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Deformation of modified couple stress thermoelastic diffusion in a thick circular plate due to heat sources

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Abstract: The aim of this study is to present a mathematical model for predicting the results for displacements, stress components, temperature change and chemical potential with considering independently a particular type of heat source. The general solution for the two-dimensional problem of a thick circular plate with heat sources in modified couple stress thermoelastic diffusion has been obtained in the context of one and two relaxation times. Laplace and Hankel transforms technique is applied to obtain the solutions of the governing equations. Resulting quantities are obtained in the transformed domain. The numerical inversion technique has been used to obtain the solutions in the physical domain. Effects of time on the resulting quantities are shown graphically.

Key words: thick circular plate, heat sources, modified couple stress theory, thermoelasticity, Laplace and Hankel transforms

I. INTRODUCTION

A couple-stress theory is an extended continuum theory that includes the effects of couple stress on a material volume, in addition to the classical direct and shear forces per unit area. This immediately admits the possibility of asymmetric stress tensor, since shear stress no longer has to be conjugate in order to ensure rotational equilibrium. The two additional constants are related to the underlying microstructure of the material and are inherently difficult to determine (e.g., Lakes [1] and Lam et al. [2]). Hence, there has been a need to develop higher-order theories involving only one additional material length scale parameter. The small length scale involved in microstructures has questioned the applicability of the classical mechanics model. The small size of the material structure, such as the lattice space between single atoms, is very important in nanotechnology problems. As this scale is ignored in the classical mechanics model, the modified couple-stress theory was developed by Yang et al. [3].

Park and Gao [4] studied the Bernoulli-Euler beam model based on a modified couple stress theory. Simsek and Reddy [5] investigated the bending and vibration of functionally graded microbeams using a new higher order beam theory and the modified couple stress theory. Recently, Shaat et al. [6] studied the size-dependent bending analysis of Kirchhoff nano-plates based on a modified couple-stress theory including surface effects. Ghorbanpour et al. [7] discussed the problem of vibration of bioliquid-filled microtubules using the modified couple stress theory. In this problem, the modified couple stress theory is applied to consider the small scale effects while motion equations are derived using the energy method and Hamilton's principle for both Euler-Bernoulli beam (EBB) and Timoshenko beam (TB) models. Drijani and Shahdadi [8] investigated the effect of shear deformation on the static bending and vibration responses of a simply supported microplate by using the modified couple stress theory and the governing equations and related boundary conditions are solved simultaneously using Hamilton's principle. Recently, Gang et al. [9] presented a nonlinear bending and post-buckling of extensible microscale beams based on the modified couple stress theory where the effects of the material length scale parameter and the Poisson ratio on the bending and thermal post-buckling behaviors of microbeams are discussed in detail and the size-dependent governing differential equations are solved numerically using a shooting method.

Diffusion plays an important role in geophysics, metal oxide semiconductor improvement in crude oil extraction from oil deposits, and at present, diffusion imaging is essentially used for brain exploration in clinical practice. Nevertheless, new applications are emerging outside neuroradiology (cancerology, musculoskeletal radiology), etc. Diffusion can be defined as the movement of molecules from a high concentration to a low concentration. Thermodiffusion in an elastic solid is due to the coupling of the fields of temperature, mass diffusion and strain. Heat and mass exchange with the environment during the process of the thermodiffusion in an elastic solid. The concept of thermodiffusion is used to describe the processes of thermomechanical treatment of metals (carboning, nitriding steel, etc.) and these processes are thermally activated, with their diffusing substances including nitrogen, carbon, etc. They are accompanied by deformations of the solid.

Podstrigach [10] and Nowacki [11–14], developed theories of thermodiffusion elastic solid in which the coupled thermoelastic model is used and implies infinite speeds of propagation of thermoelastic waves. Sherief et al. [15] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. Sherief and Saleh [16] worked on a problem of a thermoelastic half space with a permeating substance in contact with the bounding plane in the context of the theory of generalized thermoelastic diffusion with one relaxation time. Recently, Kumar and Kansal [17] derived the basic equations in generalized thermoelastic diffusion for Green Lindsay (GL-model) theory and discussed the Lamb waves.

Maghraby et al. [18] studied a problem of generalized thermoelasticity in Lord-Shulman theory [19] for a half space subjected to known axisymmetric temperature distributions by using Laplace and Hankel transforms technique. Tripathi et al. [20] investigated temperature distribution and thermal stresses in a semi-infinite cylinder with heat sources in the thermoelastic theory with one relaxation time. Recently, Tripathi et al. [21] have discussed the problem of a thick circular plate with axisymmetric heat supply in a generalized thermoelastic diffusion by using the integral transform technique.

The present paper deals with the two-dimensional problem of a thick circular plate with heat sources in the modified couple stress theory of thermoelastic with mass diffusion by applying the integral transform technique. The normal stress, tangential stress, couple stress, temperature change and chemical potential are computed and presented graphically for different values of time with respect to radial distance. Some deduced cases are also derived from the present investigation.

II. GOVERNING EQUATIONS

Following (Yang et al. [3], Kumar and Kansal [17]), the constitutive relations and the equations of motion in a modified couple-stress generalized thermoelastic elastic with mass diffusion in the absence of body forces, body couples and mass diffusion sources are given by:

(i) Constitutive relations

$$\begin{aligned} t_{ij} =& \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \frac{1}{2}e_{kij}m_{lk,l} + \\ & -\beta_1\left(1+\tau_1\frac{\partial}{\partial t}\right)T\delta_{ij} - \beta_2\left(1+\tau^1\frac{\partial}{\partial t}\right)C\delta_{ij}, \end{aligned} \tag{1}$$

$$m_{ij} = 2\alpha\chi_{ij}, \tag{2}$$

$$\chi_{ij} = \frac{1}{2}\left(\omega_{ij}+\omega_{ji}\right), \tag{3}$$

$$\omega_i = \frac{1}{2}e_{ipq}u_{q,p}, \tag{4}$$

$$P = -\beta_2 e_{kk} + b\left(1+\tau^1\frac{\partial}{\partial t}\right)C - a\left(1+\tau_1\frac{\partial}{\partial t}\right)T, \tag{5}$$

(ii) Equations of motion

$$\begin{aligned} &\left(\lambda+\mu+\frac{\alpha}{4}\Delta\right)\nabla\left(\nabla.u\right) + \left(\mu-\frac{\alpha}{4}\Delta\right)\nabla^2 u + \\ &-\beta_1\left(1+\tau_1\frac{\partial}{\partial t}\right)\nabla T - \beta_2\left(1+\tau^1\frac{\partial}{\partial t}\right)\nabla C = \rho\ddot{u}, \end{aligned} \tag{6}$$

(iii) Equation of heat conduction

$$\begin{aligned} &K\Delta T - \rho c_e\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)T - aT_0\left(\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)C + \\ &+\left(1+\tau_0\frac{\partial}{\partial t}\right)Q = T_0\beta_1\left(\frac{\partial}{\partial t}+\tau_0\eta_0\frac{\partial^2}{\partial t^2}\right)\left(\nabla.u\right), \end{aligned} \tag{7}$$

(iv) Equation of mass diffusion

$$D\beta_2\Delta\left(\nabla.u\right)+Da\left(1+\tau_1\frac{\partial}{\partial t}\right)\Delta T+ \\ +\left(\frac{\partial}{\partial t}+\tau^0\eta_0\frac{\partial^2}{\partial t^2}\right)C-Db\Delta\left(1+\tau^1\frac{\partial}{\partial t}\right)C=0, \tag{8}$$

where t_{ij} are the components of stress tensor, λ and μ are material constants, δ_{ij} is Kronecker's delta, e_{ij} are the components of strain tensor, e_{ijk} is alternate tensor, m_{ij} are the components of couple-stress, $\beta_1 = (3\lambda+2\mu)\,\alpha_t$, $\beta_2 = (3\lambda+2\mu)\,\alpha_c$. Here α_t, α_c are the coefficients of linear thermal expansion and diffusion expansion, respectively, T is the temperature change, C is the mass concentration, α is the couple stress parameter, χ_{ij} is symmetric curvature, ω_i is the rotational vector, P is the chemical potential of the material per unit mass, b is the coefficient describing the measure of mass diffusion effects, a is the coefficient describing the measure of thermoelastic diffusion, $u = (u_1, u_2, u_3)$ is the displacement vector, ρ is the density, Δ is the Laplacian operator, ∇ is the nabla (gradient) operator, K is the coefficient of the thermal conductivity, c_e is the specific heat at constant strain, Q is the heat source, T_0 is the reference temperature assumed to be such that $T/T_0 << 1$. D is the thermoelastic diffusion constant. Here τ^0, τ^1 are the diffusion relaxation times with $\tau^1 \geq \tau^0 \geq 0$ and τ_0, τ_1 are thermal relaxation times with $\tau_1 \geq \tau_0 \geq 0$. Here $\tau_1 = \tau^1 = 0, \eta_0 = 1, \gamma = \tau_0$, for Lord-Shulman (L-S) model and $\eta_0 = 0, \gamma = \tau^0$, for Green Lindsay (G-L) model.

III. FORMULATION OF THE PROBLEM

We consider a homogeneous isotropic, modified couple stress generalized thermodiffusion elastic thick plate of thickness $2d$ occupying the region defined by $0 \leq r \leq \infty$, $-d \leq z \leq d$. Cylindrical polar coordinates (r, ϕ, z) having origin on the surface $z = 0$, between the lower and upper surfaces of the plate and the z-axis is assumed to be the axis of symmetry. Due to symmetry about z-axis, all the field quantities depend only on (r, z, t).

The initial temperature in the thick plate is given by a constant temperature T_0 and the heat flux $g_0F(r, z)$ is prescribed on the upper and lower boundary surfaces. For $t > 0$, heat is generated within the plate at the rate $Q(r, z, t)$. Under these conditions, thermoelastic quantities in a semi-infinite thick circular plate are required to be determined. The displacement vector takes the form

$$u = (u_r, 0, u_z)\,. \tag{9}$$

We introduce the dimensionless quantities

$$\left(r', z'\right)=\frac{\omega^*}{c_1}\left(r, z\right),\ \left(u_r', u_z'\right)=\frac{\omega^*}{c_1}\left(u_r, u_z\right),$$
$$t_{ij}'=\frac{t_{ij}}{\beta_1T_0},\ m_{ij}'=\frac{\omega^*m_{ij}}{c_1\beta_1T_0}, \\ \left(t', \gamma', \tau_1', \tau_0', \tau^{0'}, \tau^{1'}\right)=\omega^*\left(t, \gamma, \tau_1, \tau_0, \tau^0, \tau^1\right), \\ \left(T', C'\right)=\frac{\left(\beta_1T, \beta_2C\right)}{\rho c_1^2}, P'=\frac{P}{\beta_2}, Q'=\frac{c_eQ}{K^*\omega^{*2}}, \\ c_1^2=\frac{\lambda+2\mu}{\rho}, \omega^{*2}=\frac{\lambda}{\left(\mu t^2+\rho\alpha\right)}, \tag{10}$$

where ω^* and c_1 are the characteristic frequency and longitudinal wave velocity. Upon introducing (9), (10) in equations (6)–(8), after suppressing the primes, we obtain

$$a_1\frac{\partial e}{\partial r}+a_2\left(\nabla^2-\frac{1}{r^2}\right)u_r+a_3\Delta\left(\frac{\partial e}{\partial r}+\right. \\ \left.-\left(\nabla^2-\frac{1}{r^2}\right)u_r\right)-\tau_t\frac{\partial T}{\partial r}-\tau_t^1\frac{\partial C}{\partial r}=\frac{\partial^2u_r}{\partial t^2}, \tag{11}$$

$$a_1\frac{\partial e}{\partial z}+a_2\nabla^2u_z+a_3\Delta\left(\frac{\partial e}{\partial z}-\nabla^2u_z\right)-\tau_t\frac{\partial T}{\partial z}+ \\ -\tau_t^1\frac{\partial C}{\partial z}=\frac{\partial^2u_z}{\partial t^2}, \tag{12}$$

$$\nabla^2T-a_4\tau_t^0T-a_5\tau_\gamma^0C+a_6\tau_t^{10}Q=a_7\tau_{\eta_0}^0e, \tag{13}$$

$$a_8\nabla^2e+a_9\tau_t\nabla^2T+\tau_t^{20}C-a_{10}\tau_t^1\nabla^2C=0, \tag{14}$$

where

$$a_1=\frac{(\lambda+\mu)}{\rho c_1^2}, a_2=\frac{\mu}{\rho c_1^2}, a_3=\frac{\alpha\omega^{*2}}{4\rho c_1^4}, a_4=\frac{\rho c_e c_1^2}{K\omega^*}, \\ a_5=\frac{aT_0\beta_1c_1^2}{\beta_2K\omega^*}, a_6=\frac{\beta_1}{\rho c_e}, a_7=\frac{T_0\beta_1^2}{\rho K\omega^*}, a_8=\frac{\beta_2^2D\omega^*}{\rho c_1^4}, \\ a_9=\frac{a\beta_2D\omega^*}{\beta_1c_1^2}, a_{10}=\frac{bD\omega^*}{c_1^2}, \tau_t=\left(1+\tau_1\frac{\partial}{\partial t}\right), \\ \tau_t^1=\left(1+\tau^1\frac{\partial}{\partial t}\right), \tau_t^0=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right), \\ \tau_{\eta_0}^0=\left(\frac{\partial}{\partial t}+\eta_0\tau_0\frac{\partial^2}{\partial t^2}\right), \tau_\gamma^0=\left(\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right), \\ \tau_t^{10}=\left(1-\tau_0\frac{\partial}{\partial t}\right), \tau_t^{20}=\left(\frac{\partial}{\partial t}+\eta_0\tau^0\frac{\partial^2}{\partial t^2}\right), \\ \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}, \\ e=\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial u_z}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{\partial u_z}{\partial z}.$$

The displacement components u_r and u_z in terms of potential functions ϕ and ψ in a dimensionless form are given by

$$u_r=\frac{\partial\phi}{\partial r}+\frac{\partial^2\psi}{\partial r\partial z}, \tag{15}$$

$$u_z = \frac{\partial \phi}{\partial z} - \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} \right). \tag{16}$$

With the aid of (15) and (16), equations (11)–(14) yield

$$\left[\delta \nabla^2 - \frac{\partial^2}{\partial t^2} \right] \phi - \tau_t T - \tau_t^1 C = 0, \tag{17}$$

$$\left[a_2 \nabla^2 - a_3 \nabla^4 - \frac{\partial^2}{\partial t^2} \right] \psi = 0, \tag{18}$$

$$a_7 \tau_{\eta_0}^0 \nabla^2 \phi - \left(\nabla^2 - a_4 \tau_t^0 \right) T + a_5 \tau_\gamma^0 C = a_6 \tau_t^{10} Q, \tag{19}$$

$$a_8 \nabla^4 \phi + a_9 \tau_t \nabla^2 T + \left(\tau_t^{20} - a_{10} \tau_t^1 \nabla^2 \right) C = 0, \tag{20}$$

where

$$\delta = (a_1 + a_2)\,, \, e = \nabla^2 \phi.$$

We define Laplace and Hankel transforms as

$$\bar{f}\,(r, z, s) = \int_0^\infty f\,(r, z, t)\;\; e^{-st} dt,$$

$$\hat{f}\,(\eta, z, s) = H\left[\bar{f}\,(r, z, s) \right] = \int_0^\infty \bar{f}\,(r, z, s)\;\; r J_n\,(\eta r)\, dr, \tag{21}$$

where s is the Laplace transform parameter, η is the Hankel transform parameter and $J_n()$ is the Bessel function of the first kind of order n.

Applying the Laplace and Hankel transforms defined by (21) on equations (17)–(20), after simplification, we obtain

$$\left[D_1 D^6 + D_2 D^4 + D_3 D^2 + D_4 \right] \left(\hat{\phi}, \hat{T}, \hat{C} \right) = a_6 \tau_t^{10} Q, \tag{22}$$

$$\left[D^4 - G_1 D^2 + G_2 \right] \hat{\psi} = 0, \tag{23}$$

where

$$D_1 = a_8 + \delta a_{10} \tau_t^{22},$$

$$D_2 = \left\{ \begin{matrix} - \left(\eta^2 \delta + s^2 \right) a_{10} \tau_t^{22} - \delta \left(\tau_t^{77} + a_5 a_9 \tau_t^{11} \tau_t^{44} + \left(a_4 \tau_t^{33} + 2\eta^2 \right) a_{10} \tau_t^{22} \right) - \tau_t^{11} \left(a_7 a_{10} \tau_t^{22} \tau_t^{55} + a_5 a_8 \tau_t^{44} \right) \\ - \tau_t^{22} \left(a_7 a_8 \tau_t^{11} \tau_t^{55} - a_4 a_8 \tau_t^{33} \right) - 3 a_8 \eta^2 \end{matrix} \right\},$$

$$D_3 = \left\{ \begin{matrix} \left(\eta^2 \delta + s^2 \right) \left(\tau_t^{77} + a_5 a_9 \tau_t^{11} \tau_t^{44} + \left(a_4 \tau_t^{33} + 2\eta^2 \right) a_{10} \tau_t^{22} \right) + \delta \left(a_{10} \tau_t^{22} \eta^4 + \left(\tau_t^{77} + a_5 a_9 \tau_t^{11} \tau_t^{44} + a_4 a_{10} \tau_t^{22} \tau_t^{33} \right) \eta^2 + a_4 \tau_t^{33} \tau_t^{77} \right) \\ + a_7 \tau_t^{11} \tau_t^{55} \tau_t^{77} + 2\eta^2 \left(a_7 a_{10} \tau_t^{22} \tau_t^{55} + a_5 a_8 \tau_t^{44} + a_8 \tau_t^{22} \left(a_7 \tau_t^{11} \tau_t^{55} - a_4 \tau_t^{33} \right) \right) + 3 \eta^2 a_8 \end{matrix} \right\},$$

$$D_4 = \left\{ \begin{matrix} - \left(\eta^2 \delta + s^2 \right) \left(a_{10} \tau_t^{22} \eta^4 + \left(\tau_t^{77} + a_5 a_9 \tau_t^{11} \tau_t^{44} + a_4 a_{10} \tau_t^{22} \tau_t^{33} \right) \eta^2 + a_4 \tau_t^{33} \tau_t^{77} \right) \\ - a_7 \tau_t^{11} \tau_t^{55} \tau_t^{77} \eta^2 - \eta^4 \left(a_7 a_{10} \tau_t^{22} \tau_t^{55} + a_5 a_8 \tau_t^{44} + \tau_t^{22} \left(a_7 a_8 \tau_t^{11} \tau_t^{55} - a_4 a_8 \tau_t^{33} - a_8 \eta^6 \right) \right) \end{matrix} \right\},$$

$$G_1 = \frac{a_2 + 2 a_3 \eta^2}{a_3}, G_2 = \frac{a_3 \eta^4 + a_2 \eta^2 + s^2}{a_3}, \tau_t^{11} = (1 + \tau_1 s)\,, \tau_t^{22} = \left(1 + \tau^1 s \right), \tau_t^{33} = \left(s + \tau_0 s^2 \right),$$

$$\tau_t^{44} = \left(s + \gamma s^2 \right), \tau_t^{55} = \left(s + \eta_0 \tau_0 s^2 \right), \tau_t^{66} = (1 - \tau_0 s)\,, \tau_t^{77} = \left(s + \eta_0 \tau^0 s^2 \right).$$

The general solution of equation (22) can be written as

$$\hat{\phi} = \hat{\phi}_1 + \hat{\phi}_2 + \hat{\phi}_3 + \hat{\phi}_p, \tag{24}$$

where $\hat{\phi}_i$ (i=1, 2, 3) is a solution of the homogeneous differential equation given by

$$\left(D^2 - m_i^2 \right) \hat{\phi}_i = 0, \quad i = 1, 2, 3. \tag{25}$$

The solution of the equation (25) can be written as

$$\hat{\phi}_i = \sum_{i=1}^{3} A_i \cosh\,(m_i z)\,, \tag{26}$$

where m_1, m_2 and m_3 are the roots of the characteristic equation given by

$$\left[D_1 D^6 + D_2 D^4 + D_3 D^2 + D_4 \right] = 0. \tag{27}$$

Also $\hat{\phi}_p$ is the particular solution satisfying the equation

$$\left(D^2 - m_1^2 \right) \left(D^2 - m_2^2 \right) \left(D^2 - m_3^2 \right) \hat{\varphi}_p = a_6 \tau_t^{66} \hat{Q}. \tag{28}$$

Let the heat source $Q\,(r, z, t)$ be taken as

$$Q\,(r,\, z,\, t) = \frac{H\,(t)\,\sqrt{r} e^{-z}}{\sqrt{1 + r^2}}, \tag{29}$$

where $H()$ is the Heaviside unit step function. Applying Laplace and Hankel transforms on equation (30), yield

$$\hat{Q}(\eta, z, s) = \frac{e^{-(\eta+z)}}{s\sqrt{\eta}}. \tag{30}$$

The solution of the equation (28) take the form

$$\hat{\phi}_p = \frac{a_6\tau_t^{66}\hat{Q}}{(1-m_1^2)(1-m_2^2)(1-m_3^2)}, \tag{31}$$

the solution of the equation (22), with the aid of (26) and (31) in (24) can be written as

$$\left(\hat{\phi}, \hat{T}, \hat{C}\right)(\eta, z, s) = \sum_{i=1}^{3}(1, b_i, d_i)\left(A_i\cosh(m_i z) + \frac{a_6\tau_t^{66}\hat{Q}}{(1-m_i^2)}\right), \tag{32}$$

where

$$b_i = \sum_{i=1}^{3}\frac{a_7\tau_t^{55}\left(m_i^2-\eta^2\right)\left(\tau_t^{77} - a_{10}\left(m_i^2-\eta^2\right)\tau_t^{22}\right) - a_5a_8\tau_t^{44}\left(m_i^2-\eta^2\right)^2}{\left[\left(-\left(m_i^2-\eta^2\right)+a_4\tau_t^{33}\right)\left(\tau_t^{77} - \left(m_i^2-\eta^2\right)\left(a_{10}\tau_t^{22}+a_5a_9\tau_t^{11}\tau_t^{44}\right)\right)\right]},$$

$$d_i = \sum_{i=1}^{3}\frac{\left(m_i^2-\eta^2\right)^2\left[a_7a_9\tau_t^{11}\tau_t^{55} + a_8\left(\left(m_i^2-\eta^2\right) - a_4\tau_t^{33}\right)\right]}{\left[\left(-\left(m_i^2-\eta^2\right)+a_4\tau_t^{33}\right)\left(\tau_t^{77} - \left(m_i^2-\eta^2\right)\left(a_{10}\tau_t^{22}+a_5a_9\tau_t^{11}\tau_t^{44}\right)\right)\right]}, \quad i = 1, 2, 3.$$

Following the same procedure, we take the solution of (23) as

$$\hat{\psi} = \sum_{i=4}^{5}A_i\sinh(m_i z), \tag{33}$$

where m_4 and m_5 are the roots of the characteristic equation (23).

IV. BOUNDARY CONDITIONS

$$\frac{\partial T}{\partial z} = \pm g_0 F(r, z) \quad \text{at } z = \pm d, \tag{34}$$

$$t_{zz} = t_{zr} = m_{z\phi} = 0 \quad \text{at } z = \pm d, \tag{35}$$

$$P = \delta(t) f(r) \quad \text{at } z = \pm d, \tag{36}$$

where

$$F(r, z) = z^2 e^{-\omega r}, \omega > 0, \tag{37}$$

$$f(r) = H(a - r), \tag{38}$$

and $\delta()$ is the Dirac delta function.

The non-dimensional values of t_{zz}, t_{zr}, $m_{z\phi}$ and P are given by

$$t_{zz} = a_{11}e + 2a_{12}\left(\frac{\partial u_z}{\partial z}\right) + - a_{13}\left\{\left(1+\tau_1\frac{\partial}{\partial t}\right)T + \left(1+\tau^1\frac{\partial}{\partial t}\right)C\right\}, \tag{39}$$

$$t_{zr} = a_{12}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) + - a_{14}\left\{\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\right\}, \tag{40}$$

$$m_{z\phi} = 2a_{14}\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{\partial^2 u_z}{\partial r\partial z}\right), \tag{41}$$

$$P = -e + a_{15}\tau_t^1 C - a_{16}\tau_t T, \tag{42}$$

where $a_{11} = \frac{\lambda}{\beta_1 T_0}$, $a_{12} = \frac{\mu}{\beta_1 T_0}$, $a_{13} = \frac{\rho c_1^2}{\beta_1 T_0}$, $a_{14} = \frac{\alpha\omega^{*2}}{4c_1^2\beta_1 T_0}$, $a_{15} = \frac{b\rho c_1^2}{\beta_2^2}$, $a_{16} = \frac{a\rho c_1^2}{\beta_1\beta_2}$.

Applying Laplace and Hankel transforms defined by (21) on (37) and (38), we obtain

$$\hat{F}(\eta, z) = \frac{z^2\omega}{s\left(\omega^2+\eta^2\right)^{3/2}}, \tag{43}$$

$$\hat{f}(\eta) = \frac{aJ_1(\eta a)}{\eta}. \tag{44}$$

Substituting the values of $\hat{\phi}$, $\hat{T}$, $\hat{C}$ and $\hat{\psi}$ from (32) and (33) in the boundary conditions (34)–(36) and with the aid of (16), (17), (21) and (39)–(44), we obtain the expressions for displacement components, stresses, temperature change, chemical potential and mass concentration as

$$\hat{u}_r = -\eta\left[\sum_{i=1}^{3}A_i\cosh(m_i d) + \sum_{i=4}^{5}A_i\sinh(m_i d) + H\cosh(d)\right], \tag{45}$$

$$\hat{u}_z=\left[\sum_{i=1}^{3}m_iA_i\sinh(m_id)+\eta^2\sum_{i=4}^{5}A_i\cosh(m_id)+H\sinh(d)\right], \tag{46}$$

$$\left(\hat{T},\hat{C}\right)=\left[\sum_{i=1}^{3}(b_i,\,d_i)A_i\cosh(m_id)+\sum_{i=4}^{5}(b_i,\,d_i)H\cosh(d)\right], \tag{47}$$

$$\hat{t}_{zz}=\left[\sum_{i=1}^{3}M_iA_i\cosh\left(m_id\right)+\sum_{i=4}^{5}M_iA_i\cosh\left(m_id\right)+P_2\right], \tag{48}$$

$$\hat{t}_{zr}=-\eta\left[\sum_{i=1}^{3}N_iA_i\sinh\left(m_id\right)+\sum_{i=4}^{5}N_iA_i\sinh\left(m_id\right)+P_3\right], \tag{49}$$

$$\hat{P}=\left[\sum_{i=1}^{3}K_iA_i\cosh\left(m_id\right)+\sum_{i=4}^{5}K_iA_i\cosh\left(m_id\right)\;+P_4\right], \tag{50}$$

$$\hat{m}_{z\phi}=-2\eta a_{14}\left[\sum_{i=4}^{5}T_iA_i\cosh\left(m_id\right)\right], \tag{51}$$

where

$$A_1=\frac{\Delta_1}{\Delta},A_2=\frac{\Delta_2}{\Delta},A_3=\frac{\Delta_3}{\Delta},A_4=\frac{\Delta_4}{\Delta},A_5=\frac{\Delta_5}{\Delta},$$

$$\begin{aligned}\Delta=R_1m_1g_1&\left[\begin{array}{l}M_2h_2\left(N_3K_5g_3h_5-N_5K_3g_5h_3+N_3K_4g_3h_4-N_4K_3g_4h_3\right)\\-M_3h_3\left(N_2K_5g_2h_5-N_5K_2g_5h_2+N_2K_4g_2h_4-N_4K_2g_4h_2\right)\\+M_4h_4\left(N_2K_3g_2h_3-N_3K_2g_3h_2\right)+M_5h_5\left(N_2K_3g_2h_3-N_3K_2g_3h_2\right)\end{array}\right]+\\+R_2m_2g_2&\left[\begin{array}{l}M_1h_1\left(N_3K_5g_3h_5-N_5K_3g_5h_3-N_3K_4g_3h_4+N_4K_3g_4h_3\right)\\-M_3h_3\left(N_1K_5g_1h_5-N_5K_1g_5h_1-N_1K_4g_1h_4+N_4K_1g_4h_1\right)\\+M_4h_4\left(N_3K_1g_3h_1-N_1K_3g_1h_3\right)+M_5h_5\left(N_1K_3g_1h_3-N_3K_1g_3h_1\right)\end{array}\right]+\\+R_3m_3g_3&\left[\begin{array}{l}M_1h_1\left(N_2K_5g_2h_5-N_5K_2g_5h_2-N_4K_2g_4h_2+N_2K_4g_2h_4\right)\\-M_2h_2\left(N_1K_5g_1h_5-N_5K_1g_5h_1+N_1K_4g_1h_4-N_4K_1g_4h_1\right)\\+M_4h_4\left(N_1K_2g_1h_2-N_2K_1g_2h_1\right)\\+M_5h_5\left(N_1K_2g_1h_2-N_2K_1g_2h_1\right)\end{array}\right],\end{aligned}$$

$$g_1=\sinh(m_1d),g_2=\sinh(m_2d),g_3=\sinh(m_3d),g_4=\sinh(m_4d),g_5=\sinh(m_5d),$$

$$h_1=\cosh(m_1d),h_2=\cosh(m_2d),h_3=\cosh(m_3d),h_4=\cosh(m_4d),h_5=\cosh(m_5d),$$

$\Delta_i\;(i=1,\ldots,5)$ are obtain by replacing 1st, 2nd, 3rd, 4th and 5th column by

$$\left[\left(g_0\hat{F}\left(\eta z\right)-P_1\right),-P_2,-P_3,0,\left(\hat{F}\left(\eta\right)-P_4\right)\right]^T\;\text{ in }\Delta_i,\;\text{ and}$$

$$H=\frac{a_6\tau_t^{66}\hat{Q}}{\left(1-m_1^2\right)\left(1-m_2^2\right)\left(1-m_3^2\right)},P_1=\sum_{i=1}^{3}b_iH\sinh(d),P_2=H\left[a_{11}\left(1+\eta^2\right)+2a_{12}-a_{13}\left(\tau_t^{11}+\tau_t^{22}\right)\right]\cosh(\mathrm{d}),$$

$$P_3=-2Ha_{12}\eta\sinh\left(d\right),P_4=\left[-\left(1+\eta^2+\sum_{i=1}^{3}\left(a_{15}\tau_t^{22}d_i-a_{16}\tau_t^{66}b_i\right)\right)\right]H\cosh(d),$$

$$\sum_{i=1}^{3}M_i=\sum_{i=1}^{3}\left[a_{11}\left(m_i^2-\eta^2\right)+2a_{12}m_i^2-a_{13}\left(\tau_t^{11}b_i+\tau_t^{22}d_i\right)\right],\sum_{i=4}^{5}M_i=\sum_{i=4}^{5}\left[2\left(a_{11}+a_{12}\right)m_i\eta^2\right],$$

$$\sum_{i=1}^{3}N_i=\sum_{i=1}^{3}2a_{12}m_i,\sum_{i=4}^{5}N_i=\sum_{i=4}^{5}\left[a_{12}\left(m_i^2+\eta^2\right)-a_{14}\left(m_i^2-\eta^2\right)^2\right]\;,\sum_{i=4}^{5}T_i=\sum_{i=4}^{5}m_i\left(m_i^2-\eta^2\right),$$

$$\sum_{i=1}^{3}K_i=\sum_{i=1}^{3}\left[-\left(m_i^2+\eta^2\right)+a_{15}\tau_t^{22}d_i-a_{16}\tau_t^{66}b_i\right]\;,\sum_{i=4}^{5}K_i=\sum_{i=4}^{5}-2\eta^2m_i.$$

V. DEDUCED CASES

(i) If $\alpha = 0$, in equations (45)–(51), we obtain the components of displacement, stresses and temperature change for a generalized thermoelastic with mass diffusion medium. The results obtained here are similar as given by Tripathi et al. [21] with the changed value of

$$\hat{F}(\eta, z) = \frac{z^2\omega}{s(\omega^2+\eta^2)^{3/2}}.$$

(ii) In the absence of diffusion $\left(a = D = \tau_t^1 = 0\right)$, in equations (45)–(51), we obtain the components of displacement, stresses and temperature change in a modified couple stress thermoelastic medium.

(iii) If $\tau_1 = \tau^1 = 0$, $\eta_0 = 1$, $\gamma = \tau_0$, in equations (45)–(51), we obtain the corresponding results for modified couple stress thermoelastic with mass diffusion for Lord-Shulman (L-S) model.

(iv) If $\eta_0 = 0$, $\gamma = \tau^0$, in equations (45)–(51), we obtain the corresponding results for modified couple stress thermoelastic with mass diffusion for Green-Lindsay (G-L) model.

VI. NUMERICAL INVERSION OF THE TRANSFORMS

To obtain a solution of the problem in the physical domain we must invert the transforms in (45)–(51) for all the theories. Here the displacement components, normal and tangential stresses, temperature change, chemical potential and mass concentration are functions of z, the parameters of Laplace and Hankel transforms s and η, respectively, and hence are of the form $\hat{f}(\eta, z, s)$. We first invert the Hankel transform, which gives the Laplace transform expression $\bar{f}(r, z, s)$ of the function $f(r, z, t)$ as

$$\bar{f}(r,z,s) = \int_0^\infty \eta \hat{f}(\eta, z, s) J_n(\eta r)\, d\eta. \qquad (52)$$

Now for fixed values of η, r and z, the function $\bar{f}(r, z, s)$ in (52) can be considered as the Laplace transform $\bar{g}(s)$ of the same function $g(t)$. Following Honig and Hirdes [22], the Laplace transformed function $\bar{g}(s)$ can be inverted as given below:

$$g(t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st}\bar{g}(s)\, ds, \qquad (53)$$

where C is an arbitrary real number greater than all the real parts of the singularities of $\bar{g}(s)$. Taking $s = C + iy$, we get

$$g(t) = \frac{e^{Ct}}{2\pi}\int_{-\infty}^{\infty} e^{ity}\bar{g}(C+iy)\, dy. \qquad (54)$$

Now, taking $e^{-Ct}g(t)$ as $h(t)$ and expanding it as Fourier series in $[0, 2L]$, we obtain approximately

$$g(t) = g_\infty(t) + E_D, \qquad (55)$$

$$g_\infty(t) = (C_0/2) + \sum_{k=1}^{\infty} C_k, \text{ for } 0 \le t \le 2L, \qquad (56)$$

and

$$C_k = \left(e^{Ct}/L\right) Re\left[e^{ik\pi t/L}\bar{g}\left(C + (ik\pi/L)\right)\right]. \qquad (57)$$

E_D is the discretization error that can be made arbitrarily small by choosing a large enough C. The values of C and L are chosen according to the criteria outlined by Honig and Hirdes [22].

Since the infinite series in (57) can be summed up only to a finite number of N terms, the approximate value of $g(t)$ becomes

$$g_N(t) = (C_0/2) + \sum_{k=1}^{N} C_k, \text{ for } 0 \le t \le 2L. \qquad (58)$$

We now introduce a truncation error E_T that must be added to the discretization error to produce the total approximate error in evaluating $g(t)$ using the above formula. Two methods are used to reduce the total error. The discretization errors is reduced by using the 'Korrecktur' method Honig and Hirdes [22] and then the 'ϵ-algorithm' is used to reduce the truncation error and hence to accelerate the convergence. The Korrecktur method formula to evaluate the function $g(t)$ is

$$g(t) = g_\infty(t) - e^{-2CL}g_\infty(2L+t) + E_D^{'},$$

where $\left|E_D^{'}\right| >> |E_D|$. Thus the approximate value of $g(t)$ becomes

$$g_{N_k}(t) = g_N(t) - e^{-2CL}g_{N'}(2L+t), \qquad (59)$$

where $N^{'}$ is an integer such that $N^{'} < N$.

We shall now describe the ε-algorithm, which is used to accelerate the convergence of the series in (58). Let N be an odd natural number and $s_m = \sum_{k=1}^{m} C_k$ be the sequence of partial sums of (59). We define the ε-sequence by

$$\varepsilon_{0,m} = 0, \varepsilon_{1,m} = s_m,$$
$$\varepsilon_{n+1,m}\varepsilon_{n-1,m+1} + \frac{1}{\varepsilon_{n,m+1} - \varepsilon_{n,m}} = (s_m + 1); \qquad (60)$$
$$n, m = 1, 2, 3, \ldots$$

Following Press et al. [23], the sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, \ldots, \varepsilon_{N,1}$ converges to $g(t) + E_D - (C_0/2)$ faster than the sequence of partial sums s_m, $m = 1, 2, 3, \ldots$ The actual procedure to invert the Laplace transform consists of (60) together with the ε-algorithm.

The last step is to calculate the integral in equation (52). The method for calculating this integral is described by Press et al. [23]. It involves the use of Romberg's integration with adaptive step size. This also uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

VII. NUMERICAL DATA

For numerical computations, following Daliwal and Singh [24], we take the magnesium material (thermoelastic diffusion solid) as:

λ=2.696×10^{10} kg m^{-1}s^{-2}, μ=1.639×10^{10} kg m^{-1}s^{-2},
$\rho = 1.74\times10^3$ kg m^{-3}, $T_0 = 0.293 \times 10^3$ K,
$a = 1.02 \times 10^4$ m^{2}s^{-2}K^{-1}, $\alpha_t = 1.78 \times 10^{-5}$ K^{-1},
$c_e = 1.04 \times 10^3$ J kg^{-1} K^{-1}, α_c=1.98 × 10^{-4}m^{3}kg^{-1},
$b = 9 \times 10^5$ kg^{-1}m^{5}s^{-2}, $D = 0.85 \times 10^{-8}$ kg s m^{-3},
$\alpha = 1$ kg m s^{-2}, $K = 1.7\times10^2$ Wm^{-1}K^{-1}, $t = 1$ s,
$\omega = 10$ s^{-1}, $d = 1$, $\tau_0 = 0.01$ s, $\tau^0 = 0.03$ s, $\tau_1 = 0.02$ s,
$\tau^1 = 0.04$ s.

VIII. RESULTS AND DISCUSSION

MATLAB software has been used to determine the normal stress, tangential stress, couple stress, temperature change and chemical potential for different values of time for both L-S and G-L theories are computed numerically and shown graphically in Figs. 1–5, respectively.

In Figs. 1–5, the solid line (—) corresponds to L-S $(t = 0.5)$, the solid line with a centre symbol $(- * -)$ corresponds to L-S $(t = 1.0)$, the solid line with a centre symbol $(-o-)$ corresponds to L-S $(t = 1.5)$, the small dash line (- - - -) corresponds to G-L $(t = 0.5)$, the small dash line with a centre symbol (- - -∗- - -) corresponds to G-L $(t = 1.0)$ and the small dash line with a centre symbol (—o—) corresponds to G-L $(t = 1.5)$.

Figs. 1, 2, 3, 4 and 5 show normal stress, tangential stress, couple stress, temperature change and chemical potential in the radial direction. Fig. 1 depicts the variation of normal stress t_{zz} with radial distance r. It is observed from the figure that the value of normal stress initially increases, decreases for the range $0 \leq r \leq 0.6$ and then increases in the remaining range. The value of normal stress t_{zz} is higher for L-S theory than that of G-L theory for all cases. Fig. 2 represents the variation of tangential stress t_{zr} with radial distance r. It is noted that the values of tangential stress

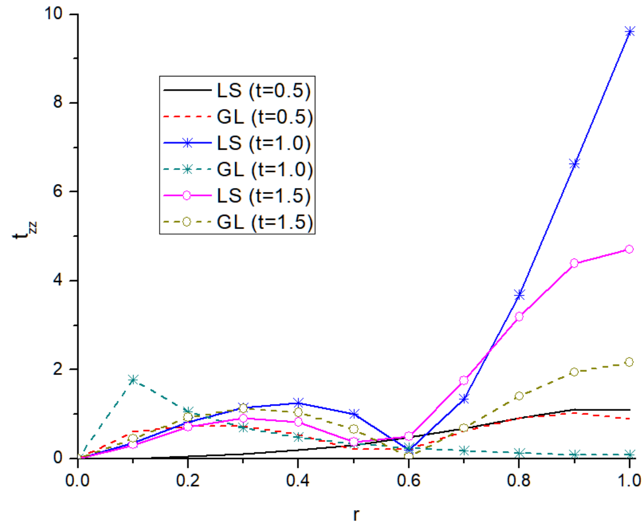

Fig. 1. Variation of normal stress with radial distance

for L-S theory are higher in comparison to G-L theory for $t = 0.5, 1.0$ and opposite behavior is observed for $t = 1.5$.

Fig. 3 depicts the variation of tangential stress $m_{z\phi}$ with radial distance r. It is evident that the values of couple stress oscillate in the whole range with all values of time and for both theories of thermoelastic diffusion. Fig. 4 shows the variation of temperature change T with radial distance r. The values of temperature change T initially increases and then decreases in the whole range for all values of time and both theories of thermoelastic diffusion. It is observed that the values of temperature change for L-S theory is higher in comparison to G-L theory for $t = 0.5, 1, 1.5$. Fig. 5 shows that variation of chemical potential P with radial distance r. It is noted from the figure that the value of chemical potential P is oscillatory in nature for all cases.

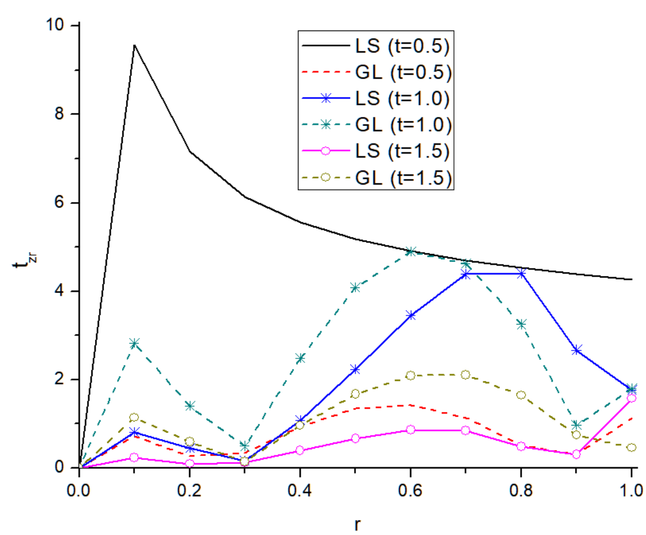

Fig. 2. Variation of tangential stress with radial distance

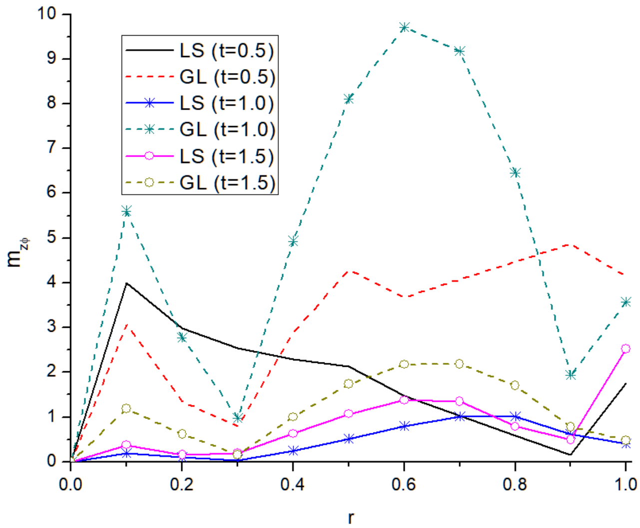

Fig. 3. Variation of couple stress with radial distance

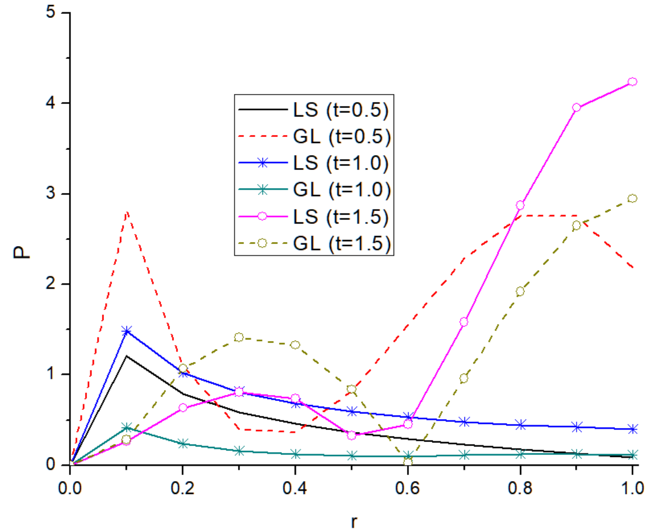

Fig. 5. Variation of chemical potential with radial distance

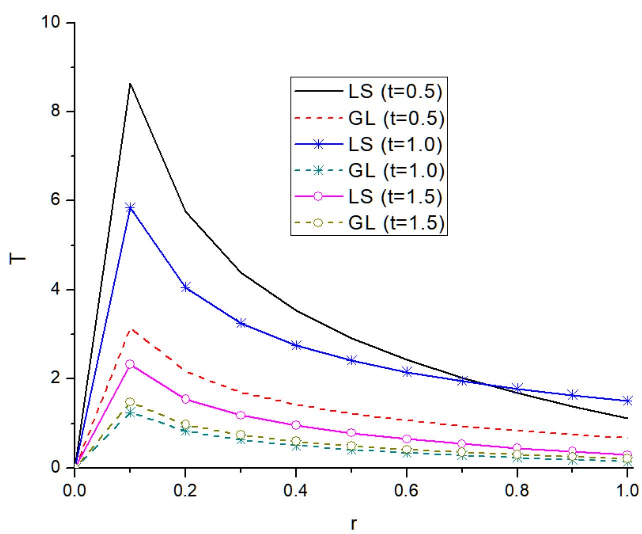

Fig. 4. Variation of temperature change with radial distance

IX. CONCLUSIONS

The problem of a thick circular plate with heat generation in modified couple stress thermoelastic with mass diffusion is a significant problem of continuum mechanics. It is noted from Figs. 1–5 that time effects on stress components, temperature change and chemical potential depend upon the radial distance r. Also, oscillatory effects are obtained for tangential stress, couple stress and chemical potential. As time increases, the values of tangential stress and temperature change decrease with respect to radial distance for both theories of thermoelastic with mass diffusion.

The effect of diffusion is used to improve the conditions of oil extractions (seeking ways of more efficiently recovering oil from oil deposits). The study of thermal and diffusion effects also plays an important role in understanding many seismological processes. Nowadays, there is a great deal of interest in the study of this phenomenon due to its application in geophysics and electronic industry. These results are useful in engineering problems, particularly in the determination state of stresses in a thick circular plate subjected to transient heat inside.

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