CMST 20(2) 49-58 (2014) DOI:10.12921/cmst.2014.20.02.49-58

Boundary Integral Equations Formulation for Fractional Order Thermoelasticity

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Received: 10 March 2014; revised: 16 May 2014; accepted: 20 May 2014; published online: 24 June 2014

Abstract: The present work is concerned with the boundary integral equation formulation for the solutions of equations under fractional order thermo elasticity in a three dimensional Euclidean space. A mixed initial-boundary value problem is considered and the fundamental solutions of the corresponding coupled differential equations are obtained in the Laplace transform domain. We employ one reciprocal relation in the present context and formulate the boundary integral equations on the basis of our fundamental solutions.Then the formulation is illustrated with a suitable example.

Key words: fundamental solutions, boundary integral equation method, thermoelasticity, fractional order thermoelasticity

I. INTRODUCTION

Most often numerical techniques are employed as an alternative tool to solve practical engineering problems that are intractable to solve by any analytical method. Moreover, the advent of high-speed computers in today's time has drawn the attention towards versatile and accurate numerical methods in engineering analysis. In the recent years, the Boundary Element Method (BEM) or Boundary Integral Equation Method (BIEM) has been playing a very crucial role for solving linear partial differential equations due to its efficiency with respect to the computer time and storage, its simplicity and the ease of its implementation as compared to other numerical techniques. As mentioned by Brebbia [1], Brebbia and Walker [2] and Fenner [3], it becomes very methodical as compared to other numerical methods like the Finite Element Method (FEM) for obtaining the solutions of the same accuracy. Particularly, the BIEM/BEM method is easily applicable to solve the elasticity problem in an infinite region. BIEM can be applied in many more areas of engineering and sciences, including fluid mechanics, acoustics, electro magnetics and fracture mechanics. The first numerical treatment of the BIE method was formulated by Jawson [4] and Symm [5]. Rizzo and Shippi [6] introduced the boundary element method for steady state thermo elasticity and showed numerical results for three dimensional linear homogeneous isotropic medium. Chen and Dargush [7] reported the BIE formulations for the dynamic coupled poroelasticity and thermo elasticity with relaxation times by using a unified approach. Work carried out by the researchers like Cruse and Rizzo [8], Banerjee and Butterfield [9], Brebbia et al. [10], Ziegler and Irschik [11] are also worth mentioning in this direction. Application of the boundary element method for three dimensional problems of coupled thermo elasticity was analyzed by Tanaka et al. [12]. Anwar and Sherief [13] presented the boundary integral equation formulation for generalized thermo elasticity with relaxation times. Subsequently, several researchers like Kogl and Gaul [14], El-Karamany and Ezzat [15, 16] El-Karamany [17], Prasad et al. [18], Kothari and Mukhopadhyay [19] have reported BIE formulations in various thermoelasticity theories. Remillat et al. [20] have given viscoelastic testing and fractional derivative modelling to describe the thermally induced transformation.

In the last few decades, various models have been established by using fractional calculus to study the properties of various real materials like polymers, etc. Furthermore, fractional calculus has also been proved to be very useful in the areas of diffusion, heat conduction, viscoelasticity, continuum mechanics and electromagnetics, etc. As we know, the first application of fractional calculus was made by Abel (1802-1829) for the solution of an integral equation that arises in the formulation of the tautochronous problem [21]. Liouville has done a rigorous study of fractional calculus, and later on Caputo played a very important role in employing the fractional order derivatives for the description of viscoelastic material.

The classical heat conduction equation (diffusion equation) that is based on Fourier's law of heat conduction has been widely and successfully applied to the conventional heat conduction problems. However, it has been realized in recent years that it is applicable to the problems that involve large spatial dimension and long-time response. It yields unacceptable results in situations that involve extreme thermal gradients, high heat flux conduction and short time behaviour (Tzou [22], Chandrasekharaiah [23, 24]). Moreover, this heat conduction implies that if the material is subjected to any thermal disturbance then its effect will be felt instantaneously at distances infinitely far from its source. It is a physically unrealistic behaviour, particularly for transient process of heat conduction at extremely short time (Chandrasekharaiah [23]). In order to overcome the inadequacy of Fourier's law, serious efforts have been made during the last few decades. It must be mentioned that Cattaneo [25] and Vernotte [26, 27] proposed for the first time a model of heat conduction (CV model) for a homogenous and isotropic solid in the form

$$-k\nabla T = q(x,t) + \tau \frac{\partial q}{\partial t}. \tag{1}$$

As a modification of Fourier's law where τ, a nonnegative parameter and referred to as the thermal relaxation time, is interpreted as the time lag needed to establish the steady state of heat conduction at a point in a material when a temperature gradient is suddenly imposed on it. Very recently, Sherief et al. [28] has introduced a theory of fractional order thermoelasticity in the frame work of the CV model and proposed a new form of the heat conduction model in the form

$$-k\nabla T = q(x,t) + \tau \frac{\partial^\alpha q}{\partial t^\alpha}, \tag{2}$$

where $(0 < \alpha < 1)$is the fractional order parameter and Caputo's definition of fractional order derivative is employed here.

The main objective of the present work is to formulate the boundary integral equations for the solutions of equations under fractional order thermoelasticity in a three dimensional Euclidean space. We consider a mixed initial-boundary value problem and derive the expressions of fundamental solutions of the corresponding coupled and time-fractional order differential equations in the Laplace transform domain. We formulate the boundary integral equations on the basis of our fundamental solutions and one reciprocal relation in the present context. In the present formulation, we have considered the model of fractional order thermoelasticity by introducing the fractional order derivatives and the parameter, whose value lies between 0 and 1, in the heat conduction equation. The concept of fractional calculus has been applied in the present formulation. This formulation is believed to be helpful for the solution of problems under fractional order thermoelasticity by using the boundary element method.

II. MATHEMATICAL FORMULATION: BASIC GOVERNING EQUATIONS

We consider a homogeneous isotropic elastic body occupying the region V and bounded by a smooth surface S. We employ a three dimensional rectangular Cartesian coordinate system. The basic governing equations that describe the physical components of the thermo elastic system in the context of fractional order thermo elasticity can therefore be considered as follows (Sherief [28]):
The equations of motion:

$$\mu u_{i,kk} + (\lambda+\mu)u_{k,ki} + \rho F_i - \gamma\theta_{,i} = \rho \ddot{u}_i. \tag{3}$$

The equation of energy:

$$\begin{aligned} k\bar{\theta}_{,kk} &= \rho C_e\left(\dot{\theta} + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\dot{\theta}\right) + \gamma T_0\left(\dot{e} + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\dot{e}\right) \\ &\quad - \rho C_e\left(Q + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}Q\right), \end{aligned} \tag{4}$$

where $(0 < \alpha < 1)$, α being the fractional order parameter.
The constitutive equations:

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e - \gamma\theta)\delta_{ij}, \tag{5}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{6}$$

In the above equations, the superposed dot and comma notations are used for time derivative and the material derivative, respectively. Summation convention has been used here and δ_{ij} denotes Kronecker delta. i, j, k varies from 1 to 3. σ_{ij} and e_{ij} are the components of stress and strain tensors respectively; $e = e_{kk}$ is the dilatation; u_i are the components of the displacement vector u. λ and μ are the Lame's elastic constants. C_e is the specific heat at constant strain, θ is the temperature, T_0 is the reference temperature and

$$\gamma = (3\lambda + 2\mu)\alpha_t, \tag{7}$$

where α_t is the coefficient of thermal expansion, Q is the heat source and F_i are the components of the body force vector. k is the thermal conductivity, τ_0 is thermal relaxation parameter. We assume that all the functions are to be the function of x and t, where $x = (x_1,\ x_2,\ x_3)$.

III. BOUNDARY CONDITIONS

(S_1 ,S_2) and (S_3,S_4) are the partitions of the surface S s.t.

$$S_1 \cup S_2 = S = S_3 \cup S_4, \tag{8}$$

$$S_1 \cap S_2 = S_3 \cap S_4 = \phi. \tag{9}$$

Now we assume the following boundary conditions.
Mechanical conditions: The traction vector component

$$p_i = \sigma_{ij} n_j \tag{10}$$

is specified on the part S_1 of S. u_i is specified on S_2 . Here n_j are the components of the outward normal n on the surface S. These conditions can be written as:

$$\sigma_{ij} n_j = p_{i0}(x,t) \text{ on } S_1, \tag{11}$$

$$u_i = u_{i0}(x,t) \text{ on } S_2. \tag{12}$$

Thermal conditions: Thermal conditions are taken as:

$$\theta = \theta_0(x,t) \text{ on } S_3, \tag{13}$$

$$\theta_{,n} = \theta_{n0}(x,t) \text{ on } S_4. \tag{14}$$

We consider that the initial conditions are homogeneous. Clearly, using equation (5), the components of the traction vector on the surface S is obtained in the form

$$\begin{aligned} p_i(x,t) &= [2\mu e_{ij} + (\lambda e - \gamma\theta)\delta_{ij}] n_j(x) = \\ &= \mu n_j(x) u_{i,j} + \mu n_j(x) u_{j,i} + \lambda n_i(x) u_{j,j} - \gamma\theta n_i(x). \end{aligned} \tag{15}$$

IV. FORM OF ABOVE EQUATIONS IN LAPLACE TRANSFORM DOMAIN

The Laplace transform of a function $f(t)$ is given by

$$\bar{f}(p) = \int_0^{+\infty} e^{-pt} f(t) dt. \tag{16}$$

Applying the Laplace transform on both sides of equation (3)-(5), we get

$$\mu \bar{u}_{i,kk} + (\lambda+\mu)\bar{u}_{k,ki} + \rho \bar{F}_i - \gamma\bar{\theta}_{,}i = \rho p^2 \bar{u}_i, \tag{17}$$

$$k\bar{\theta}_{,kk} = \rho C_e p(1+\tau_0 p^\alpha)\left(\bar{\theta} + \frac{\gamma T_0}{\rho C_e}\bar{u}_{,kk} - \frac{\bar{Q}}{p}\right), \tag{18}$$

$$\bar{\sigma}_{ij} = 2\mu \bar{e}_{ij} + (\lambda \bar{u}_{k,k} - \gamma\bar{\theta})\delta_{ij}. \tag{19}$$

Now, we introduce Helmholtz decomposition of the displacement and body force vectors in the following way:

$$u_i = \phi_{,i} + \epsilon_{ijk}\psi_{k,j}, \tag{20}$$

where

$$\psi_{i,i} = 0, \tag{21}$$

$$F_i = X_{,i} + \epsilon_{ijk} Y_{k,j}, \tag{22}$$

where

$$Y_{i,i} = 0. \tag{23}$$

In equations (20) and (22), ϕ, X are scalar potentials and ψ_k and Y_k are vector potentials. Therefore, by substituting (20) and (22) in (17) and (18), we get

$$\Box_1^2 \bar{\phi} - m\bar{\theta} = -\frac{\bar{X}}{c_1^2}, \tag{24}$$

$$\Box_2^2 \bar{\psi}_i = -\frac{\bar{Y}_i}{c_2^2}, \tag{25}$$

$$D\bar{\theta} - ap(1+\tau_0 p^\alpha)\nabla^2\bar{\phi} = -\frac{\rho C_e}{k}(1+\tau_0 p^\alpha)\bar{Q}, \tag{26}$$

where we have introduced the notations:

$$m = \frac{\gamma}{(\lambda+2\mu)}, \quad c_1^2 = \frac{(\lambda+2\mu)}{\rho},$$
$$c_2^2 = \frac{\mu}{\rho}, \qquad a = \frac{\gamma T_0}{k},$$
$$\Box_i^2 \equiv \nabla^2 - p^2/c_i^2 \ \text{ for } \ i = 1,2,3;$$

and

$$D \equiv \left[\nabla^2 - \frac{\rho C_e}{k} p\,(1+\tau_0 p^\alpha)\right]$$

V. FUNDAMENTAL SOLUTIONS IN LAPLACE TRANSFORM DOMAIN

In order to describe the action of body force, the source of heat of very large magnitude that act for a very short period of time upon the body, we shall consider the following two cases:
Case 1: We assume that an instantaneous source of heat located at $x_i = y_i$ where $y \in (V \cup S)$ is acting upon an elastic body in the absence of the body forces i.e. $Q = \delta(x-y)\delta(t)$, $F_i = 0$

Let us denote the corresponding fundamental solutions by primes. Now, under the above assumptions, the equations (24)-(26) reduce to

$$\Box_1^2 \bar{\phi}' - m\bar{\theta}' = 0 \tag{27}$$

$$\Box_2^2 \bar{\psi}_i' = 0 \tag{28}$$

$$\begin{aligned} D\bar{\theta}' - ap(1+\tau_0 p^\alpha)\nabla^2\bar{\phi}' &= \\ = -\frac{\rho C_e}{k}(1+\tau_0 p^\alpha)\delta(x-y). \end{aligned} \tag{29}$$

From equation (28), we can conclude that

$$\bar{\psi}_i' = 0. \tag{30}$$

Decoupling equations (27) and (29), we arrive at

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\bar{\phi}' = -m\frac{\rho C_e}{k}(1+\tau_0 p^\alpha)\delta(x-y), \tag{31}$$

where k_1^2, k_2^2 are the solutions of characteristic equation

$$\begin{aligned} &kv^4 - \left[(\rho C_e + mak)p(1+\tau_0 p^\alpha) + p^{\alpha+1}\frac{k}{c_1^2}\right]v^2 \\ &+p^{\alpha+2}\rho C_e \frac{(1+\tau_0 p^\alpha)}{c_1^2} = 0. \end{aligned} \tag{32}$$

The solution $\bar{\phi}'$ of equation (31) is given by

$$\bar{\phi}'(x,p) = -m\rho C_e \frac{(1+\tau_0 p^\alpha)}{(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)}\delta(x-y). \tag{33}$$

By using the Helmholtz equation

$$\frac{1}{(\nabla^2 - k^2)}\delta(r) = -\frac{1}{4\pi r}e^{-kr}, \tag{34}$$

we obtain from equation (33)

$$\bar{\phi}'(x,p) = \frac{m\rho C_e}{4\pi k(k_1^2 - k_2^2)}(1+\tau_0 p^\alpha)(e^{-k_1 r} - e^{-k_2 r}). \tag{35}$$

Equations (20), (30) yield

$$\bar{u'}_i(x,p) = \bar{\phi}'_{,i}(x,p). \tag{36}$$

Now, using $r = \sqrt{(x_i - y_i)(x_i - y_i)}$,

$$r_{,i} = \frac{(x_i - y_i)}{r}, \tag{37}$$

$$\begin{aligned} \bar{u}'_i(x,p) = &-\frac{\rho C_e}{kr}\frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)} \\ &\times \left[e^{-k_1 r}\left(1 + \frac{k_1}{r}\right) - e^{-k_2 r}\left(1 + \frac{k_2}{r}\right)\right] r_{,i} \end{aligned} \tag{38}$$

in view of equation (27) we find

$$\bar{\theta}'(x,p) = \frac{1}{m}(\nabla^2 - \frac{p^2}{c_1^2})\bar{\phi}'(x,p). \tag{39}$$

Hence, we can obtain $\bar{\theta}'(x,p)$ as

$$\begin{aligned} \bar{\theta}'(x,p) = &\frac{\rho C_e}{kr}\frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)} \\ &\times \left[e^{-k_1 r}\left(k_1^2 - \frac{p^2}{c_1^2}\right) - e^{-k_2 r}\left(k_2^2 - \frac{p^2}{c_2^2}\right)\right]. \end{aligned} \tag{40}$$

Taking Laplace transform of traction vector, we get from equation (15)

$$\begin{aligned} \bar{p}_i(x,t) = &\mu n_j(x)\bar{u}_{i,j} + \mu n_j(x)\bar{u}_{j,i} \\ &+\lambda n_i(x)\bar{u}_{j,j} - \gamma\bar{\theta} n_i(x). \end{aligned} \tag{41}$$

Equation (38) yields

$$\bar{u}'_{i,j} = -\frac{\rho C_e}{k}\left[\frac{r_{,i}}{r}g_{1,j} - \frac{(r_{,j}\, r_{,i}\, g_1 - r r_{,ij}\, g_1)}{r^2}\right], \tag{42}$$

where

$$\begin{aligned} g_1 = &\frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)} \\ &\times \left[e^{-k_1 r}\left(1 + \frac{k_1}{r}\right) - e^{-k_2 r}\left(1 + \frac{k_2}{r}\right)\right] \end{aligned} \tag{43}$$

and

$$\begin{aligned} g_{1,j} = &\frac{m\,(1+\tau_0 p^\alpha)}{4\pi\,(k_1^2 - k_2^2)}\left[- k_1 e^{-k_1 r} r_{,j}\left(1 + \frac{1}{r^2}\right)\right. \\ &-k_1^2 e^{-k_1 r}\frac{r_{,j}}{r} + k_2 e^{-k_2 r} r_{,j}\left(1 + \frac{1}{r^2}\right) \\ &\left.-k_2^2 e^{-k_2 r}\frac{r_{,j}}{r}\right]. \end{aligned} \tag{44}$$

Equation (42) can be simplified as

$$\begin{aligned} \bar{u}'_{i,j} = &\frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)} \\ &\times\left[r_{,i}\, r_{,j}\,\frac{g_3}{r} - \frac{1}{r^2}(\delta_{ij} - 3r_{,i}\, r_{,j}\,)g_1\right], \end{aligned} \tag{45}$$

where

$$g_3 = \frac{(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)}\left[k_1^2 e^{-k_1 r} - k_2^2 e^{-k_2 r}\right], \tag{46}$$

therefore from equations (41) and (45), we get

$$\begin{aligned} \bar{p}'_l(x,p) = &\frac{\rho C_e}{kr}\left\{2\mu n_k\left[r_{,l}\, r_{,k}\left(g_3 + 3\frac{g_1}{r}\right)\frac{\delta_{lk}}{r}\right]\right. \\ &\left.+n_l\left[\lambda r_{,k}\, r_{,k}\left(g_3 + 3\frac{g_1}{r}\right) - 3\frac{\lambda}{r}\right]\right\}. \end{aligned} \tag{47}$$

Here case (1) is completed.

Case 2: In this case, we assume that in absence of heat source i.e. when $Q = 0$, an instantaneous concentrated body force is acting at the point $x_i = y_i$ in the direction of x_j axis. Therefore, we take

$$\bar{F}_i = \bar{F}_i^{(j)} = \delta_{ij}\delta(x-y).$$

Let $\bar{u}_i^{(j)}, \bar{\theta}^{(j)}$ denote the corresponding fundamental solutions. We use Helmholtz Resolution for the vectors $u_i^{(j)}$ and $F_i^{(j)}$ and so we can write

$$\bar{u}_i^{(j)} = \bar{\phi}_{,i}^{(j)} + \epsilon_{ilk}\bar{\psi}_{l,k}^{(j)}, \tag{48}$$

$$\bar{F}_i^{(j)} = \bar{P}_{,i}^{(j)} + \epsilon_{ilk}\bar{R}_{l,k}^{(j)} \tag{49}$$

The potentials in R.H.S. of above equations satisfy the equations

$$\begin{aligned} &(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\bar{\phi}^{(j)} = \\ &= -\frac{1}{c_1^2}\left[\nabla^2 - \frac{\rho C_e}{k}p\,(1+\tau_0 p^\alpha)\right]\bar{P}^{(j)}, \end{aligned} \tag{50}$$

$$\bar{\theta}^{(j)} = \frac{1}{m}\left[\left(\nabla^2 - \frac{p^2}{c_1^2}\right)\bar{\phi}^{(j)} + \frac{\bar{P}^{(j)}}{c_1^2}\right], \tag{51}$$

$$\left(\nabla^2 - \frac{p^2}{c_1^2}\right)\bar{\psi}_l^{(j)} = -\frac{1}{c_2^2}\bar{R}_i^{(j)}, \tag{52}$$

where k_1^2 and k_2^2 are the solutions of the same characteristic equation as given by (32). In view of the body forces as chosen above, the corresponding Helmholtz decomposition leads to

$$\bar{P}^{(j)} = -\frac{1}{4\pi}\left(\frac{\delta_{ij}}{r}\right)_{,i} \tag{53}$$

$$\bar{R}_l^{(j)} = \frac{1}{4\pi}\epsilon_{ilk}\left(\frac{\delta_{ij}}{r}\right)_{,k}. \tag{54}$$

By using Helmholtz equation, solution of equation (50) can be written as

$$\begin{aligned}\bar{\phi}^{(j)} &= \frac{1}{4\pi c_1^2}\frac{r_{,i}}{r^2}\frac{\delta_{ij}}{(k_1^2-k_2^2)}\\ &\times\left[\sum_{n=1}^{2}(-1)^{n-1}E(1+k_n r)e^{-k_n r}\right] + \delta_{ij}\frac{r_i}{r^2}\frac{1}{4\pi p^2},\end{aligned} \tag{55}$$

where

$$E = \frac{[k_n^2 - \frac{\rho C_e p(1+\tau_0 p^\alpha)}{k}]}{k_n^2}, \tag{56}$$

$$\bar{\psi}_l^{(j)} = \epsilon_{ijl}\left(\frac{r_{,i}}{4\pi p^2 r^2}\right)\left[\left(1+\frac{pr}{c_2}\right)e^{-\frac{pr}{c_2}} - 1\right], \tag{57}$$

$$\bar{u}_l^{(j)} = \frac{U_1\delta_{ij}}{r} + \frac{U_2 r_{,i} r_{,j}}{r}, \tag{58}$$

where

$$\begin{aligned}U_1 &= \frac{1}{4\pi p^2}\left(\frac{p^2}{c_2^2} + \frac{p}{rc_2} + \frac{1}{r^2}\right)e^{-\frac{pr}{c_2}}\\ &+ \sum_{n=1}^{2}(-1)^{n-1}\frac{B_n}{r^2}(1+k_n r)e^{-k_n r},\end{aligned} \tag{59}$$

$$\begin{aligned}U_2 &= \left[-\frac{1}{4\pi p^2}\left(\frac{p^2}{c_2^2} + 3\frac{p}{rc_2} + 3\frac{1}{r^2}\right)e^{-\frac{pr}{c_2}}\right]\\ &+ \sum_{n=1}^{2}(-1)^{n-1}\frac{B_n}{r^2}\left(k_n^2 + 3\frac{k_n}{r} + \frac{3}{r^2}\right)e^{-k_n r},\end{aligned} \tag{60}$$

$$B_n = \frac{1}{k_n^2 c_1^2}\frac{1}{(k_1^2-k_2^2)}\left(k_n^2 - \frac{\rho C_e p(1+\tau_0 p^\alpha)}{k}\right). \tag{61}$$

With the help of equations (26) and (50) with the condition that the source of heat is absent, we find

$$\bar{\theta}^{(j)} = G\sum_{n=1}^{2}(-1)^{n-1}e^{-k_n r}(1+k_n r)\frac{r_{,i}}{r^2}, \tag{62}$$

where

$$G = \frac{\delta_{ij}}{4\pi c_1^2}\frac{p(1+\tau_0 p^\alpha)}{(k_2^2-k_1^2)}.$$

Here we achieve the conclusion that the temperature for Case 2 is related with the expression of the displacement for Case 1 as

$$\bar{\theta}^{(j)} = \frac{\rho p\epsilon}{m\gamma}u'_j, \tag{63}$$

where

$$\epsilon = \frac{mak}{\rho C_e}.$$

Then the component of the traction vector can be obtained in a similar way as in Case (1).

$$\bar{p}_l^{(j)}(x,p) = \mu n_k[(\bar{u}_{l,k}^{(j)} + \bar{u}_{k,l}^{(j)}) + n_l(\lambda\bar{u}_{k,k}^{(j)} - \gamma\bar{\theta}^{(j)})], \tag{64}$$

where

$$\begin{aligned}\bar{u}_{i,k}^{(j)} &= \frac{U_{1,k}}{r}\delta_{ij} - \frac{U_1\delta_{ij}}{r^2} + \frac{U_{2,k}}{r^2}r_{,i}r_{,j}\\ &-\frac{U_2}{r^2}r_{,i}r_{,j}r_{,k} + \frac{U_2}{r}r_{,ik}r_{,j} + \frac{U_2}{r}r_{,i}r_{,jk},\end{aligned} \tag{65}$$

$$\begin{aligned}U_{1,k} &= \sum_{n=1}^{2}(-1)^{n-1}B_n e^{-k_n r}\frac{r_{,k}}{r}\left[-k_n^2 - 2\frac{(1+k_n r)}{r^2}\right]\\ &+\frac{r_{,k}}{4\pi p^2}e^{\frac{-pr}{c_2}}\left[-\frac{p^3}{c_2^3} - 2\frac{p}{r^2 c_2} - \frac{2}{r^3} - \frac{p^2}{c_2^2 r}\right],\end{aligned} \tag{66}$$

$$\begin{aligned}U_{2,k} &= -\frac{r_{,k}}{4\pi p^2}e^{\frac{-pr}{c_2}}\left[-\frac{p^3}{c_2^3} - 6\frac{p}{r^2 c_2} - \frac{6}{r^2} - 3\frac{p^2}{c_2^2 r}\right]\\ &+\sum_{n=1}^{2}(-1)^{n-1}B_n e^{-k_n r}r_{,k}\left[-k_n^2 - 6\frac{k_n}{r^2} - 3\frac{k_n^2}{r} - \frac{6}{r^3}\right].\end{aligned} \tag{67}$$

U_1, U_2 are given by equation (59) and (60), respectively. This completes Case 2.

VI. RECIPROCITY THEOREM

When a body is under the action at two different thermoelastic loadings, the reciprocal relation states the relation between two sets of thermoelastic loadings and corresponding thermoelastic configurations.

In order to find out the integral representations of the displacement and temperature distributions in terms of boundary values, we will now employ a reciprocity theorem for the linear theory of fractional order thermo elasticity. For this we consider the body with volume V subjected to two different systems of thermoelastic loadings

$$L^{(\beta)} = (F_i^{(\beta)}, Q^{(\beta)}, p_i^{(\beta)}, u_{i0}^{(\beta)}, \theta_0^{(\beta)}, \theta_{n0}^{(\beta)}) \text{ for } \beta = 1,2$$

and the corresponding thermoelastic configurations are denoted as

$$I^\beta = (u_i^\beta, \theta^\beta) \text{ for } \beta = 1,2.$$

Now by following Sherief et al. [28], we can derive the following reciprocal relation:

$$\begin{aligned}&\frac{\rho T_0 p(1+\tau_0 p^\alpha)}{k}\int_V\left[\bar{F}_i^{(1)}\bar{u}_i^{(2)} - \bar{F}_i^{(2)}\bar{u}_i^{(1)}\right]\mathrm{d}V\\ &-\frac{\rho C_e(1+\tau_0 p^\alpha)}{k}\int_V\left[\bar{Q}^{(1)}\bar{\theta}^{(2)} - \bar{Q}^{(2)}\bar{\theta}^{(1)}\right]\mathrm{d}V =\\ &= \int_{S_3}\left[\bar{\theta}^{(1)}\bar{\theta}_{,n}^{(2)} - \bar{\theta}^{(2)}\bar{\theta}_{,n}^{(1)}\right]\mathrm{d}S\\ &+\int_{S_4}\left[\bar{\theta}^{(1)}\bar{\theta}_{,n}^{(2)} - \bar{\theta}^{(2)}\bar{\theta}_{,n}^{(1)}\right]\mathrm{d}S - \frac{T_0 p(1+\tau_0 p^\alpha)}{k}\\ &\times\left\{\int_{S_1}\left[\bar{\sigma}_{ij}^{(1)}\bar{u}_i^{(2)} - \bar{\sigma}_{ij}^{(2)}\bar{u}_i^{(1)}\right]n_j\mathrm{d}S\right.\\ &\left.+\int_{S_2}\left[\bar{\sigma}_{ij}^{(1)}\bar{u}_i^{(2)} - \bar{\sigma}_{ij}^{(2)}\bar{u}_i^{(1)}\right]n_j\mathrm{d}S\right\}.\end{aligned} \tag{68}$$

Clearly, for an infinite isotropic medium, the body forces, heat sources act only in a bounded region and the surface integral in equation (68) will be absent.

VII. BOUNDARY INTEGRAL EQUATIONS

In order to obtain the integral representation of the transformed displacement and temperature inside the bounded region V in terms of the prescribed functions $\bar{p}_{i0}$, $\bar{u}_{i0}$, $\bar{\theta}_0$, $\bar{\theta}_{n0}$ on the surface S, the fundamental solutions $\bar{u}'_i$, $\bar{\theta}'$, $\bar{u}_i^{(j)}$, $\bar{\theta}^{(j)}$ in the infinite region and their values $\bar{u}'_{i0}$, $\bar{\theta}'_0$, $\bar{u}_{i0}^{(j)}$, $\bar{\theta}_0^{(j)}$ on the surface S, we consider the following two cases:

First, as we have considered earlier

$$\bar{F}_i = 0 \text{ and } \bar{Q} = \delta(x-y) \text{ where } y \in (V \cup S) \tag{69}$$

thus equation (68) becomes

$$\begin{aligned}(1+\tau_0 p^\alpha)\Delta(x)\bar{\theta}(x,p) = \frac{k}{\rho C_e}\left\{\int_{S_3}\left[\bar{\theta}'_0\bar{\theta}_{,n} - \bar{\theta}_0\bar{\theta}'_{n0}\right]\mathrm{d}S + \int_{S_4}\left[\bar{\theta}'\bar{\theta}_{n0} - \bar{\theta}\bar{\theta}'_{,n}\right]\mathrm{d}S\right\} - \frac{T_0}{\rho C_e}p(1+\tau_0 p^\alpha)\\ \times\left\{\int_{S_1}[\bar{p_{i0}}\bar{u}'_i - \bar{p}'_{i0}\bar{u}_i]\mathrm{d}S + \int_{S_2}\left[\bar{\sigma}_{ij}\bar{u}'_{i0} - \bar{\sigma}'_{ij}\bar{u}_{i0}\right]n_j\mathrm{d}S\right\} - \frac{T_0}{C_e}(p+\tau_0 p^\alpha)\int_V \bar{F}_i\bar{u}'_i\mathrm{d}V + (1+\tau_0 p^\alpha)\int_V \bar{Q}\bar{\theta}'\mathrm{d}V,\end{aligned} \tag{70}$$

where $\bar{u}'_i$,$\bar{\theta}'$ are the fundamental solutions obtained previously in Case 1 and we denote

$$\int_V \delta(x-y)\mathrm{d}V(y) = \Delta(x),$$

where

$$\Delta(x) = 1, x \in V \qquad \Delta(x) = 0, \text{ if } x \notin (V \cup S) \qquad \Delta(x) = \frac{1}{2}, x \in S.$$

Next, we assume

$$\bar{F}_i = \delta_{ij}\delta(x-y) \quad \text{and} \quad \bar{Q} = 0.$$

Then from equation (68) we get as previously,

$$\begin{aligned}p(1+\tau_0 p^\alpha)\Delta(x)\bar{u}_j(x,p) = \frac{k}{\rho T_0}\left\{\int_{S_3}\left[\bar{\theta}_0\bar{\theta}_{,n}^{(j)} - \bar{\theta}_0^{(j)}\bar{\theta}_{,n}\right]\mathrm{d}S\right.\\ \left. + \int_{S_4}\left[\bar{\theta}\bar{\theta}_{n0}^{(j)} - \bar{\theta}^{(j)}\bar{\theta}_{n0}\right]\mathrm{d}S\right\} + \frac{1}{\rho}p(1+\tau_0 p^\alpha)\left\{\int_{S_1}\left[\bar{p}_{i0}\bar{u}_i^{(j)} - \bar{p}_{i0}^{(j)}\bar{u}_{i0}\right]\mathrm{d}S\right.\\ \left. + \int_{S_2}\left[\bar{\sigma}_{il}\bar{u}_{i0}^{(j)} - \bar{\sigma}_{il}^{(j)}\bar{u}_{i0}\right]n_j\mathrm{d}S\right\} - p(1+\tau_0 p^\alpha)\int_V \bar{F}_i\bar{u}_i^{(j)}\mathrm{d}V - \frac{C_e}{T_0}(1+\tau_0 p^\alpha)\int_V \bar{Q}\bar{\theta}^{(j)}\mathrm{d}V\end{aligned} \tag{71}$$

where $\bar{u}_i^{(j)}$,$\bar{\theta}^{(j)}$ denote the fundamental solutions which have been obtained previously in case 2. Now applying the inverse Laplace transform to equation (70) and using the Convolution theorem of Laplace transform as

$$L^{-1}(\bar{F}_1(p)\bar{F}_2(p)) = \int_0^t F_1(\tau)F_2(\tau)d\tau$$

we arrive at

$$\Delta(x)\left[\theta(x,t) + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\theta(x,t)\right] = M_1(x,t), \tag{72}$$

where

$$
\begin{aligned}
M_1(x,t) = {} & \frac{k}{\rho C_e}\int_0^t \left\{ \int_{S_3} \left[\theta_0'\left(y,t-\tau\right)\theta,_n\left(y,x,\tau\right) - \theta_0(y,x,\tau)\theta',_n\left(y,t-\tau\right)\right]\mathrm{d}S \right.\\
& \left. + \int_{S_4} \left[\theta'\left(y,t-\tau\right)\theta_{n0}\left(y,x,\tau\right) - \theta(y,x,\tau)\theta_{n0}'\left(y,t-\tau\right)\right]\mathrm{d}S \right\} d\tau \\
& + \frac{T_0}{\rho C_e}\int_0^t \left\{ \int_{S_1} \left[p_{i0}\left(y,x,\tau\right)\left(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}}\right)u_i'\left(y,t-\tau\right)\right]\mathrm{d}S \right.\\
& \left. \times \int_{S_2} \left[\sigma_{ij}\left(y,t-\tau\right)\left(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}}\right)u_{i0}'\left(y,x,\tau\right)\right]\mathrm{d}S \right\} d\tau \\
& - \frac{T_0}{\rho C_e}\int_0^t \left\{ \int_{S_1} u_i\left(y,x,\tau\right)\left(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}}\right)p_{i0}'\left(y,t-\tau\right)\mathrm{d}S \right.\\
& \left. + \int_{S_2} u_{i0}\left(y,t-\tau\right)\left(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}}\right)\sigma_{ij}'\left(y,x,\tau\right)\mathrm{d}S \right\} d\tau \\
& + \frac{T_0}{C_e}\int_0^t\int_V F_i(y,t-\tau)\left(\frac{\partial}{\partial\tau} + \frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}}\right)u_i'\left(y,x,\tau\right)\mathrm{d}V d\tau \\
& + \int_0^t\int_V Q(y,t-\tau)\left(1 + \frac{\partial^{\alpha}}{\partial\tau^{\alpha}}\right)\theta'\left(y,x,\tau\right)\mathrm{d}V d\tau.
\end{aligned}
\tag{73}
$$

Solving equation (72) we get the expression of temperature in the following manner:

$$
\theta(x,t) = \frac{1}{\Delta x\tau_0}\left[\int_0^t E_{\alpha,\alpha}(-\frac{1}{\tau_0}\tau^{\alpha})M_1(x,t-\tau)d\tau\right], \tag{74}
$$

where $E_{\alpha,\alpha}(z)$ is known as "Mittag-Leffler" function which is defined as (Podlubny [21]) :

$$
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{75}
$$

Similarly, from equation (71) we obtain

$$
\Delta(x)\left[u_j(x,t) + \tau_0\frac{\partial^{\alpha}}{\partial t^{\alpha}}u_j(x,t)\right] = M_2(x,t), \tag{76}
$$

where

$$
\begin{aligned}
M_2(x,t) = {} & \frac{k}{\rho T_0}\int_0^t \left\{ \int_{S_3} [\theta_0\left(y,x,\tau\right)\frac{\partial u_j'\left(y,t-\tau\right)}{\partial n} - u_j'\left(y,t-\tau\right)\frac{\partial\theta\left(y,x,\tau\right)}{\partial n}]\mathrm{d}S + \int_{S_4}\left[\theta\left(y,x,\tau\right)\frac{\partial u_j'\left(y,t-\tau\right)}{\partial n}\right.\right.\\
& \left.\left. - u_j'\left(y,t-\tau\right)\theta_{n0}\left(y,x,\tau\right)\right]\mathrm{d}S\right\} d\tau + \frac{1}{\rho}\int_0^t \left\{ \int_{S_1}\left[p_{i0}\left(y,x,\tau\right)\left(1+\tau_0\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}\right)u_i^{(j)}\left(y,t-\tau\right)\mathrm{d}S\right]\right.\\
& \left. + \int_{S_2}\left[\sigma_{il}\left(y,t-\tau\right)\left(1+\tau_0\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}\right)u_{i0}^{(j)}\left(y,x,\tau\right)n_l\mathrm{d}S\right]\right\} d\tau - \frac{1}{\rho}\left\{\int_0^t\int_{S_1} u_i\left(y,x,\tau\right)\left(1+\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}\right)p_{i0}^{(j)}\left(y,t-\tau\right)\mathrm{d}S\right.\\
& \left. + \int_{S_2} u_{i0}\left(y,t-\tau\right)\left(1+\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}\right)\sigma_{il}^{(j)}\left(y,x,\tau\right)n_l\mathrm{d}S\right\} d\tau + \int_V F_i(y,t-\tau)\left(1+\tau_0\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}\right)u_i^{(j)}\left(y,x,t-\tau\right)\mathrm{d}V d\tau \\
& - \int_0^t\int_V Q\left(y,t-\tau\right)\left(1+\tau_0\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}\right)u_j'\left(y,x,t-\tau\right)\mathrm{d}V d\tau
\end{aligned}
\tag{77}
$$

As before, the solution of equation (76) yield displacement as

$$
u_j(x,t) = \frac{1}{\Delta x\tau_0}\left[\int_0^t E_{\alpha,\alpha}\left(-\frac{1}{\tau_0}\tau^{\alpha}\right)M_2(x,t-\tau)d\tau\right] \tag{78}
$$

Taking the limit $x \to \xi$, where ξ is a point on the boundary S, we get from equation (74) and (78) as

$$\theta(\xi,t) = \frac{2}{\tau_0}\left[\int_0^t E_{\alpha,\alpha}\left(-\frac{1}{\tau_0}\tau^\alpha\right) M_1(\xi,t-\tau)d\tau\right] \tag{79}$$

$$u_j(\xi,t) = \frac{2}{\tau_0}\left[\int_0^t E_{\alpha,\alpha}\left(-\frac{1}{\tau_0}\tau^\alpha\right) M_2(\xi,t-\tau)d\tau\right] \tag{80}$$

This completes our formulation.

The above two equations together with the prescribed boundary conditions which are taken in the beginning and the limiting behaviour of fundamental solutions as $r \to 0$ can be used to set up linear equations of the boundary integral equation method.

VIII. EXAMPLE

Now, we will consider an example in order to illustrate the present formulation. We consider a formulation in which we determine the primary variables $u_i(x,t)$ and $\theta(x,t)$ as the solution of the field equations (3) and (4), subjected to the homogeneous initial and boundary conditions as follows:

$$\sigma_{ij}(x_B,t)n_j(x_B) = p_{i0}(x_B,t) = 0 \tag{81}$$

$$\theta_{,n}(x_B,t) = \theta_{n0}(x_B,t) = 0 \tag{82}$$

where x_B is a point on $S_1 = S_4$ and

$$\theta(x_B,t) = \theta_0(x_B,t) \tag{83}$$

$$u_i(x_B,t) = u_{i0}(x_B,t) \tag{84}$$

where x_B is a point on $S_2 = S_3$

Now, using equations (69) and (70), we achieve

$$\begin{aligned}\bar{\theta}(x,p) = &-\frac{k}{\rho C_e(1+\tau_0 p^\alpha)}\left\{\int_{S_3}\left[\bar{\theta}_0(y,x,p)\bar{\theta}'_{,n}(y,p) - \bar{\theta}'_0(y,p)\bar{\theta}_{,n}(y,x,p)\right]\mathrm{d}S\right.\\
&\left.+\int_{S_4}\left[\bar{\theta}(y,x,p)\bar{\theta}'_{n0}(y,p) - \bar{\theta}'(y,p)\bar{\theta}_{n0}(y,x,p)\right]\mathrm{d}S\right\} + \frac{T_0 p}{\rho C_e}\left\{\int_{S_1}[\bar{p_{i0}}'(y,x,p)\bar{u}_i(y,p) - \bar{p}_{i0}(y,p)\bar{u}'_i(y,x,p)]\mathrm{d}S\right.\\
&\left.+\int_{S_2}\left[\bar{\sigma}'_{ij}(y,p)\bar{u}_{i0}(y,x,p) - \bar{\sigma}_{ij}(y,x,p)\bar{u}'_{i0}(y,p)\right]n_j\mathrm{d}S\right\} - \frac{T_0 p}{C_e}\int_V \bar{F}_i(y,p)\bar{u}'_i(y,x,p)\mathrm{d}V\\
&+\int_V \bar{Q}(y,p)\bar{\theta}'(y,x,p)\mathrm{d}V,\end{aligned} \tag{85}$$

$$\begin{aligned}p\bar{u}_j(x,p) = &\frac{k}{\rho T_0(1+\tau_0 p^\alpha)}\left\{\int_{S_3}\left[\bar{\theta}_0(y,x,p)\bar{\theta}_{,n}^{(j)}(y,p) - \bar{\theta}_0^{(j)}(y,p)\bar{\theta}_{,n}(y,x,p)\right]\mathrm{d}S\right.\\
&\left.+\int_{S_4}\left[\bar{\theta}(y,p)\bar{\theta}_{n0}^{(j)}(y,x,p) - \bar{\theta}^{(j)}(y,x,p)\bar{\theta}_{n0}(y,p)\right]\mathrm{d}S\right\} - \frac{p}{\rho}\left\{\int_{S_1}\left[\bar{p}_{i0}^{(j)}(y,p)\bar{u}_i(y,x,p) - \bar{p}_{i0}(y,x,p)\bar{u}_i^{(j)}(y,p)\right]\mathrm{d}S\right.\\
&\left.+\int_{S_2}\left[\bar{\sigma}_{ij}^{(j)}(y,x,p)\bar{u}_{i0}(y,p) - \bar{\sigma}_{ij}(y,x,p)\bar{u}_{i0}^{(j)}(y,p)\right]n_j\mathrm{d}S\right\} - p\int_V \bar{F}_i(y,p)\bar{u}_i^{(j)}(y,x,p)\mathrm{d}V + \int_V \bar{Q}(y,p)\bar{\theta}^{(j)}(y,x,p)\mathrm{d}V.\end{aligned} \tag{86}$$

Here we have taken $\Delta(x) = 1$.

In view of (83)-(84), the functions $\theta(x_B,t)$ and $u_i(x_B,t)$ are unknowns on the part $S_1 = S_4$ of the surface S. We further assume that the fundamental solutions satisfy the conditions

$$\bar{\theta}'_0(x_B,t) = \bar{\theta}_0^{(j)}(x_B,t) = 0 \text{ on } S_2 = S_3$$

and

$$\bar{u}'_{i0}(x_B,t)=\bar{u}^{(j)}_{i0}(x_B,t)=0 \text{ on } S_2=S_3.$$

Therefore by using these conditions and taking $x\to\xi$ we get the equations (85) and (86) as the system of two Fredholm's integral equations:

$$\begin{aligned}0=&-\frac{k}{\rho C_e(1+\tau_0 p^\alpha)}\left\{\int_{S_3}\left[\bar{\theta}_0(y,p)\frac{\partial}{\partial n'(\xi)}\bar{\theta}'_{,n}(y,\xi,p)\right]\mathrm{d}S\right.\\&\left.+\int_{S_4}\left[\bar{\theta}(y,p)\frac{\partial}{\partial n'(\xi)}\bar{\theta}'_{n0}(y,\xi,p)\right]\mathrm{d}S\right\}+\frac{T_0p}{\rho C_e}\left\{\int_{S_4}\left[\bar{p_{i0}}'(y,p)\frac{\partial}{\partial n'(\xi)}\bar{u}_i(y,\xi,p)\right]\mathrm{d}S\right.\\&\left.+\int_{S_3}\left[\bar{\sigma}'_{ij}(y,p)\frac{\partial}{\partial n'(\xi)}\bar{u}_{i0}(y,\xi,p)\right]n_j\mathrm{d}S\right\}-\frac{T_0p}{C_e}\int_V\bar{F}_i(y,p)\frac{\partial}{\partial n'(\xi)}\bar{u}'_i(y,\xi,p)\mathrm{d}V\\&+\int_V\bar{Q}(y,p)\frac{\partial}{\partial n'(\xi)}\bar{\theta}'(y,\xi,p)\mathrm{d}V,\end{aligned}\tag{87}$$

$$\begin{aligned}p\bar{u}_j(\xi,p)=&\frac{k}{\rho T_0(1+\tau_0p^\alpha)}\left\{\int_{S_3}\left[\bar{\theta}_0(y,\xi,p)\bar{\theta}_{,n}^{(j)}(y,p)\right]\mathrm{d}S+\int_{S_4}\left[\bar{\theta}(y,p)\bar{\theta}^{(j)}_{n0}(y,\xi,p)\right]\mathrm{d}S\right\}\\&-\frac{p}{\rho}\left\{\int_{S_4}\left[\bar{p}^{(j)}_{i0}(y,p)\bar{u}_i(y,\xi,p)\right]\mathrm{d}S+\int_{S_3}\left[\bar{\sigma}^{(j)}_{ij}(y,\xi,p)\bar{u}_{i0}(y,p)\right]n_j\mathrm{d}S\right\}\\&-p\int_V\bar{F}_i(y,p)\bar{u}^{(j)}_i(y,\xi,p)\mathrm{d}V+\int_V\bar{Q}(y,p)\bar{\theta}^{(j)}(y,\xi,p)\mathrm{d}V,\end{aligned}\tag{88}$$

where $n'(\xi)$ is the outer normal to the surface S_4. By employing suitable numerical techniques, the integrals involved in equations (87) and (88) can be discretized and the problem then reduces to finding the solution of a system of linear equations. The final solution can therefore be determined by using a suitable numerical method of Laplace inversion.

IX. SUMMARY

In the present paper, we achieve fundamental solutions in the Laplace transform domain for fractional order thermoelasticity. Then by employing a suitable reciprocal relation, we formulate boundary integral equations for a mixed boundary initial value problem.

At last, we have given an example which represents a better explanation for our formulation. We believe that the present formulation will help to find the numerical solution of a concrete problem under fractional order thermoelasticity by the BEM/BIEM method.

There are many numerical methods for solving the BIE system like the point collocation method etc. In the point collocation method, the boundary is discretized at some points (collocation points) and we select a solution that satisfies the given equation at those collocation points. This method is believed to be accurate for several one and two-dimensional problems and the convergence rate of the point collocation method can be applied for better results.

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