I. INTRODUCTION

In statistical literature it is possible to find numerous methods of inference relating to one variable. However, objects are often described by means of a large number of variables. The table, which is created by dividing data according to two variables, is called a two-way contingency table and ranks among basic statistical tools. In statistical inference it is not possible to answer a lot of questions in the analytical way, and one should use computer simulations to do so.

A wide collection of tests for exponentiality (including the Kolmogorov-Smirnov test) was discussed and compared in [1]. In [2] a new goodness-of-fit test was proposed. The test utilizes a novel characterization of the exponential distribution through its characteristic function. In [3] the goodness-of-fit technique based on the use of transformed empirical processes was applied to the construction of a test of exponentiality, focused on Weibull alternatives. In [11] the procedure of generating TT using two-dimensional normal distribution was presented. However, this method was not successful as the generator of two-way table of large sizes, because the corner cells of table often remained empty. For this reason in [12] the method of generating of two-way tables was suggested, making use of uniform random numbers, which was named the “bar method”.

This publication presents the TT which was used as a goodness-of-fit test defined as the TT-test. For creating TT the empirical and theoretical distribution functions were used. This test is devoted to the exponential distribution. However, samples subjected to the test come from the Generalized Gamma Distribution. The aim is to determine the power of the TT-test and to compare the obtained results with the Kolmogorov-Smirnov goodness-of-fit test.

In the second section a random numbers generator of the GG was presented in the form of the user function “GenGG”, which was implemented in VBA. The third section focuses on creating TT on the base of the empirical distribution function and the exponential distribution function (exponential distribution is a special case of the GG). In the fourth section a computer implementation of the TT-test using of the Pearson $\chi^2$-statistic was presented. The last section was devoted to the analysis of the obtained results. For a sample of size $n \in \{15, 20, 30\}$ and at significance level $\alpha = 0.05$ the power of the TT-test was determined. It turns out that for analyzed values of the GG parameters the
power of the discussed test is almost always greater than the power of a very popular KS goodness-of-fit test.

II. GENERATOR OF GENERALIZED GAMMA RANDOM NUMBERS

To execute simulation, random numbers of given probability distribution are necessary. The GG has a complex analytic form, which gives it the desired flexibility. The density function of the GG is given by [10]

\[
f(z; a, b, c) = \frac{b}{a \Gamma(c)} \left(\frac{z}{a}\right)^{b-1} \exp\left[-\left(\frac{z}{a}\right)^b\right], \quad (z > 0),
\]

(1)

where \(b > 0, c > 0\) are shape parameters, \(a > 0\) is a scale parameter. It is possible to write the CDF of the GG by means of the incomplete gamma-function

\[
\Gamma_n(c, x) = \int_0^x u^{c-1} \exp(-u) \, du
\]

in the form

\[
T_n(z) = \frac{\Gamma_n\left(c, \frac{z}{a}\right)^b}{\Gamma(c)}.
\]

(2)

Then \(f(x; a, 1, c)\) is a gamma density function, which for \(c = 1\) becomes an exponential density function. The random variables \(X\) and \(Z\) are interrelated

\[
X = \left(\frac{Z}{a}\right)^b \Rightarrow Z = a \cdot X^{1/b}.
\]

Thus, a generator of gamma random variables \(X\) will be sufficient to construct. A theoretical bases for creating user’s function “GenGG” written in VBA are possible to find in [13].

**Function GenGG(a As Single, b As Single, c As Single) As Single**

'declaration of variables
Dim z As Single, x As Single
Dim rand1 As Single, rand2 As Single
Dim d As Single, e As Single, f As Single
Dim g As Single, h As Single, k As Single
Dim m As Single, p As Single, accept As Boolean

'main part
Randomize Timer
If (c = 1) Then
    x = -Log(Rnd)
    z = a * x ^ (1 / b)
End If
If (c < 1) Then
    d = 1 / c
    e = 0.07 + 0.75 * Sqr(1 - c)
    f = 1 + c / e / Exp(e)
    Do
        rand1 = Rnd
        rand2 = Rnd
        g = f * rand1
        If g < 1 Then
            x = e * g ^ d
            accept = (rand2 <= (2 - x) / (2 + x))
            If accept = False Then accept = (rand2 <= Exp(-x))
            Else
                h = x / e
                accept = (rand2 * (c + h - c * h) <= 1)
                If accept = False Then accept = (rand2 <= (x / e) ^ (c - 1))
        End If
    Loop Until accept
    z = a * x ^ (1 / b)
End If
If (c > 1) Then
    k = 1 / Sqr(2 * c - 1)
    d = c + 1 / k
    f = c - Log(4)
    Do
        again:
            rand1 = Rnd
            rand2 = Rnd
            g = k * Log(rand1 / (1 - rand1))
            x = c * Exp(g)
            m = rand1 ^ 2 * rand2
            If m = 0 Then GoTo again
            p = f * d ^ * g - x
            accept = (p + (1 + Log(k) - k * m >= 0)
            If accept = False Then accept = (p >= Log(m))
        Loop Until accept
    z = a * x ^ (1 / b)
End If
GenGG = z
End Function

The random values \(z_i\) obtained by means of function “GenGG” were sorted and then on the basis of (2) a theoretical distribution function of the GG were calculated as well as an empirical distribution function

\[
F_i = \frac{i}{n + 1} \quad i = 1, \ldots, n.
\]

(3)
Graphs of both empirical and theoretical distribution functions are quite the same (Fig. 1), therefore \( z_i^* \) are generalized gamma random numbers.

III. GENERATOR OF TWO-BY-TWO TABLE

Let \( z_i^* \) will be sorted generalized gamma random values. To create TT values exponential distribution function \( T_2 \) as well as values of the empirical distribution \( z_i^* \) were used

\[
T_2\left(z_i^*; a^*\right) = 1 - \exp\left(-\frac{z_i^*}{a^*}\right).
\]  

(4)

An unknown value of the parameter of the exponential distribution was estimated by means of MLS, thanks to which the empirical distribution function fits better the theoretical distribution function (Fig. 2) than in MLM or MM (Fig. 3).

An estimate \( a^* \) of parameter \( a \) is a value which minimizes a function [6]

\[
M(a) = \sum_{i=1}^{n} \left[ T_2\left(z_i^*; a^*\right) - F_i \right]^2.
\]  

(5)

In order to find a minimum of the function (5) the supplement of the spreadsheet, which is called Solver, was used. It was written in the form of the procedure, so that it can be applied in the loop.

On the basis of (3) and (4) a difference of CDF was marked

\[
D_i = F_i - T_2\left(z_i^*; a^*\right), \quad i = 1, \ldots, n,
\]

whose sign decides on rows size of TT. A number \( M = DP_{\left(DP_{l(b)\in\mathbb{Z}}\right)} \) assuring best results of TT-test was calculated on the basis of sorted values \( DP_i = |D_i| \) and decides on the columns size (Table 1).

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( D_1 &gt; 0 \land D_1 \leq M )</td>
<td>( D_1 &gt; 0 \land D_1 &gt; M )</td>
</tr>
<tr>
<td>2</td>
<td>( D_1 \leq 0 \land</td>
<td>D_1</td>
</tr>
</tbody>
</table>

In the statistical literature requirements for the minimal number of objects in cells of the two-way table are diversified. The author of this publication assumed that every cell should have at least one element. The VBA implementation of TT generation, written in VBA, was introduced in the “GenTable” procedure which is using the “Solver” procedure. After apostrophes very exact comments were placed in order to make it easier for the reader to understand the codes of procedures.

```
Sub GenTable()
'declaration of tables
Dim Z() As Single           'random numbers of the GG
Dim T() As Single           'empirical distribution function
Dim X() As Single           'exponential distribution function
Dim X2, 2 As Integer        'TT
Dim D() As Single           'difference CDF
Dim Dp() As Single          'absolute values of difference of CDF
'declaration of variables
Dim a As Single                  'scale parameter of the GG
Dim b, c As Single     'shape parameter of the GG
Dim aEmp As Single            'sample parameter a
Dim i As Integer                 'index of rows
Dim j As Integer                 'index of columns
Dim n As Long                    'simple size
Dim r As Integer              'accessory variables
Dim l, 1, m, r As Single          'accessory variables
Dim hor, ro As Single          'accessory variables
```
Randomize Timer
Let a = 1
Let b = InputBox("Give b", "GG")
Let c = InputBox("Give c", "GG")
Let n = InputBox("Give a sample size", "n")
'redeclaration of tables
ReDim Z(n): ReDim T(n): ReDim F(n)
ReDim D(n): ReDim DP(n)
'preparing formulas for Solver
For r = 1 To n
ro = r + 20
Range("K" & ro).Select
ActiveCell.FormulaR1C1 = =1-EXP(-RC[-1]/R21C14)
Range("L" & ro).Select
ActiveCell.FormulaR1C1 = =RC[-3]/n + 1
Next r
Range("N22").Select
continue:
'Zeroing of cells of TT
For i = 1 To 2
For j = 1 To 2
X(i, j) = 0
Next j
Next i
'generation of generalized gamma random numbers
For r = 1 To n
Z(r) = GenGG(a, b, c)
Cells(20 + r, 9) = r
Cells(20 + r, 10) = Z(r)
Next r
'sorting the table Z
back:
hor = 0
For r = 1 To n - 1:
If (Z(r) <= Z(r + 1)) Then GoTo further
b1 = Z(r)
Z(r) = Z(r + 1)
Z(r + 1) = b1
hor = 1
further:
Next r
If hor = 1 Then GoTo back1
'calculating M
M = DP(int(r / 2) + 1)
'creating a TT
For r = 1 To n
If D(r) > 0 And D(r) <= M Then X(1, 1) = X(1, 1) + 1
If D(r) > 0 And D(r) > M Then X(1, 2) = X(1, 2) + 1
If D(r) <= 0 And Abs(D(r)) <= M Then X(2, 1) = X(2, 1) + 1
If D(r) < 0 And Abs(D(r)) > M Then X(2, 2) = X(2, 2) + 1
Next r
'checking the size of cells of TT
For i = 1 To 2
For j = 1 To 2
If X(i, j) < 1 Then GoTo continue
Next j
Next i
End Sub
Sub Solver()
Dim wynik As Long
SolverOptions MaxTime:=100, Iterations:=200,
Precision:=0.00000001, AssumeLinear:=False, StepThru:=False, Estimates:=1,
Derivatives:=1, SearchOption:=2, IntTolerance:=5, Scaling:=False,
Convergence:=0.0001, AssumeNonNeg:=False
SolverOk SetCell:='$N$22', MaxMinVal:=2, ValueOf:="1",
ByChange:='$N$21'
wynik = SolverSolve(True)
End Sub

IV. THE GOODNESS-OF-FIT TEST FOR EXPONENTIALITY

As mentioned in the introduction, this section presents the TT test that uses TT. Let $X_{ij}$ be the observed cells size of TT ($\sum \sum X_{ij} = n$). By $XR_{ij} = \sum X_{ij}$ indicated a number of sampling units belonging to $i$-th row, however, by $XC_{ij} = \sum X_{ij}$ a number of sampling units belonging to $j$-th column. Expected sizes of cells of TT were fixed by means of the formula [7]

$$NE_{ij} = \frac{XR_{ij} \cdot XC_{ij}}{n}.$$  

Pearson $\chi^2$-statistic for TT is given by [2]

$$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{(X_{ij} - NE_{ij})^2}{NE_{ij}}.$$  

Null hypothesis $H_0$ says that distribution in general population is exponential. Pearson $\chi^2$-statistic were calculated one thousand times for a sample size $n \in \{15, 20, 30\}$ and at significance level $\alpha = 0.05$. The generalized gamma sample value was determined by means of a function „GenGG". On the base of the received results the power of the test was obtained, that is the probability to reject the null hypothesis $H_0$, when it is not true.
Number $U$ is a number of cases when the value of $\chi^2$-statistic is smaller than critical value $CV$. Critical values for a given sample size $n$ on the base of estimates of quantiles were marked. Unknown values of quantiles were replaced by appropriate order statistics [5]. For shape parameter values of the GG $b = c = 1$ (the GG is becoming the exponential distribution) one thousand values of $\chi^2$-statistic were calculated. $K$-th order statistic of $\chi^2$-statistics as a critical value $CV$ was accepted, where $k = \text{int}[10000 \cdot (1 - \alpha)]$ (Table 2).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$CV$</th>
<th>$n$</th>
<th>$CV$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>3.233</td>
<td>30</td>
<td>3.453</td>
</tr>
<tr>
<td>20</td>
<td>3.300</td>
<td>35</td>
<td>3.540</td>
</tr>
<tr>
<td>25</td>
<td>3.381</td>
<td>40</td>
<td>4.821</td>
</tr>
</tbody>
</table>

The computer procedure of the TT-test written in VBA was introduced below.

```vba
Sub TT-test()
    Dim X(2, 2) As Integer 'observed size of TT cells
    Dim XR(2) As Single 'rows sizes
    Dim XC(2) As Single 'columns sizes
    Dim NE(2, 2) As Single 'expected size of TT cells
    Dim Chi(1000) As Single 'χ²-statistic
    Dim i As Integer 'index of row
    Dim j As Integer 'index of column
    Dim n As Long 'sample size
    Dim CV As Single 'critical value
    Dim U As Integer 'number of accepted null hypotheses
    Dim w As Integer 'accessory variable
    Dim b1, M, r As Single 'accessory variables
    Select Case n
        Case 15
            CV = 3.233
        Case 20
            CV = 3.300
        Case 30
            CV = 3.453
    End Select
    For i = 1 To 2
        For j = 1 To 2
            XR(i) = XR(i) + X(i, j)
        Next j
    Next i
    For i = 1 To 2
        For j = 1 To 2
            XC(j) = XC(j) + X(i, j)
        Next i
    Next j
    For j = 1 To 2
        For i = 1 To 2
            NE(i, j) = XC(j) * XR(i) / n
        Next i
    Next j
    For w = 1 To 1000
        For j = 1 To 2
            For i = 1 To 2
                Chi(w) = Chi(w) + ((X(i, j) - NE(i, j)) ^ 2) / NE(i, j)
            Next i
        Next j
        If Chi(w) < CV Then
            U = U + 1
        End If
    Next w
    MsgBox "Power of the TT-test is \((1000 - U) / 1000\)"
End Sub
```

V. OBTAINED RESULTS

For $a = 1$ and different values of shape parameters of the GG the power of the TT-test was calculated. On the basis of the obtained results for $n \in \{15, 20, 30\}$ surface graphs were prepared (Fig. 4-6). For values close to two the power of the test equals zero, next a tendency of the increase is visible with bending for $b \cdot c = 1$ (particularly visible in Fig. 6).

![Fig. 4. Relation between shape parameters of the GG and power of the TT-test for $n = 15$ and $\alpha = 0.05$](image-url)
Fig. 5. Relation between shape parameters of the GG and power of the TT-test for $n = 20$ and $\alpha = 0.05$

Fig. 6. Relation between shape parameters of the GG and power of the TT-test for $n = 30$ and $\alpha = 0.05$

In order to compare the results received by means of the TT-test, the author using the same samples also carried out the KS goodness-of-fit test (a parameter of the exponential distribution was estimated from a sample) and marked it power [9]. This test is well-known and therefore its results in the graphical form in Fig. 7-9 were shown only. The received surfaces have the bathtub shape and the real significance level of this test is equal to zero.

Fig. 7. Relation between shape parameters of the GG and power of the KS goodness-of-fit test for $n = 15$ and $\alpha = 0.05$

Fig. 8. Relation between shape parameters of the GG and power of the KS goodness-of-fit test for $n = 20$ and $\alpha = 0.05$
Two-by-Two Contingency Table as a Goodness-of-Fit Test

Fig. 9. Relation between shape parameters of the GG and power of the KS goodness-of-fit test for \( n = 30 \) and \( \alpha = 0.05 \)

As it was mentioned in section two, the GG is a very flexible distribution. By changing the parameter values one can get all possible shapes of the GG risk function (Table 3), which is given by [4]

\[
h(z) = \frac{f(z)}{1 - T_1(z)},
\]

where \( f(z) \), \( F(z) \) are appropriately a density function and a CDF of the GG.

Table 3. The shapes of the GG risk function

<table>
<thead>
<tr>
<th>Set</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A ) = {(b,c): (b = 1) \land (c = 1) }</td>
<td>constant</td>
</tr>
<tr>
<td>( B = {(b,c): (c &gt; \frac{1}{b}) \land (b &lt; 1) }</td>
<td>having a maximum</td>
</tr>
<tr>
<td>( C = {(b,c): (c \geq \frac{1}{b}) \land (b \geq 1) } - G_1</td>
<td>strictly increasing</td>
</tr>
<tr>
<td>( D = {(b,c): (b \leq 1) \land (c \leq \frac{1}{b}) } - G_1</td>
<td>strictly decreasing</td>
</tr>
<tr>
<td>( E = {(b,c): (c &lt; \frac{1}{b}) \land (b &gt; 1) }</td>
<td>having a minimum</td>
</tr>
</tbody>
</table>

In Tables 4-6 the powers of the TT-test and KS goodness-of-fit test were compared. In most of the obtained results the power of the proposed test is greater than the power of KS goodness-of-fit test and it is exemplified by cells of table marked with the letter T. In several cases the KS goodness-of-fit test has the greater power and it was marked with the letter K. Colors signify different shapes of the GG risk function:

- red – constant,
- purple – having a maximum,
- blue – strictly increasing,
- yellow – strictly decreasing,
- green – having a minimum.

The proposed TT-test in most of the analyzed cases is more powerful than the KS goodness-of-fit test and it is particularly visible for the little sample. Together with the sample increase this situation is changing for the better.

Table 4. The comparison of power of the TT-test and the KS goodness-of-fit test for \( n = 15 \) and \( \alpha = 0.05 \)

| \( c \) | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   | 1.1  | 1.2  | 1.3  | 1.4  | 1.5  | 1.6  | 1.7  | 1.8  | 1.9  | 2    |
|--------|-----|-----|-----|-----|-----|-----|------|------|------|------|------|------|------|------|-----|
| 0.5    | K   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 0.6    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 0.7    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 0.8    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 0.9    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1      | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.1    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.2    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.3    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.4    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.5    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.6    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.7    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.8    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 1.9    | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
| 2      | T   | T   | T   | T   | T   | T   | T    | T    | T    | T    | T    | T    | T    | T    | T    |
of the KS goodness-of-fit test. For \( n = 15 \) the TT-test has a greater power in 91% of the received results, for \( n = 20 \) in 80%, but for \( n = 30 \) in 72%.

### References


Two-by-Two Contingency Table as a Goodness-of-Fit Test


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