Study of memory effect in an economic order quantity model with quadratic type demand rate

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Abstract: The study of memory effect in an economic order quantity model has a great impact on the inventory system. Although business policy almost depends on the past experiences of the system, usually the classical inventory model does not include the past experience or memory effect, i.e. one important part of the system is ignored. Our purpose is to include memory or past experience in the inventory model. The purpose of this paper is to incorporate the existence of dynamic memory in an inventory model with shortage via fractional calculus. To derive the memory dependent inventory model associated with inventory holding cost, shortage cost has been developed. Analytical solution of the proposed inventory model has been solved via primal geometric programming method. Numerically long memory effect or short memory effect of the inventory system has been established. In this paper, an effort has also been made to compare the memory effect on the minimized total average cost and the optimal ordering interval using different numerical examples.

Key words: fractional Laplace transforms method, differential equation with memory kernel, classical inventory model, memory dependent Inventory model

I. INTRODUCTION

The origin of the fractional calculus goes back to L'Hospital and Leibnitz in the seventeenth century. The modern work devoted exclusively to the subject of fractional calculus has been published [1–8]. Since that moment many monographs dedicated to this subject have been published [1–8]. Until the present arms fractional calculus have been discussed as a comparatively esoteric mathematical theory without applications but in the recent few years fractional calculus has appeared with its application backgrounds in different mathematical problems [1, 5, 6, 22–24], Physics [8], Economics [2–5], etc. Many well-known scientists like Leibniz, Euler, Lagrange, Fourier, Abel, Liouville,

Riemann, Grunwald, Holmgren, Peacock, Tarday, Cauchy, Hadamard, Hardy, Riesz, Weyl, etc. [9] worked on the fractional calculus to develop fractional derivative and integration. Fractional derivative and integration are generalized as non-integer order. Riemann-Liouville (R-L) fractional integration, Caputo fractional derivative [9, 10], Jumarie fractional derivative [9, 10] play a crucial role in developing any real life problems. It is worth noting that the standard classical mathematical models of the integer order derivatives do not work as a generalized version, but as a particular case of generalized version and are also not able to take into account memory of the system.

On the other hand, the first order ordinary derivative is equivalent to the physical meaning of speed (the physical

meaning of speed is path length traveler per unit of time). But the physical interpretation of the fractional order derivative and integration is an index of memory [1]. In the present paper, the fractional calculus has been applied to illustrate the memory effect of the inventory system. The authors believe that the research in the area of the fractional calculus will constitute an important tool in the scientific progress of mankind.

It is known that having the ability to measure inventory in a time with accurate policy is critical without arranging the inventory model from a company's balance sheet chart. Owing to the above problem, inventory literature was born by a cited article of Harris-Willsons [11, 12] square root formulae. Most of these inventory models are proposed on the economic order (EOQ) model by many researchers [13–16]. In this paper, the classical inventory model has been developed on the basis of the fixed shortage time. The objective of this paper is to generalize the classical model using fractional derivatives and fractional integration as well as taking into account memory effect of the inventory model. Authors expect that memory dependent inventory model has more importance compared to the memory less inventory model or classical model.

In a real market or real inventory system the demand of the product changes with respect to time as well as space or situation. It is known that when an object gets popularity in the market then its demand will increase or if it makes bad impression then its demand will decrease. In some sense demand of any object is not the same in all shops: increase or decrease of the selling depends dealings of the shopkeeper, i.e. the selling of any product depends on the quality as well as the shopkeeper's attitude. Thus, this system depends on previous history, not only on the present state of the process, so it is legal to include the memory effect in the inventory system. We think that the memory effect should be incorporated in different human related physical problems and observe the different situation for low or strong memory effect or long memory effect, short memory effect.

Here, we have suggested fractional order derivative to take into account the memory effect. But why?

It is known that the time rate of change of integer orders is determined by the property of differentiable functions of time only in infinitely small neighborhood of the considered point of time. Hence, an instant change of the marginal output is assumed, when the input level changes. Therefore, the dynamic memory effect is not present in classical calculus and it is not able to discuss all state of the system i.e. (the present system depends on the past) [1–7]. But in fractional derivative the rate of change is affected by all points of the considered interval, so it is able to incorporate memory of the system and fractional order is physically treated as an index of memory. So, it can remove amnesia from the system.

Our analysis has established the existence of the memory effect in the EOQ model. It is observed from the numerical example that in the long memory effect the minimized to-

tal average cost is very low compared to the memory less or short memory effect. In the long memory effect, the optimal ordering interval is much higher compared to the short memory effect. Hence, in the long memory effect business takes much more time to reach the minimum value of the total average cost compared to the short memory effect. When the demand rate is proportional to square of the time, the minimized total average cost with the memory effect is low compared to the quadratic type demand rate, constant demand rate, and linear type demand rate. Hence, when the system is strong with its past experience, profit is maximum compared to short gained past experience.

The rest part of the paper has been furnished as follows, In Sec. II., the review of fractional calculus has been discussed, the classical inventory model has been given in Sec. III., the memory dependent inventory model with analytic calculation has been proposed in Sec. IV., numerical examples and sensitivity analysis have been conducted in Sec. V., graphical presentation has been arranged in Sec. VI., and finally some conclusions are given to show the effect of the work in Sec. VII.

II. REVIEW OF FRACTIONAL CALCULUS

There are several definitions of the fractional derivative, the mostly used to definitions are given below.

II. 1. Riemann Liouville fractional derivative

Left Riemann-Liouville (R-L) definition of fractional derivative of αth order (where $m \leq \alpha < m+1$) for any integrable function $f(x)$ on $[a, b]$ is denoted $_a^{\text{RL}}D_x{}^\alpha (f(x))$ and defined as

$$
_a^{\text{RL}}D_x{}^\alpha (f(x)) =
$$
$$
= \frac{1}{\Gamma(m+1-\alpha)} \left(\frac{d}{dx} \right)^{m+1} \int_a^x (x-\xi)^{(m-\alpha)} f(\xi)\, d\xi.
$$
(1)

The above results create a difference between ordinary derivative and fractional derivative. To eradicate this difficulty, M. Caputo [9, 10] proposed a new definition of fractional order derivative for any differentiable functions.

II. 2. Caputo fractional order derivative

For any differentiable function $f(x)$ on $[a, b]$ Caputo fractional order derivative [15] of αth order is denoted $_a^{\text{C}}D_x^\alpha (f(x))$ and defined as

$$
_a^{\text{C}}D_x^\alpha (f(x)) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\xi)^{(m-1-\alpha)} f^{(m)}(\xi)\, d\xi,
$$
(2)

where: $m-1 \leq \alpha < m$.

The Caputo definition contains both merit and demerit. Demerit of this definition is that the function should be (m) times differentiable; otherwise this definition will not be valid. On the other hand, merit of this definition is, Caputo

fractional order derivative of any constant function is zero as in ordinary derivative. In the main work of the paper, the Caputo fractional derivative operator has been symbolized as $_0^C D_t^\alpha$.

II. 3. Fractional Laplace Transform Method

Laplace transformation plays an important role in integer and fractional order differential equations. The Laplace transform of the function [9, 10] $f(t)$ is denoted by $F(s)$ and defined as

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) dt, \qquad (3)$$

where s>0 and s is called the transform parameter. The Laplace transformation of integer order derivative is defined as

$$L\left(f^m(t)\right) = s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1} f^k(0), \qquad (4)$$

where $f^m(t)$ denotes m^{th} order ordinary derivative of f with respect to t. For non-integer order α, it is defined in generalized form as

$$L\left(f^\alpha(t)\right) = s^\alpha F(s) - \sum_{k=0}^{m-1} s^k f^{\alpha-k-1}(0), \qquad (5)$$

where $(m-1) < \alpha \le m$.

II. 4. Differential equation with memory kernel

If we consider any first order differential equation in the form as

$$\frac{d\left(f_1(x)\right)}{dx} = -f_2(x), \qquad (6)$$

then in terms of the memory kernel it can be written as follows [1]

$$\frac{d\left(f_1(x)\right)}{dx} = -\int_a^x K(x-\xi) f_2(\xi) d\xi. \qquad (7)$$

To derive the concept of memory effect via fractional calculus, we consider $K(x-\xi) = \frac{(x-\xi)^{(\alpha-2)}}{\Gamma(1-\alpha)}$ and substitute it in (7) and we get

$$\frac{d\left(f_1(x)\right)}{dx} = -\int_a^x \frac{(x-\xi)^{(\alpha-2)}}{\Gamma(1-\alpha)} f_2(\xi) d\xi = \\ = -_a D_x^{-(\alpha-1)}\left(f_2(x)\right). \qquad (8)$$

Now, taking both sides $(\alpha - 1)$ derivative and we get the required fractional order memory dependent differential equation

$$_a D_x^\alpha\left(f_1(x)\right) = -f_2(x). \qquad (9)$$

Here, the rate of memory kernel decays depending on the fractional parameter α. The strength of the memory is controlled by the fractional order α. When $\alpha \to 1$, the system

becomes week in the sense of memory and when $\alpha = 1$, the system becomes memory less.

Long Memory: The strength of memory is controlled by the order of fractional derivative or fractional integration. If the order of fractional derivative or fractional integration lies in the range $(0, 0.5)$, then the system may be called to have long memory effect.

Short Memory: If the order of fractional derivative or fractional integration is in the range $[0.5, 1)$, then the system is affected by short memory.
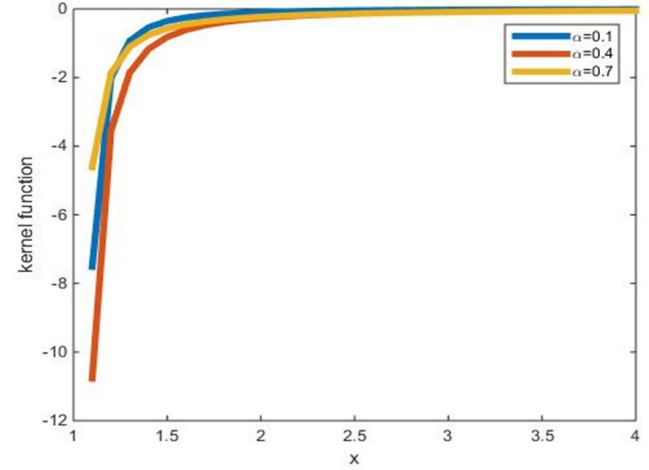
Fig. 1. Plot of memory kernel function $\left(k(x-\xi) = \frac{(x-\xi)^{(\alpha-2)}}{\Gamma(1-\alpha)}, \text{for } \xi = 3\right)$ for different value of α

It is clear from Fig. 1 that the pick of the curve gradually decreases depending on α. Hence, memory strongly depends on the fractional order α.

III. CLASSICAL MODEL

The classical or memory less inventory model has been developed [13] on the basis of the assumptions below. First, we have formulated the classical inventory model.

III. 1. Assumptions

Here, the following assumptions are made in developing the inventory model.
- The inventory consists of only one type of items.
- The lead time is zero or negligible.
- Demand rate is $\begin{cases} (a + bt + ct^2) \text{ for } 0 \le t \le t_1 \\ D_0 \text{ for } t_1 \le t \le T \end{cases}$
- The planning horizon is infinite.
- Shortage occurs during $t_1 \le t \le T$.
- During the stock out period or shortage time, there is complete backlogging.

III. 2. Notations

The notations which are used to develop the inventory model are listed below.

Tab. 1. Symbols and items which are used in the inventory model

Symbols	Items	Symbols	Items
(i) $I_1(t)$	Positive inventory level at time t.	(ii) $I_2(t)$	Negative inventory level at time t.
(iii) C_1	Holding cost per unit per unit time.	(iv) C_2	Shortage cost per unit per unit time.
(v) T	Length of the ordering cycle.	(vi) $TC_{\alpha,\beta}^{av}$	Total average cost per unit per unit time with fractional effect.
(vii) T^*	Optimal ordering interval.	(viii) TC^*	Minimized total average cost.
(ix) C_3	Ordering cost per unit per order.	(x) $(\Gamma, .)$	Gamma function
(xi) $HC_{\alpha,\beta}$	Inventory holding cost with memory effect.	(xii) $(B, .)$	Beta function
(xiii) $PC_{\alpha,\beta}$	Purchasing cost per unit time with memory effect.	(xiv) $SC_{\alpha,\beta}$	Shortage cost per unit with memory effect.
(xv) $T_{\alpha,\beta}^*$	Optimal ordering interval with memory effect.	(xvi) $TC_{\alpha,\beta}^*$	Minimized total average cost with memory effect.

III. 3. Mathematical Formulation

As the positive inventory level $I_1(t)$ reduces due to quadratic type demand during the time interval $[0, t_1]$ and negative inventory level $I_2(t)$ undergoes due to constant demand in the time interval $[t_1, T]$. During the positive inventory $[0, t_1]$ the demand rate is $\left(a + bt + ct^2\right)$ and during the shortage time interval $[t_1, T]$ demand rate is D_0. The classical inventory system is governed by the two ordinary differential equations as in the following form

$$\frac{d(I_1(t))}{dt} = -(a + bt + ct^2)$$
$$\text{for } 0 \leq t \leq t_1 \text{ with } I_1(t_1) = 0, \tag{10}$$

$$\frac{d(I_2(t))}{dt} = -(D_0)$$
$$\text{for } t_1 \leq t \leq T \text{ with } I_2(t_1) = 0. \tag{11}$$

We do not want to describe the derivation of the whole classical inventory model because our focus is to develop the memory dependent inventory model or fractional order inventory model.

IV. FRACTIONAL ORDER INVENTORY MODEL FORMULATION WITH MEMORY KERNEL

To establish the influence of memory effects, first the differential equations (10, 11) can be written using the kernels function as follows [1]

$$\frac{dI_1(t)}{dt} = -\int_0^t k(t - t')(a + bt' + c(t')^2)dt', \tag{12}$$

$$\frac{dI_2(t)}{dt} = -\int_0^t k(t - t')D_0 dt', \tag{13}$$

in which $k(t - t')$ is the kernel function. This type of kernel guarantees the existence of scaling features as it is often intrinsic in most natural phenomena. Thus, to generate the fractional order model we consider $k(t - t') = \frac{1}{\Gamma(\alpha-1)}(t - t')^{\alpha-2}$ where $0 < \alpha \leq 1$ and, $\Gamma(\alpha)$ denotes the gamma function.

Using the definition of fractional derivative [9, 10] we can re-write the Eq. (12, 13) to the form of fractional differential equations with the Caputo-type derivative in the following form

$$\frac{dI_1(t)}{dt} = -{_0D_t^{-(\alpha-1)}}\left(a + bt + ct^2\right), \tag{14}$$

$$\frac{dI_2(t)}{dt} = -{_0D_t^{-(\alpha-1)}}\left(D_0\right). \tag{15}$$

Now, applying fractional Caputo derivative of order $(\alpha - 1)$ on both sides of the Eq. (14, 15) and using the fact the Caputo fractional order derivative and fractional order integral are inverse operators, the following fractional differential equations can be obtained for the model

$${_0^C D_t^\alpha}(I_1(t)) = -(a + bt + ct^2), \tag{16}$$

$${_0^C D_t^\alpha}(I_2(t)) = -D_0, \tag{17}$$

along with boundary condition $I_1(t_1) = 0$, $I_2(t_1) = 0$.

IV. 1. Analysis of the fractional order inventory model

Thus the fractional order inventory model can be governed by the two fractional order differential equations in the following form as

$$\frac{d^\alpha(I_1(t))}{dt^\alpha} = -\left(a + bt + ct^2\right),$$
$$\text{where } 0 \leq t \leq t_1 \text{ with } I_1(t_1) = 0, \tag{18}$$

$$\frac{d^\alpha (I_2(t))}{dt^\alpha} = -D_0, \tag{19}$$

where $t_1 \leq t \leq T$ with $I_2(t_1) = 0$,

(α is considered as differential memory index).

After solving the equation (18, 19), we get the inventory level in the following form:

$$I_1(t) = \frac{a}{\Gamma(1+\alpha)} (t_1^\alpha - t^\alpha) +$$
$$+ \frac{b}{\Gamma(2+\alpha)} (t_1^{\alpha+1} - t^{\alpha+1}) + \frac{2c}{\Gamma(3+\alpha)} (t_1^{\alpha+2} - t^{\alpha+2}), \tag{20}$$

$$I_2(t) = \frac{D_0}{\Gamma(1+\alpha)} (t_1^\alpha - t^\alpha). \tag{21}$$

As the inventory level decreases with respect to time t, so the maximum positive inventory level is

$$M = I_1(0) = \frac{a}{\Gamma(1+\alpha)} (t_1^\alpha) + \frac{b}{\Gamma(2+\alpha)} (t_1^{\alpha+1}) + \frac{2c}{\Gamma(3+\alpha)} (t_1^{\alpha+2}). \tag{22}$$

The maximum backorder units during the shortage period is

$$S = -I_2(T) = \left(\frac{D_0}{\Gamma(1+\alpha)} (T^\alpha - t_1^\alpha) \right). \tag{23}$$

Therefore, the order size during $[0, T]$, is denoted by Q and defined as

$$Q = \big((\text{maximum positive inventory level}) +$$
$$+ (\text{maximum backlogged demand}$$
$$\text{during shortage period}) \big) =$$
$$= M + S = \left(\frac{a}{\Gamma(1+\alpha)} (t_1^\alpha) + \frac{b}{\Gamma(2+\alpha)} (t_1^{\alpha+1}) +$$
$$+ \frac{2c}{\Gamma(3+\alpha)} (t_1^{\alpha+2}) \right) + \left(\frac{D_0}{\Gamma(1+\alpha)} (T^\alpha - t_1^\alpha) \right). \tag{24}$$

Inventory holding Cost with the memory effect is denoted by $HC_{\alpha,\beta}$ [6] and defined as

$$HC_{\alpha,\beta} = C_1 D^{-\beta} (I_1(t)) =$$
$$= \frac{C_1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - t)^{(\beta-1)} I_1(t) \, dt =$$
$$= \frac{C_1}{\Gamma(\beta)} \left(\frac{at_1^{\alpha+\beta}}{\Gamma(1+\alpha)} \left(\left(\frac{1}{\beta}\right) - B(\alpha+1, \beta) \right) + \right. \tag{25}$$
$$+ \frac{bt_1^{\alpha+\beta+1}}{\Gamma(2+\alpha)} \left(\left(\frac{1}{\beta}\right) - B(\alpha+2, \beta) \right) +$$
$$\left. + \frac{2ct_1^{\alpha+\beta+2}}{\Gamma(3+\alpha)} \left(\left(\frac{1}{\beta}\right) - B(\alpha+3, \beta) \right) \right),$$

where $0 \leq t \leq t_1$ (β is considered as integral memory index).

Shortage cost with the memory effect is denoted by $SC_{\alpha,\beta}$ and defined as follows

$$SC_{\alpha,\beta}(T) = -C_2{}_{t_1} D_T^{-\beta} (I_2(t)) =$$
$$= \frac{C_2}{\Gamma(\beta)} \int_{t_1}^T (T-t)^{(\beta-1)} \left(\frac{D_0}{\Gamma(1+\alpha)} (t^\alpha - t_1^\alpha) \right) dt =$$
$$= -\frac{C_2 D_0 t_1^\alpha}{\Gamma(\beta+1)\Gamma(1+\alpha)} T^\beta + \frac{C_2 D_0 \alpha t_1^{\alpha+1}}{(1+\alpha)\Gamma(\beta)\Gamma(1+\alpha)} T^{\beta-1} +$$
$$+ \frac{C_2 D_0}{\Gamma(\beta)\Gamma(1+\alpha)} \left(\frac{1}{(1+\alpha)} - \frac{(\beta-1)}{(2+\alpha)} \right) T^{\beta+\alpha} +$$
$$+ \frac{C_2 D_0 (\beta-1) t_1^{\alpha+2}}{(2+\alpha)\Gamma(\beta)\Gamma(1+\alpha)} T^{\beta-2}, \tag{26}$$

$\left(\text{expanding } (T-t)^{(\beta-1)} \cong T^{\beta-1} \left(1 - (\beta-1)\frac{t}{T} \right), \text{ neglecting higher term as } \left(\frac{t}{T}\right) \right).$

Purchasing cost for the fractional order model is denoted by $PC_{\alpha,\beta}$ and defined

$$PC_{\alpha,\beta} = P \times Q = P \left(\frac{a}{\Gamma(1+\alpha)} (t_1^\alpha) + \frac{b}{\Gamma(2+\alpha)} (t_1^{\alpha+1}) +$$
$$+ \frac{2c}{\Gamma(3+\alpha)} (t_1^{\alpha+2}) + \frac{D_0}{\Gamma(1+\alpha)} (T^\alpha - t_1^\alpha) \right). \tag{27}$$

Therefore, total average cost for the fractional order inventory model is as

$$TC_{\alpha,\beta} = \frac{(HC_{\alpha,\beta} + SC_{\alpha,\beta} + PC_{\alpha,\beta} + C_3)}{T} =$$
$$= \left(HC_{\alpha,\beta} + C_3 + P \left(\frac{a}{\Gamma(1+\alpha)} (t_1^\alpha) + \frac{b}{\Gamma(2+\alpha)} (t_1^{\alpha+1}) + \right. \right.$$
$$\left. + \frac{2c}{\Gamma(3+\alpha)} (t_1^{\alpha+2}) \right) - \frac{PD_0 t_1^\alpha}{\Gamma(1+\alpha)} \right) T^{-1} + \frac{PD_0}{\Gamma(1+\alpha)} T^{\alpha-1} +$$
$$- \frac{C_2 D_0 t_1^\alpha}{\beta\Gamma(\beta)\Gamma(\alpha+1)} T^{\beta-1} + \left(\frac{C_2 D_0 \alpha t_1^{\alpha+1}}{\Gamma(\beta)\Gamma(\alpha+1)(\alpha+1)} \right) T^{\beta-2} +$$
$$+ \left(\frac{C_2 D_0}{\Gamma(\beta)\Gamma(\alpha+1)(\alpha+1)} - \frac{C_2 D_0 (\beta-1)}{\Gamma(\beta)\Gamma(\alpha+1)(\alpha+2)} \right) T^{\alpha+\beta-1} +$$
$$+ \frac{C_2 D_0 (\beta-1) t_1^{\alpha+2}}{(\alpha+2)\Gamma(\beta)\Gamma(\alpha+1)} T^{\beta-3}.$$

We consider different cases of the total average cost depending on the different values of the memory indexes α, β: (i) $0 < \alpha \leq 1.0, 0 < \beta \leq 1.0$. (ii) $\beta = 1.0, 0 < \alpha \leq 1.0$. (iii) $\alpha = 1.0, 0 < \beta \leq 1.0$.

(i) Case-1: $0 < \alpha \leq 1.0, 0 < \beta \leq 1.0$.
In this case, the inventory model can be written as follows

$$\begin{cases} \text{Min} TC_{\alpha,\beta}(T) = AT^{-1} + B_1 T^{\alpha-1} + CT^{\beta-1} + \\ \qquad\qquad + DT^{\beta-2} + ET^{\alpha+\beta-1} + FT^{\beta-3} \\ \text{Subject to } T \geq 0 \end{cases} \tag{28}$$

where, $A = P\left(\frac{at_1^\alpha}{\Gamma(1+\alpha)} + \frac{bt_1^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{2ct_1^{\alpha+2}}{\Gamma(3+\alpha)} - \frac{D_0t_1^\alpha}{\Gamma(\alpha+1)}\right) +$

$+ C_3 + HC_{\alpha,\beta}$, $C = -\frac{C_2D_0t_1^\alpha}{\beta\Gamma(\beta)\Gamma(\alpha+1)}$, $B_1 = \frac{PD_0}{\Gamma(\alpha+1)}$,

$D = \frac{C_2D_0\alpha t_1^{\alpha+1}}{(\alpha+1)\Gamma(\beta)\Gamma(\alpha+1)}$, $E = \frac{C_2D_0}{\Gamma(\beta)\Gamma(\alpha+1)}\left(\frac{1}{(\alpha+1)} - \frac{(\beta-1)}{(\alpha+2)}\right)$,

$F = \frac{C_2D_0(\beta-1)t_1^{\alpha+2}}{(\alpha+2)\Gamma(\beta)\Gamma(\alpha+1)}$.

(a) Primal Geometric Programming Method

To solve (28) analytically, the primal geometric programming method has been applied. The dual form of (28) has been introduced by the dual variable (w). The corresponding primal geometric programming problem has been constructed in the following form as

$$\text{Max}\, d(w) = \left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B_1}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5}\left(\frac{F}{w_6}\right)^{w_6}.$$
(29)

Normalized condition is as

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = 1.$$ (30)

Orthogonal condition is as

$$-w_1 + (\alpha-1)w_2 + (\beta-1)w_3 + (\beta-2)w_4 +$$
$$+ (\alpha+\beta-1)w_5 + (\beta-3)w_6 = 0,$$
(31)

and the primal-dual relations are as follows

$$\left.\begin{array}{l} AT^{-1} = w_1 d(w)\ ,\ \ B_1T^{\alpha-1} = w_2 d(w) \\ CT^{\beta-1} = w_3 d(w)\ ,\ \ DT^{\beta-2} = w_4 d(w) \\ ET^{\alpha+\beta-1} = w_5 d(w)\ ,\ \ FT^{\beta-3} = w_6 d(w) \end{array}\right\}.$$ (32)

Using the above primal-dual relations the followings are evaluated as

$$\frac{Aw_2}{B_1w_1} = \left(\frac{Dw_3}{Cw_4}\right)^\alpha,\ \ \frac{Cw_2}{B_1w_3} = \left(\frac{Dw_3}{Cw_4}\right)^{\alpha-\beta},$$
$$\frac{Dw_5}{Ew_4} = \left(\frac{Dw_3}{Cw_4}\right)^{\alpha+1},\ \ \frac{Fw_5}{Ew_6} = \left(\frac{Dw_3}{Cw_4}\right)^{\alpha+2},$$
(33)

along with

$$T = \frac{Dw_3}{Cw_4}.$$ (34)

Solving (30), (31) and (33) the critical value w_1^*, w_2^*, w_3^*, w_4^*, w_5^*, w_6^* of the dual variables w_1, w_2, w_3, w_4, w_5, w_6 can be obtained and finally the optimum value T^* of T has been calculated from the equation of (34) substituting the critical values. Now the minimized total average cost $TC_{\alpha,\beta}^*$ has been calculated by substituting T^* in (28) analytically. The minimized total average cost and the optimal ordering interval is evaluated from (28) numerically.

(ii) Case-2: $\beta = 1.0,\ \ 0 < \alpha \le 1.0$.
In this case, the inventory model is

$$\begin{cases} \text{Min}\,TC_{\alpha,1}(T) = AT^{-1} + B_1T^{\alpha-1} + CT^0 + \\ \qquad\qquad + DT^{-1} + ET^\alpha + FT^{-2} \\ \text{Subject to}\, T \ge 0 \end{cases}$$ (35)

where, $A = P\left(\frac{at_1^\alpha}{\Gamma(1+\alpha)} + \frac{bt_1^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{2ct_1^{\alpha+2}}{\Gamma(3+\alpha)} - \frac{D_0t_1^\alpha}{\Gamma(\alpha+1)}\right) + C_3 +$

$+ HC_{\alpha,1}$, $B_1 = \frac{PD_0}{\Gamma(\alpha+1)}$, $C = -\frac{C_2D_0t_1^\alpha}{\Gamma(\alpha+1)}$, $D = \frac{C_2D_0\alpha t_1^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}$,

$E = \frac{C_2D_0}{(\alpha+1)\Gamma(\beta)\Gamma(\alpha+1)}$, $F = 0$.

Using the similar analogy as previously, the minimized total average cost and optimal ordering interval has been solved from (35).

(iii) Case-3: $\alpha = 1.0,\ \ 0 < \beta \le 1.0$.
In this case, the inventory model becomes,

$$\begin{cases} \text{Min}\,TC_{1,\beta}(T) = AT^{-1} + B_1T^0 + CT^{\beta-1} + \\ \qquad\qquad + DT^{\beta-2} + ET^\beta + FT^{\beta-3} \\ \text{Subject to}\, T \ge 0 \end{cases}$$ (36)

where, $A = P\left(\frac{at_1}{\Gamma(2)} + \frac{bt_1^2}{\Gamma(3)} + \frac{2ct_1^3}{\Gamma(4)} - \frac{D_0t_1}{\Gamma(2)}\right) + C_3 + HC_{1,\beta}$,

$B_1 = \frac{PD_0}{\Gamma(2)}$, $C = -\frac{C_2D_0t_1}{\beta\Gamma(\beta)\Gamma(2)}$, $D = \frac{C_2D_0t_1^2}{2\Gamma(\beta)\Gamma(2)}$, $E = \frac{C_2D_0}{\Gamma(\beta)\Gamma(2)} \times$

$\times\left(\frac{1}{2} - \frac{(\beta-1)}{3}\right)$, $F = \frac{C_2D_0(\beta-1)t_1^3}{3\Gamma(\beta)\Gamma(2)}$.

Using the similar way of Case-1, the minimized total average cost and the optimal ordering interval has been solved from (36).

V. NUMERICAL EXAMPLES

(i) To establish the memory effect in the inventory model, we have found the optimal ordering interval $\left(T_{\alpha,\beta}^*\right)$, minimized total average cost $\left(TC_{\alpha,\beta}^*\right)$ for different values of the memory index α, β considering the model parameters as given in Tab. 2.

Tab. 2. Fixed values of the parameter in the numerical example

Parameter	P	C_3	C_1	C_2	D_0	a	b	c	t_1
Value	8	250	0.5	1.0	15	2	6	13	0.3456

It is observed from Tab. 3 that when memory of the system is gradually increasing, minimized total average cost is gradually decreasing, i.e. profit is high in long memory effect compared to short memory effect. In long memory effect, minimized total average cost is very low compared

Tab. 3. Optimal ordering interval and minimized total average cost for $\beta=1.0$, and α varies from 0.1 to 1.0 as defined in Sec. IV.1

α	β	$T^*_{\alpha,\beta}$	$TC^*_{\alpha,\beta}$
0.1	**1.0**	**679.3798 (maximum value)**	**16.1243**
0.2↑ (growing memory effect)	1.0	230.6127	35.6275
0.3↑	1.0	114.7478	58.6401
0.4↑	1.0	68.0305	84.6849
0.5↑	1.0	44.6124	113.2369
0.6↑	1.0	31.1730	143.6906
0.7↑	1.0	22.7070	175.2772
0.8↑	1.0	17.0060	206.9958
0.9↑	1.0	12.9827	237.5986↓ (decreasing)
1.0	**1.0**	**10.0566↑ (increasing)**	**265.6647**

to short memory effect or memory less system. In long memory effect, business takes a long time to reach the minimum value of the total average cost compared to low memory effect or memory less system.

(ii) Consider $c = 0$ keeping the other parameters fixed as in the Tab. 2, i.e. demand rate is linear type with initial demand.

Tab. 4. Optimal ordering interval and minimized total average cost for $\beta=1.0$, and α varies from 0.1 to 1.0 as defined in Sec. IV.1

α	β	$T^*_{\alpha,\beta}$	$TC^*_{\alpha,\beta}$
0.1	**1.0**	**149.6354 (maximum value)**	**11.9781**
0.2↑ (growing memory effect)	1.0	65.6815	25.4371
0.3↑	1.0	39.5173	41.2693
0.4↑	1.0	26.9637	59.5881
0.5↑	1.0	19.5944	80.2670
0.6↑	1.0	14.7296	102.9165
0.7↑	1.0	11.2780	126.8481
0.8↑	1.0	8.7276	151.0789
0.9↑	1.0	6.8151	174.1198
1.0	**1.0**	**5.3904↑ (increasing)**	**195.6716↓ (decreasing above)**

It is found from Tab. 4 that the same nature of changes of minimized total average cost and optimal ordering interval has been found like Tab. 3 but the numerical value is low compared to Tab. 3.

(iii) Consider $a = 0$ keeping other parameters fixed as in the Tab. 2, i.e. quadratic type demand rate with no initial demand.

Tab. 5. Optimal ordering interval and minimized total average cost for $\beta=1.0$, and α varies from 0.1 to 1.0 as defined in Sec. IV.1

α	β	$T^*_{\alpha,\beta}$	$TC^*_{\alpha,\beta}$
0.1	**1.0**	**674.3177 (maximum value)**	**16.1020**
0.2↑ (growing memory effect)	1.0	229.1122	35.5660
0.3↑	1.0	114.0888	58.5263
0.4↑	1.0	67.6811	84.5101
0.5↑	1.0	44.4043	112.9967
0.6↑	1.0	31.0388	143.3838
0.7↑	1.0	22.6150	174.9045
0.8↑	1.0	16.9400	206.5590
0.9↑	1.0	12.9338	237.1002
1.0	**1.0**	**10.0195↑ (increasing)**	**265.1079↓ (decreasing above)**

It is observed from Tab. 5 that minimized total average cost is gradually decreasing with gradually increasing memory effect but the numerical value of the optimal ordering interval is gradually increasing with gradually increasing memory effect. The numerical values of the minimized total average cost and optimal ordering interval are low compared to Tab. 3.

(iv) Consider $b = 0$ keeping other parameters fixed as in the Tab. 2, i.e. demand rate is square of the time with initial demand.

Tab. 6. Optimal ordering interval and minimized total average cost for $\beta=1.0$, and α varies from 0.1 to 1.0 as defined in Sec. IV.1

α	β	$T^*_{\alpha,\beta}$	$TC^*_{\alpha,\beta}$
0.1	**1.0**	**674.5641 (maximum value)**	**16.1031 (minimum value)**
0.2↑ (growing memory effect)	1.0	229.3058	35.5739
0.3↑	1.0	114.2185	58.5487
0.4↑	1.0	67.7702	84.5547
0.5↑	1.0	44.4678	113.0701
0.6↑	1.0	31.0856	143.4910
0.7↑	1.0	22.6507	175.0490
0.8↑	1.0	16.9679	206.7433
0.9↑	1.0	12.9559	237.3258
1.0	**1.0**	**10.0373↑ (increasing)**	**265.3753↓ (decreasing above)**

It is observed from the Tab. 6, that in long memory effect (here $\alpha = 0.1$), minimized total average cost is very low compared to the short memory effect or memory less system. In long memory effect, business stay long time compared to the short memory effect. The numerical value of the minimized total average cost and optimal ordering interval are low compared to the Tab. 3 but more compared to the Tab. 4, 5.

(v) Consider $a = 0, b = 0$ keeping other parameters fixed as in Tab. 2, i.e. demand rate is square of the time proportional.

Tab. 7. Optimal ordering interval and minimized total average cost for $\beta=1.0$, and α varies from 0.1 to 1.0 as defined in Sec. IV.1

α	β	$T^*_{\alpha,\beta}$	$TC^*_{\alpha,\beta}$
0.1	**1.0**	**669.4979** (maximum value)	**16.0805** (minimum value)
0.2↑ (growing memory effect)	1.0	227.8003	35.5121
0.3↑	1.0	113.5585	58.4343
0.4↑	1.0	67.4202	84.3792
0.5↑	1.0	44.2594	112.8291
0.6↑	1.0	30.9511	143.1833
0.7↑	1.0	22.5585	174.6755
0.8↑	1.0	16.9018	206.3056
0.9↑	1.0	12.9069	236.8263
1.0	**1.0**	**10.0001↑** (increasing)	**264.8174↓** (decreasing above)

It is found from Tab. 7, in long memory effect or short memory effect, the nature of the changing of the minimized total average cost is same like Tab. 3, 5, 6 but the numerical value is low compared to Tab. 3, 5, 6.

(vi) Consider $a = 0, c = 0$ keeping other parameters fixed as in Tab. 2, i.e. demand rate is time proportional with no initial demand.

Tab. 8 shows that the nature of changing of the minimized total average cost and the optimal ordering interval are the same as Tab. 3–7 but the numerical values are low compared to Tab. 3–7, 9.

(vii) Consider $b = 0, c = 0$ keeping other parameters fixed as in Tab. 2, i.e demand rate is constant which was initially established.

Tab. 9 clears that the nature of changing of the minimized total average cost and optimal ordering interval are the same, i.e. they are sequentially decreasing with respect to memory index as Tab. 3–8 but the numerical values are low compared to Tab. 3–7 except Tab. 8.

Hence, it is observed from Tab. 3–9 that when the demand rate is the proportional time the numerical value of the minimized total average cost is low compared to the other

Tab. 8. Optimal ordering interval and minimized total average cost for $\beta=1.0$, and α varies from 0.1 to 1.0 as defined in Sec. IV.1

α	β	$T^*_{\alpha,\beta}$	$TC^*_{\alpha,\beta}$
0.1	**1.0**	**143.4975** (maximum value)	**11.8747** (minimum value)
0.2↑ (growing memory effect)	1.0	63.5948	25.2183
0.3↑	1.0	38.5067	40.9354
0.4↑	1.0	26.3886	59.1434
0.5↑	1.0	19.2334	79.7164
0.6↑	1.0	14.4869	102.2633
0.7↑	1.0	11.1065	126.0936
0.8↑	1.0	8.6022	150.2233
0.9↑	1.0	6.7217	173.4655
1.0	**1.0**	**5.208↑** (increasing)	**194.6280↓** (decreasing above)

Tab. 9. Optimal ordering interval and minimized total average cost for $\beta=1.0$, and α varies from 0.1 to 1.0 as defined in Sec. IV.1

α	β	$T^*_{\alpha,\beta}$	$TC^*_{\alpha,\beta}$
0.1	**1.0**	**143.7971** (maximum value)	**11.8799** (minimum value)
0.2↑ (growing memory effect)	1.0	63.8650	25.2469
0.3↑	1.0	38.7065	41.0017
0.4↑	1.0	26.5358	59.2576
0.5↑	1.0	19.3440	79.8854
0.6↑	1.0	14.5719	102.4923
0.7↑	1.0	11.1732	126.3872
0.8↑	1.0	8.6553	150.5855
0.9↑	1.0	6.7641	173.8986
1.0	**1.0**	**5.3543↑** (increasing)	**195.1303↓** (decreasing)

assumed demand rate like quadratic type demand rate, constant demand rate, square of the time proportional etc. Hence for time proportional demand rate, business gains more profit without risk because business does not take too long to reach the minimum value of the total average cost as well as minimized total average cost is low.

VI. GRAPHICAL PRESENTATION

(a) Graphical comparison between positive inventory level for long memory affected system as well as low
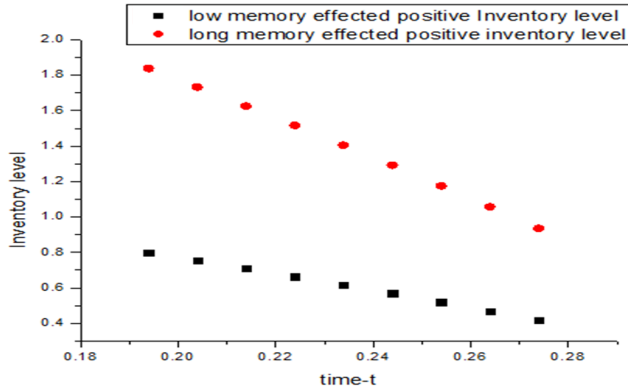
Fig. 2. Scatter diagram inventory level versus time corresponding low memory effect ($\alpha = 0.9$) and long memory effect ($\alpha = 0.1$) of the system

Fig. 3. Scatter diagram for memory index α versus minimized total average cost as numerically described in example (i)–(vii)

memory affected system with respect to time is presented in Fig. 2.

(b) Graphical comparison of the minimized total average cost for different cases as found numerically in the above tables are presented in Fig. 3.

From Fig. 2, it is observed that in long memory effect ($\alpha = 0.1$) inventory level is high but in low memory effect ($\alpha = 0.9$) inventory level is low comparatively long memory effect ($\alpha = 0.1$). It is also noted that Inventory level with respect to time is decreasing gradually for both cases for long memory effect as well as low memory effect.

From the above Fig. 3, it is observed that for some cases, minimized total average cost, which is very close, goes through one line, and for other cases minimized total average cost, which is very close, goes through another line.

VII. CONCLUSIONS

In this paper, we have established the memory effect in an inventory system via fractional calculus. Here, we have considered a memory dependent inventory model with quadratic type demand rate and shortage. The demand rate of the cosmetic product increases with time during the seasonal period like Vaseline, ponds, Olive-oil during winter season and Body spray, perfume, etc., during the summer season. The demand rate most of the time practically increases with time, which may be proportional to time or square of the time, etc. Tab. 3–9 reveal when the system has no initial demand and the demand rate is proportional to the time, profit is maximum. One important observation is found that in long memory effect, business takes more time to reach the minimum value of the total average cost compared to the short memory effect. The inventory level with respect to time is plotted corresponding long memory affected system and short memory affected system. This plot (Fig. 2) presents that the inventory level for high memory affected system as well as low memory affected system is totally different as inventory level for low memory affected system is low but for
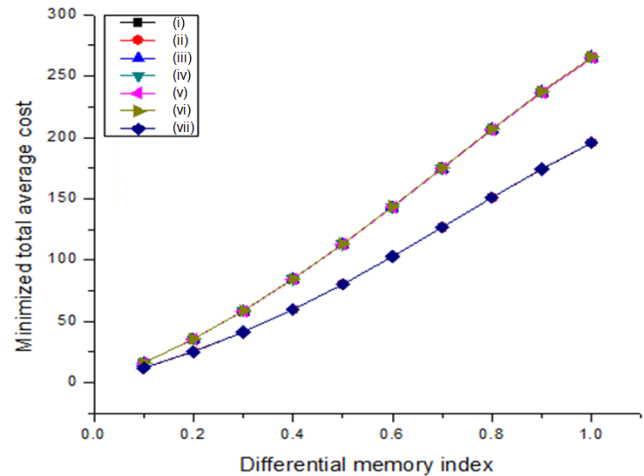
long memory affected system, inventory level is high. This paper may be extended for time delay in payment with memory effect. More new thoughts towards the memory dependent inventory model may be improved considering practical data from industries.

Acknowledgment

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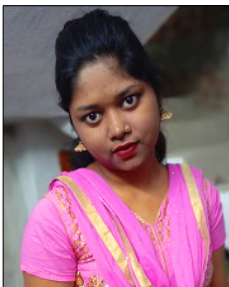
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