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The Sequence Project of the Control Plan of Reliability of the Weibull Model Distribution

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Abstract: It is no exaggeration to say that the PN-IEC 61124 standard is inapplicable in many branches e.g. electronic industry. It is because PN-IEC 61124 assumes failure rate to be constant in time. This paper rejects this assumption and presents mathematical foundation of an alternative of PN-IEC 61124 applicable in electronic industry.

Key words: reliability test, Weibull distribution, Laplace transforms, renewal process

I. INTRODUCTION

It is no exaggeration to say that the IEC 6140 standard is inapplicable in many branchesof electronic industry. It is because PN-IEC 61124 assumes failure rate to be constant in time that is an obvious relict of the past. Willy-nilly, producers are forced to applyPN-IEC 61124 to their products being not aware of consequences expressed in terms of departures of OC curves from those presented in the standard.

This paper presents mathematical foundation of an alternative to PN-IEC 61124 applicable to electronic components and parts as well as a variety of other products that fail mostly due to early failures.

II. MATHEMATICAL BASIS OF SEQUENTIAL TESTING

Let us state two hypotheses related to reliability of the item of interest:

H_0 : The item is of satisfactory reliability. Its lifetime follows the Weibull model [5]

$$f_w(t,a,b) = \frac{b}{a}\left(\frac{t}{a}\right)^{b-1} e^{-\left(\frac{t}{a}\right)^b} , \tag{1}$$

with parameter values $a = a_0, b = b_0$, $a > 0$, $b > 0$, $t \geq 0$.

H_1 : The item is of unsatisfactory reliability. Its lifetime follows the Weibull model (1) with parameter values $a = a_1$, $b = b_1$, usually $a_1 < a_0$, $b_1 < b_0$.

In a sequential test the decisions can be the following: either to accept H_0, or reject H_0 and thereby accept H_1, or, last but not least, delay the decision and continue the test.

We must keep in mind that the probability of performing exact m renewals on N laboratory positions observed in time t from the beginning of the research follows the Poisson distribution:

$$P(m,t/a,b) = \frac{\left[N \cdot H(t/a,b)\right]^m}{m!} \cdot \exp\left[-N \cdot H(t/a,b)\right] \tag{2}$$

where: $m \in (0,\infty)$, $N = 1,2,...$, $H(t/a,b)$ is the renewal function provided that interarrival times follow the Weibull model (1). Derivation of the renewal function is presented in the next section.

To derive a decision rule we define the likelihood ratio

$$L(t,m) = \frac{P_1(t,m/a_1,b_1)}{P_0(t,m/a_0,b_0)}. \tag{3}$$

When $L(t,m) << 1$ it seems reasonable to accept H_0 and reject when $L(t,m) >> 1$.

Combining (2) and (3) and transforming and simplifying the expression we get

$$L(t,m)=\left[\frac{H_1(t)}{H_0(t)}\right]^m \exp\left\{N\cdot\left[H_0(t)-H_1(t)\right]\right\}, \quad (4)$$

where:

$$H_1(t)=H(t/a_1,b_1),$$
$$H_0(t)=H(t/a_0,b_0). \quad (5)$$

A rule for making a decision is formulated as follows
Accept H_0, if

$$L\le\frac{\beta}{1-\alpha} \quad (6a)$$

Reject H_0 and accept H_1, if

$$L\ge\frac{1-\beta}{\alpha} \quad (6b)$$

Continue the test, if

$$\frac{\beta}{1-\alpha}<L<\frac{1-\beta}{\alpha} \quad (6c)$$

where α, β are producer's and consumer's risks, respectively, $\alpha,\beta\in(0,1)$.

In practice, since these values α,β are usually smaller than 0,2 this particular choice of values $\beta/(1-\alpha)$, $(1-\beta)/\alpha$, gives real values close to reality.

Introducing the logarithmic likelihood ratio we would simplify further calculation. Namely,

$$l(t,m)=\ln L(t,m)=m\cdot\ln\left[\frac{H_1(t)}{H_0(t)}\right]+ \\ +N\cdot\left[H_0(t)-H_1(t)\right]=m\cdot\Omega(t)+N\cdot\Theta(t). \quad (7)$$

Definitions of auxiliary functions

$$\Omega(t)=\ln\left[\frac{H_1(t)}{H_0(t)}\right],$$
$$\Theta(t)=H_0(t)-H_1(t). \quad (7a)$$

The test has to be continued while

$$\frac{B-N\cdot\Theta(t)}{\Omega(t)}<m<\frac{A-N\cdot\Theta(t)}{\Omega(t)}, \quad (8a)$$

where:

$$B=\ln\frac{\beta}{1-\alpha},\quad A=\ln\frac{1-\beta}{\alpha}, \quad (8b)$$

otherwise the test is stopped [1].

III. DERIVATION OF THE RENEWAL FUNCTION

The relation between renewal function $H(t)$ and the renewal density function $\Lambda(t)$ is expressed by the following formula:

$$H(t)=\int_0^t \Lambda(u)du. \quad (9)$$

The renewal density function $\Lambda(t)$ and the failure density function $f(t)$ are interrelated.

$$\Lambda(t)=f(t)+\int_0^t \Lambda(\tau)\cdot f(t-\tau)d\tau. \quad (10)$$

Applying the Laplace transform to both sides of (10) we get

$$\Lambda(s)=f(s)+f(s)\cdot\Lambda(s) \quad (11a)$$

and after simple transformation [4]

$$\Lambda(s)=\frac{f(s)}{1-f(s)}. \quad (11b)$$

Unfortunately, in case of the Weibull lifetime model this equation cannot be solved analytically. To obtain an approximate solution we employ the mixed exponential model

$$f_s(t)=\sum_{i=1}^{3}\omega_i\cdot\lambda_i\cdot e^{-\lambda_i\cdot t}\quad \sum_{i=1}^{3}\omega_i=1$$
$$\lambda_i>0,\ i=1,2,3 \quad (12)$$

as a surrogate model to replace the original one.

The surrogate model has three desired features: simplicity, flexibility and transformability in Laplace sense. Fraction parameters ω_i and scale parameters λ_i, $i=1,2,3$ are chosen in such a way that they make the surrogate model most similar to the original Weibull distribution.

The similarity measure especially developed by the authors [2] for this purpose is:

$$SM(\overline{\omega},\overline{\lambda})=\int_0^\infty \min_t\left[f_s(t,\overline{\omega},\overline{\lambda}),f_w(t,a,b)\right]dt, \quad (13)$$

where $\overline{\omega},\overline{\lambda}$ are vectors of fraction and scale parameters mentioned above.

Let $\overline{\omega}^*$, $\overline{\lambda}^*$ be vectors that maximize (13). The Laplace transform $L(s)$ corresponding to (12) is

$$f_s(s) = L\left[\sum_{i=1}^{3} \omega^*_i \cdot \lambda_i^* \cdot e^{-\lambda_i \cdot t}\right] =$$

$$= \sum_{i=1}^{3} \omega_i^* \cdot \lambda_i^* \cdot L\left[e^{-\lambda_i \cdot t}\right] = \sum_{i=1}^{3} \omega_i^* \cdot \lambda_i^* \cdot \frac{1}{s+\lambda_i^*}. \tag{14}$$

Substituting (14) to (11b) we get

$$\Lambda(s) = \frac{\sum_{i=1}^{3} \frac{\omega_i^* \cdot \lambda_i^*}{s+\lambda_i^*}}{1-\sum_{i=1}^{3} \frac{\omega_i^* \cdot \lambda_i^*}{s+\lambda_i^*}} \tag{15}$$

after simple although arduous transformations we get

$$\Lambda(s) = \frac{L_2 s^2 + L_1 s + L_0}{M_3 s^3 + M_2 s^2 + M_1 s + M_0}, \tag{16}$$

where:

$$L_2 = \omega_1^* \cdot \lambda_1^* + \omega_2^* \cdot \lambda_2^* + \omega_3^* \cdot \lambda_3^*,$$

$$L_1 = \omega_1^* \cdot \lambda_1^* \cdot \left(\lambda_2^* + \lambda_3^*\right) + \omega_2^* \cdot \lambda_2^* \cdot \left(\lambda_1^* + \lambda_3^*\right) +$$
$$+ \omega_3^* \cdot \lambda_3^* \cdot \left(\lambda_1^* + \lambda_2^*\right),$$

$$L_0 = \lambda_1^* \cdot \lambda_2^* \cdot \lambda_3^*,$$

$$M_3 = 1,$$

$$M_2 = \lambda_1^* \cdot \left(1-\omega_1^*\right) + \lambda_2^* \cdot \left(1-\omega_2^*\right) + \lambda_3^* \cdot \left(1-\omega_3^*\right),$$

$$M_1 = \lambda_1^* \cdot \lambda_2^* \cdot \omega_3^* + \lambda_1^* \cdot \lambda_3^* \cdot \omega_2^* + \lambda_2^* \cdot \lambda_3^* \cdot \omega_1^*,$$

$$M_0 = 0.$$

Derivation of the renewal density function can be simplified when we convert (16) into part fraction

$$\Lambda(s) = \frac{\Lambda_0}{s} + \frac{\Lambda_1}{s-s_1} + \frac{\Lambda_2}{s-s_2}, \tag{17}$$

where s_1, s_2 are poles of (16), namely

$$s_{1,2} = \frac{\left(-M_2 \pm \sqrt{M_2^2 - 4M_1}\right)}{2} \tag{18}$$

and

$$\Lambda_0 = \frac{L_0}{s_1 \cdot s_2}, \quad \Lambda_1 = \frac{L_2 s_1^2 + L_1 s_1 + L_0}{s_1\left(s_1 - s_2\right)},$$
$$\Lambda_2 = \frac{L_2 s_2^2 + L_1 s_2 + L_0}{s_2\left(s_2 - s_1\right)} \tag{19}$$

are corresponding residues [3].

Thus, we obtain renewal density function performing inverse Laplace transform $L^{-1}(s)$

$$\Lambda(t) = L^{-1}\left[\Lambda(s)\right] = L^{-1}\left[\frac{\Lambda_0}{s}\right] + L^{-1}\left[\frac{\Lambda_1}{s-s_1}\right] +$$
$$+ L^{-1}\left[\frac{\Lambda_2}{s-s_2}\right] = \Lambda_0 + \Lambda_1 e^{-|s_1|t} + \Lambda_2 e^{-|s_2|t}. \tag{20}$$

Finally, the renewal function we need in the sequential test takes the form

$$H(t) = \int_0^t \Lambda(t)dt = \Lambda_0 \cdot t +$$
$$+ \frac{\Lambda_1}{|s_1|} \cdot \left(1-e^{-|s_1| \cdot t}\right) + \frac{\Lambda_2}{|s_2|} \cdot \left(1-e^{-|s_2| \cdot t}\right) = \tag{21}$$
$$= H_a \cdot t + H_b \cdot \left(1-e^{-|s_1| \cdot t}\right) + H_c \cdot \left(1-e^{-|s_2| \cdot t}\right),$$

where:

$$H_a = \Lambda_0, \quad H_b = \frac{\Lambda_1}{|s_1|}, \quad H_c = \frac{\Lambda_2}{|s_2|}. \tag{22}$$

IV. EXAMPLE

Table 1. Parameters of the lifetime models ascribed to items of satisfactory and unsatisfactory reliability

Weibull parameters	Reliability	
	satisfactory	unsatisfactory
a	3	1
b	0.9	0.5

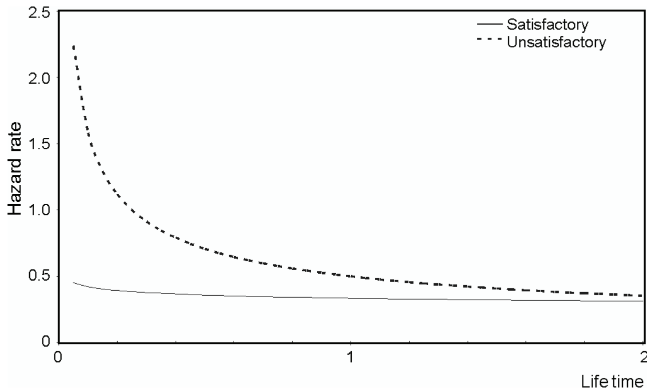

Fig. 1. Hazard rate functions ascribed to items of satisfactory and unsatisfactory reliability

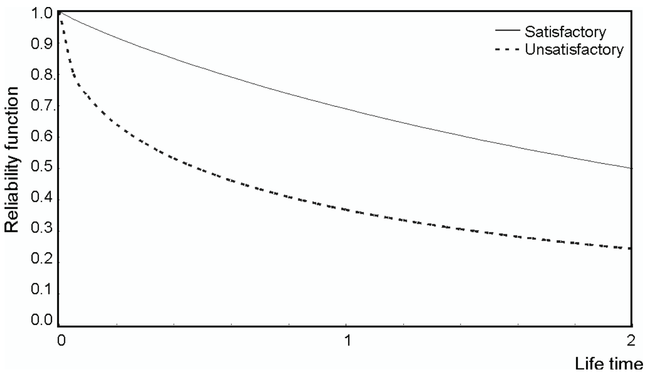

Fig. 2. Reliability functions ascribed to items of satisfactory and unsatisfactory reliability

Table 2. Parameters of the surrogate distribution

Surrogate parameters	Reliability	
	satisfactory	unsatisfactory
ω_1	0.00648	0.18974
ω_2	0.13509	0.41179
ω_3	0.85843	0.39847
λ_1	30.02119	21.79711
λ_2	1.11207	1.80952
λ_3	0.28745	0.24915

Table 3. Calculation parameters of the surrogate distribution (intermediate results)

Polynomial Coefficients	Reliability	
	Satisfactory	Unsatisfactory
L_2	0.591522551	4.980204693
L_1	12.50782126	27.28545122
L_0	9.596709336	9.827050661
M_3	1	1
M_2	30.82918745	18.87557531
M_1	29.82709909	18.03844713
M_0	0	0

Table 4. Calculation parameters of the renewal function

Renewal parameters	Reliability	
	satisfactory	unsatisfactory
s_1	–0.99992756	–1.009656861
s_2	–29.8292598	–20.72677411
Λ_1	0.321744643	0.469588746
Λ_2	0.080436722	0.635188875
Λ_3	0.178555812	3.852529843
H_a	0.321744643	0.469588746
H_b	0.08044255	0.629113612
H_c	0.005985928	0.185872139

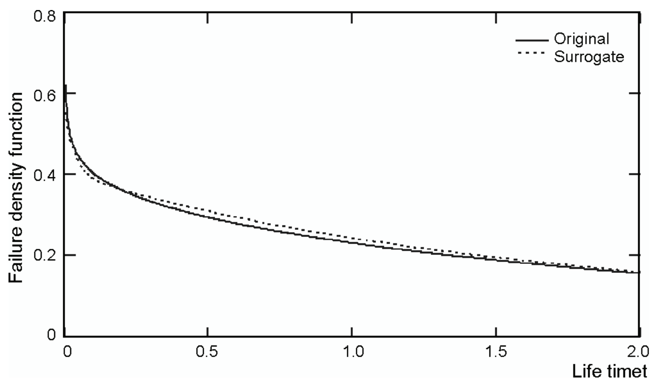

Fig. 3. The original and surrogate models in the case of unsatisfactory reliability. A comparison

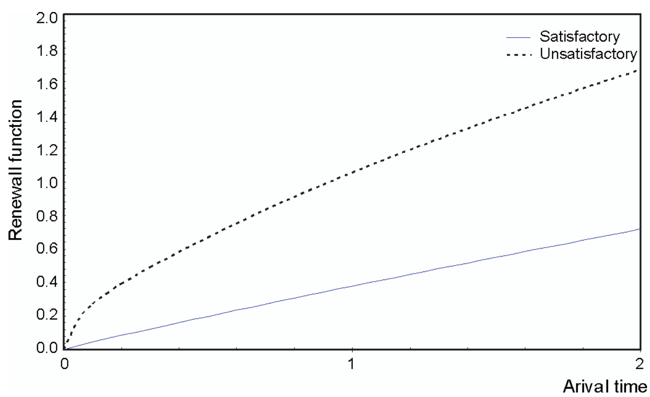

Fig. 4. Renewal functions ascribed to items of satisfactory and unsatisfactory reliability

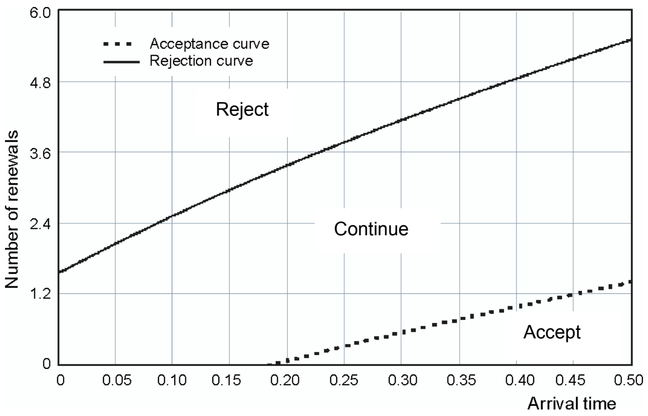

Fig. 5. A grid of the sequential compliance test

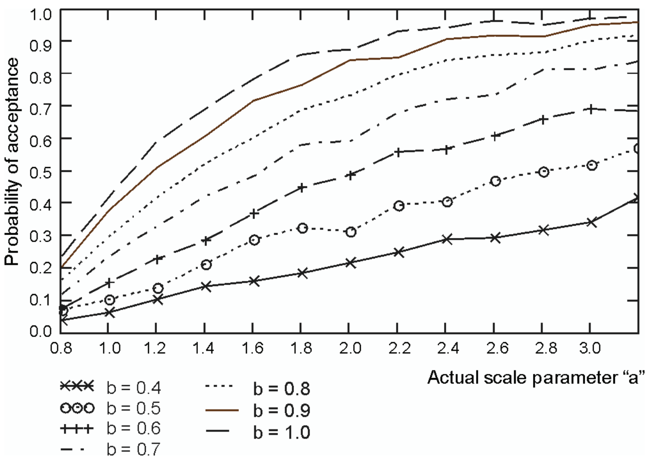

Fig. 6. Family of OC curves obtained with the Monte Carlo method

V. CONCLUSION

At first glance, the grid is similar to those of PN-IEC 61124 for constant failure rate. There is, however, a signiicant difference. Borders between "Reject", "Continue the test" and "Accept" regions are nonlinear. This is a direct consequence of nonlinearity of the renewal function. However, a decision making rule remains unchanged.

The theory presented in the article allows to design the control plans of reliability, which might be found aplicable in electronic industry.

References

[1] A. Drapella, *Lifetime models and renewal processes. Numerical treatment with Mathcad.* Pomeranian Pedagogical Academy, Słupsk, 72-73 (2002).

[2] A. Drapella, *Metoda fuzji i korekcji danych niezawodnościowych. Raport z pracy w ramach Grant KBN System ekspertowy wspomagania analiz niezawodnościowych i probabilistycznych zespołów energetycznych*. Politechnika Gdańska 1994.

[3] J. Grubicka, *Wyznaczanie gęstości odnowy metodą rozkładu zastępczego.* Warszawa 2004 (PAN, praca doktorska).

[4] J. Grubicka, *Wyznaczanie przybliżonej gęstości odnowy w przypadku weibullowskiego rozkładu czasu pracy do uszkodzenia metodą rozkładu zastępczego.* (cz. II*)* SPM-F 25 (2002).

[5] N.L Johnson, S. Kotz, *Distributions in statistics. Continuous univariate distributions*. (2 vols.) Houghton Mifflin, Boston 1970.

[6] PN-IEC 61124:2003. *Badanie nieuszkadzalności – Badania zgodności w przypadku stałej intensywności uszkodzeń i stałej intensywności strumienia uszkodzeń.*

[7] PN-EN 61124:2006. *Badanie nieuszkadzalności – Badania zgodności w przypadku stałej intensywności uszkodzeń i stałej intensywności strumienia uszkodzeń.*

[8] PN-IEC 60605-6:2002. *Badanie nieuszkadzalności urządzeń. Część 6: Weryfikowanie hipotez o stałej intensywności uszkodzeń i stałej intensywności strumienia uszkodzeń.*

[9] PN-EN 61703:2005. *Wyrażenia matematyczne dotyczące nieuszkadzalności, gotowości, obsługiwalności i zapewnienia środków obsługi.*

[10] W. Weibull, *Statistical distribution function of wide applicability*. Journal Applied Mechanics 293-317 (1951).

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