Symbols

- $h(t)$ – hazard rate function
- $f(t)$ – density function
- $F(t)$ – cumulative distribution function (CDF)
- $R(t)$ – reliability function
- MMD – modified Makeham distribution

I. INTRODUCTION

The difficulty of determining the distribution for the sum of independent random variables appears in the renewal process. The problem can be easily solved when underlying lifetime distribution is of the so-called “infinitely divisible” type. The distribution is said to be of this type if the sum of independent random variables follows the same distribution as the components do. Unfortunately, the modified Makeham distribution, which can be used in lifetime modeling [8], is not of this type. This paper shows how reliability problems in which sums of independent random variables with the modified Makeham distribution appear can be approximated.

II. MODIFIED MAKEHAM DISTRIBUTION

In the literature [1, 3, 4–6, 11] a hazard rate function of the Makeham distribution is given by:

$$ h(t) = \frac{b}{a} \exp\left(\frac{t}{a}\right), \quad t > 0 $$

(1)

where $a$ is the scale parameter. It is the distribution to which the shape parameter $b$ will be introduced:

$$ h(t) = \frac{b}{a} \left(\frac{t}{a}\right)^{b-1} \exp\left(\frac{t}{a}\right). $$

(2)

Thus defined, the distribution will be called the modified Makeham distribution (MMD) [8–10]. As a result of this modification, for parameter values $b < 1$, the MMD has a bathtub hazard rate function. This is a desired characteristics among distributions used in modeling the lifetime of technical objects. Moreover, the MMD is the distribution which changes its skewness from a positive to a negative value as the shape parameter goes from zero to infinity.

The cumulative distribution function of the MMD will be determined using the formula [1]:

$$ F(t) = 1 - \exp\left(\int_0^t h(u)\,du\right). $$

(3)

By substituting (2) into (3) we obtain:

$$ F(t) = 1 - \exp\left[1 - \exp\left(\frac{t}{a}\right)^b\right]. $$

(4)

Knowing that

$$ f(t) = \frac{dF(t)}{dt} $$

(5)

the density function is given by the formula:
Using the formula (10) let us determine the sum of two independent random variables with the MMD (4).

\[
F_2(t) = \int_0^t \left[ 1 - \exp \left( - \frac{t}{a} \right) \right] b \left( \frac{u}{a} \right)^{b-1} \exp \left( \frac{u}{a} \right) du = \\
1 - \exp \left( - \frac{t}{a} \right) - \exp \left( - \frac{t}{a} \right) \exp \left( \frac{u}{a} \right) du.
\]

Similarly, as noted above, the sum of three independent random variables having the same distribution is determined as follows [12]:

\[
F_3(t) = \int_0^t F_2(t-u) f(u) du.
\]

where \( F_2 \) is the cumulative distribution function the sum of two independent random variables with the same distribution.

In case of three independent random variables with the MMD, the cumulative distribution function has a quite complicated form:

\[
F_3(t) = \int_0^t F_2(t-u) \left( \frac{b}{a} \right)^{b-1} \exp \left[ 1 + \left( \frac{u}{a} \right)^b - \exp \left( \frac{u}{a} \right) \right] du.
\]

III. SUMS OF RANDOM VARIABLES WITH THE MODIFIED MAKEHAM DISTRIBUTION

In this paragraph mathematical equations will be determined for the distribution of the sum of two and three independent random variables with the MMD. The cumulative distribution function for the sum of two independent random variables with the same distribution is given by the convolution integral [12]:

\[
F_2(t) = \int_0^t F(t-u) f\left( \frac{u}{a} \right) du.
\]

where \( F \) is the cumulative distribution function, and \( f \) the density function for a random variable.
of the single lifetime random variable. Numerical calculation of $F_k(t)$ when $k > 3$, is extremely time consuming and may have inestimable errors.

III.1. Numerical example

Let’s calculate on formulas (11) and (13) sums of two and three random variables of MMD. The required calculations were performed in the Mathcad Professional Application.

Figures 2 and 3 show the cumulative distribution functions of two random variables of MMD for the particular shape parameter values ($a = 1$) on the modified Makeham probability paper.

Figures 4 and 5 show the cumulative distribution functions of three random variables of MMD for the particular shape parameter values ($a = 1$) on the modified Makeham probability paper.

Obtained results on the MMD probability paper appear quite linear so the sum of two and three random variables with the MMD having the same parameters can be surrogate with the MMD having different parameter values.

Let us put forward that the distribution for the sum of $k$ ($k = 2, 3$) random variables of the MMD with the same parameters can be surrogate with MMD of which the parameters are calculated from the system of equations:

$$
\alpha_{2k} = k\alpha_1, \quad (14)
$$

$$
\mu_{2k} = k\mu_2, \quad (15)
$$

where $k$ denotes the convolution of $k$ random variables, $\alpha_1$ expected value, and $\mu_2$ variance, $\alpha_{2k}$ expected value of $k$ random variables of the MMD, $\mu_{2k}$ variance of $k$ random variables of the MMD.
By numerically solving the system of equations (14) and (15) for the particular shape parameter values \((a = 1)\) the estimated values are given in Table 1.

In order to finalize the verification of that the distribution of the sum of two and three random variables from the MMD with the same parameters can be approximated by the MMD with different parameters, for each example the Mizes-Smirnov compliance test will be conducted [12]. Calculated empirical values of the test statistics are presented in Table 1.

### Table 1 Estimation parameters the author obtained by numerical calculation

<table>
<thead>
<tr>
<th>(a = 1)</th>
<th>(b)</th>
<th>(a_c)</th>
<th>(b_c)</th>
<th>(n\varpi^2)</th>
<th>(a = 1)</th>
<th>(b)</th>
<th>(a_c)</th>
<th>(b_c)</th>
<th>(n\varpi^2)</th>
</tr>
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<tr>
<td>1.0</td>
<td>1.779</td>
<td>1.544</td>
<td>0.028670</td>
<td>1.0</td>
<td>2.502</td>
<td>1.992</td>
<td>0.048792</td>
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<tr>
<td>2.0</td>
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<td>3.044</td>
<td>0.031624</td>
<td>2.0</td>
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<tr>
<td>3.0</td>
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<td>4.499</td>
<td>0.026186</td>
<td>3.0</td>
<td>2.708</td>
<td>5.663</td>
<td>0.036989</td>
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<tr>
<td>0.3</td>
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<td>0.427</td>
<td>0.021067</td>
<td>0.3</td>
<td>3.474</td>
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<tr>
<td>0.5</td>
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<td>0.756</td>
<td>0.014267</td>
<td>0.5</td>
<td>2.695</td>
<td>0.973</td>
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<td>2.508</td>
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</table>

Figures 6-9 show that surrogate distribution is accurate to approximate the sums of two and three modified Makeham random variables. It seems that errors of plots are negligible.

At the level of significance \(\alpha = 0.05\) comparing the obtained empirical values with the critical value of \(n\varpi^2 = 0.4614\), there is no reason to reject that sums of two and three independent random variables with the MMD can be approximated by surrogate distribution. Owing to this
MMD distribution can have a wide application in the reliability theory.

### III.2. Application of the Surrogate Distribution to Determine the Renewal Function

The renewal function can be approximately determined using the formula \[ H(t) = F(t) + F_1(t) + F_2(t). \] (16)

**Example:**

In this example we will compare the values of the renewal function calculated numerically from the appropriate convolution integrals (11) and (13) as well as using surrogate distributions.

The obtained results are presented in Figures 10 and 11.

![Fig. 10. Comparison between the values of the renewal function determined on the basis of convolution integrals of MMD and on the basis of the surrogate distribution for parameters \(a = 1, b = 2\).](image1)

![Fig. 11. Comparison between the values of the renewal function determined on the basis of convolution integrals of MMD and on the basis of the surrogate distribution for parameters \(a = 1, b = 0.5\).](image2)

Such property as the pseudo-infinite divisibility of the MMD can speed up the process of evaluating the renewal function. Thus instead of calculating a proper convolution integral for each point, one can just obtain approximation of the proper parameters for the surrogate MMD and put them into the renewal function.

### IV. CONCLUSIONS

There is no doubt that the surrogate MMD, which was applied to the components of the renewal function is sufficiently accurate to determine the renewal process. Comparing results obtained by computer simulation of the convolution integral and by surrogate distribution we can conclude that the presented method yields quite the same results, which are presented in Figures 10 and 11. Errors among these methods are negligible. So it seems that the presented method of approximation of sums of two and three independent random variables with the surrogate MMD distribution is sufficiently accurate for engineering purposes. The surrogate distribution turns out to be relatively easy to implement and is surprisingly accurate.

The results obtained in this paper are a valuable contribution to reliability practice.

### References

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