of the zero γ_l for which this discrepancy appeared. As it is seen from this Tabs. 1 and 2 the maximal difference between the actual com-

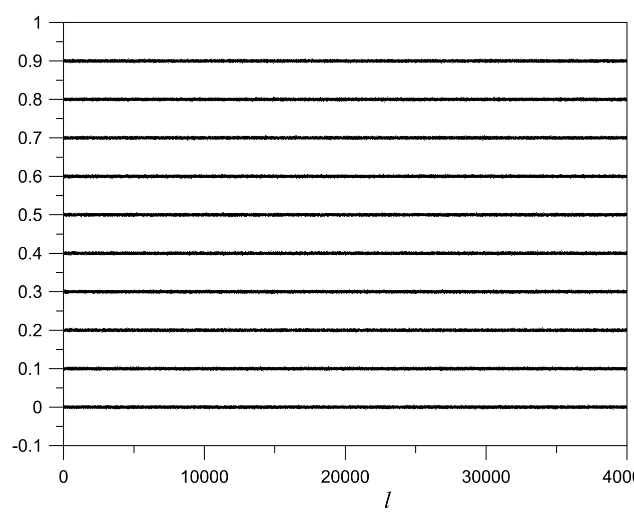

Fig. 7. The plot of the differences between 0.1 and actual frequencies of digits $0, 1, \ldots, 9$ for all 40 000 zeros. The data for digit n is plotted at y value $n \times 0.1$ for clarity

Tab. 1. In columns A and D the two digits patterns are given, ranging from 00 to 75, in columns B and E the maximal differences between 0.01 and the frequency that a given pattern ab, $a, b = 0, 1, \ldots, 9$ appears among the 40 000 digits of the nontrivial zero γ_n and in columns C and F the number n of the zero for which this maximal discrepancy appears

A	B	C	D	E	F
00	2.1024×10^{-3}	18 878	38	2.2525×10^{-3}	11 935
01	2.1734×10^{-3}	3 933	39	2.0024×10^{-3}	6 147
02	2.1024×10^{-3}	4 277	40	2.2234×10^{-3}	38 951
03	1.9734×10^{-3}	24 673	41	2.1274×10^{-3}	13 018
04	2.1274×10^{-3}	21 084	42	2.0274×10^{-3}	34 478
05	2.2024×10^{-3}	17 167	43	2.1024×10^{-3}	27 129
06	2.0524×10^{-3}	25 612	44	2.5775×10^{-3}	10 464
07	2.0734×10^{-3}	4 561	45	2.4025×10^{-3}	330
08	2.0024×10^{-3}	3 990	46	2.3025×10^{-3}	16 304
09	2.0984×10^{-3}	25 993	47	2.3775×10^{-3}	20 866
10	2.1274×10^{-3}	15 822	48	2.0484×10^{-3}	20 451
11	2.3275×10^{-3}	20 252	49	2.1774×10^{-3}	3 470
12	2.2274×10^{-3}	8 997	50	2.0984×10^{-3}	12 722
13	2.0024×10^{-3}	28 106	51	2.1774×10^{-3}	8 849
14	2.2775×10^{-3}	5 949	52	2.1984×10^{-3}	1 305
15	2.1774×10^{-3}	14 933	53	2.0774×10^{-3}	21 885
16	2.1024×10^{-3}	6 872	54	1.9734×10^{-3}	27 876
17	2.2024×10^{-3}	28 587	55	2.1024×10^{-3}	24 122
18	2.1234×10^{-3}	27 442	56	1.9774×10^{-3}	15 887
19	1.9484×10^{-3}	31 680	57	2.3525×10^{-3}	38 582
20	1.9524×10^{-3}	32 020	58	2.0774×10^{-3}	19 264
21	1.9274×10^{-3}	24 174	59	2.1734×10^{-3}	29 384
22	2.3775×10^{-3}	8 664	60	2.1274×10^{-3}	10 528
23	2.2775×10^{-3}	29 605	61	2.0274×10^{-3}	12 705
24	2.2525×10^{-3}	39 895	62	2.3485×10^{-3}	10 412
25	2.1274×10^{-3}	36 860	63	2.3025×10^{-3}	35 439
26	2.2274×10^{-3}	3 422	64	2.0774×10^{-3}	37 431
27	2.1274×10^{-3}	12 063	65	2.0774×10^{-3}	34 234
28	2.1274×10^{-3}	15 958	66	2.6025×10^{-3}	16 268
29	2.0484×10^{-3}	37 556	67	2.3525×10^{-3}	24 802
30	2.0024×10^{-3}	30 473	68	2.5025×10^{-3}	2 114
31	2.3525×10^{-3}	23 714	69	2.0524×10^{-3}	4 963
32	2.1024×10^{-3}	13 831	70	2.1734×10^{-3}	5 166
33	2.2024×10^{-3}	37 166	71	2.2234×10^{-3}	24 355
34	2.1524×10^{-3}	17 354	72	2.3775×10^{-3}	25 954
35	2.0274×10^{-3}	7 242	73	2.1274×10^{-3}	22 247
36	2.2525×10^{-3}	34 311	74	2.0774×10^{-3}	38 557
37	2.2735×10^{-3}	32 726	75	2.0984×10^{-3}	38 984

Tab. 2. Continuation of Tab. 1 for the two digits patterns ranging from 76 to 99. All other column designations are the same as in Tab. 1

A	B	C	D	E	F
76	2.2525×10^{-3}	28 568	88	2.5775×10^{-3}	35 792
77	2.5525×10^{-3}	24 645	89	2.0774×10^{-3}	22 444
78	2.3525×10^{-3}	12 616	90	2.3235×10^{-3}	27 644
79	2.5025×10^{-3}	34 043	91	2.1774×10^{-3}	16 307
80	2.2274×10^{-3}	23 791	92	2.2525×10^{-3}	31 385
81	2.1984×10^{-3}	8 795	93	2.3985×10^{-3}	31 994
82	2.0234×10^{-3}	39 690	94	2.0024×10^{-3}	5 743
83	2.2024×10^{-3}	2 712	95	2.1024×10^{-3}	1 785
84	2.1524×10^{-3}	23 952	96	2.1274×10^{-3}	4 175
85	2.2274×10^{-3}	31 623	97	2.0524×10^{-3}	23 112
86	2.3525×10^{-3}	4 808	98	2.1734×10^{-3}	28 022
87	1.9274×10^{-3}	35 793	99	2.5525×10^{-3}	34 684

puted value of frequency of two digits patterns and expected value 0.01 is typically of 2 percent.

IV. CONCLUSIONS

The results presented above do not meet strict requirements of a formal mathematical proof. But, in principle, it could happen that quantities $K_l(n(l))$, see (10), or $L_l(n(l))$ from (12) for some zeros (or even for all $40\ 000$ zeros investigated here) were equal, for example, 5 or even 10, therefore not very close to K_0 and to L_0. Hence we regard the results of our computer experiments, together with the checks of the normality reported in Sect. 3, as a strong suggestion that, as expected, imaginary parts of the nontrivial zeta zeros are indeed irrational numbers.

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