Fictitious Parts of the Load in Thermoelasticity

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Abstract: The three-dimensional coupled quasi-static problem of linear thermoelasticity is presented. The concept is based on a spatial extension of a region occupied by the considered body and on spatial formulation of a new fictitious load. All the outside objects are termed here fictitious ones. The solution of the initial-space value problem includes fictitious displacement-temperature components. Capacity values of approximate fictitious components are calculated from a boundary condition contracted to the finite time interval. The approximate solution to the primary thermoelastic problem is obtained by contracting in space the approximate solution to the initial-space value problem. It enables us to determine the thermoelastic flow.
Key words: coupled thermoelasticity, fictitious components, convolution products

I. INTRODUCTION

We consider the initial-boundary value problem of the quasi-static coupled thermoelasticity. At first a bounded region occupied by the model of the body is extending over the space in which the initial-space value problem is defined, thus corresponding with the primary initial-boundary value problem. The representation of displacement and body force vectors by means of the Helmholtz potentials led to the solution to the initial-space value problem. For some verification of the boundary condition a proposal is made to replace multidimensional integrals of the outside parts of the load with functions called fictitious components. "Fictitious" is a common term used for some ideas that do not concern the region really occupied by the model of the body. The fictitious components are determined in an approximate way in order to satisfy the boundary condition with arbitrary accuracy, at the same time, exactly satisfying the governing equations and the initial condition. Capacity values of the approximate temperature fictitious components located in some fictitious places (beyond the body region) are calculated in fictitious time defined as a discrete set of fictitious instants. Fictitious instants appear in temperature fictitious components only. To obtain an approximate solution a finite time interval is introduced in which the boundary condition is verified. The test instants divide the finite time into sec-

tions and constitute a discrete set called test time. Capacity values of the displacement fictitious components are time functions and therefore an algorithm designed for their determination concerns the current instant. An approximate solution is obtained by mathematical contraction to the body region of the solution to the initial-space value problem with the derived capacities of temperature-displacement fictitious components. It means that the stress tensor and heat flux vector should be determined as appropriate derivatives of the solution to the thermoelastic problem.

The coupled initial condition includes the initial change of dilatation. It was proposed as a consequence of the coupled current states constituting the thermoelastic process. In particular the initial condition as a description of the global initial state also has to be coupled.

II. THERMOELASTIC PROBLEM

Let the homogeneous and isotropic linearly-elastic body occupy the region D in three-dimensional Euclidean space Z. Let vector v_i denote the outward normal to the boundary ∂D. Assuming that the effect of the inertia may be omitted, the thermoelastic problem consists of the governing relations

$$\mu \cdot u_{i,jj}(\mathbf{x},t) + (\lambda + \mu) \cdot u_{j,ij}(\mathbf{x},t) - \gamma \cdot T_{,i}(\mathbf{x},t) = -X_i(\mathbf{x},t),$$

$$T_{,jj}(\mathbf{x},t) - \frac{\partial}{\partial t}\left[\frac{1}{\kappa} \cdot T(\mathbf{x},t) + \eta \cdot u_{j,j}(\mathbf{x},t)\right] = -\frac{1}{_o\lambda} \cdot W(\mathbf{x},t), \quad (1)$$

$$\mathbf{x} \in D, \quad t > 0$$

initial condition

$$T(\mathbf{x},0)/\kappa + \eta \cdot u_{j,j}(\mathbf{x},0) = g(\mathbf{x}),$$
$$\mathbf{x} \in D, \quad (2)$$

and boundary conditions

$$\mu \cdot \left[u_{i,j}(_1\mathbf{r},t) + u_{j,i}(_1\mathbf{r},t)\right] \cdot v_j(_1\mathbf{r},t) +$$

$$+ \left[\lambda \cdot u_{k,k}(_1\mathbf{r},t) - \gamma \cdot T(_1\mathbf{r},t)\right] \cdot v_i(_1\mathbf{r},t) = {}^vP_i(_1\mathbf{r},t),$$

$$u_i(_2\mathbf{r},t) = U_i(_2\mathbf{r},t),$$

$$_o\lambda \cdot T_{,j}(\mathbf{r},t) \cdot v_j(\mathbf{r},t) + \alpha \cdot T(\mathbf{r},t) = \quad (3)$$

$$= V(\mathbf{r},t) + \alpha \cdot {}_bT(\mathbf{r},t), \quad t > 0,$$

$$\mathbf{r} \in \partial D, \quad _1\mathbf{r} \in \partial_1 D, \quad _2\mathbf{r} \in \partial_2 D, \quad \partial_1 D \cup \partial_2 D = \partial D,$$

$$\partial_1 D \cap \partial_2 D = \Theta.$$

Here u_i is the displacement vector, T – the temperature, X_i – the components of body force, W – heat generated per unit volume and time, μ, λ – Lamé constants, γ – thermal expansion, κ – thermal diffusivity, and η – coefficient of thermomechanical coupling. Symbol g stands here for the thermoelastic initial state, ${}^vP_i$ – the surface traction on $\partial_1 D$, U_i – the displacement on $\partial_2 D$, V – the heat generated per unit surface and time, α – coefficient of surface heat transfer, and $_bT$ – the surrounding temperature. In the (1-3) system of equations a local thermoelastic state (which is to be determined) arises at the left-hand side of the equality signs. Therefore we will call the state the unknown. The unknown is subjected to some mathematical operations called the operator. On the other hand, the right-hand sides of the equations will be called the (thermoelastic) load.

Thermoelastic flow through a surface with the normal v_i is defined by the expressions:

$${}^vP_i(\mathbf{x},t) \equiv \sigma_{ji}(\mathbf{x},t) \cdot v_j(\mathbf{x},t),$$
$$\mathbf{x} \in D, \quad t > 0, \quad (4)$$

$${}^vq(\mathbf{x},t) \equiv q_j(\mathbf{x},t) \cdot v_j(\mathbf{x},t),$$
$$\mathbf{x} \in D, \quad t > 0. \quad (5)$$

The Cauchy formula (4) defines the mechanical part of the flow called surface traction ${}^vP_i$, while the thermal part (5) is defined by the surface heat flow ${}^vq$. The thermo-

elastic stress tensor is defined by the Duhamel-Neumann equation

$$\sigma_{ij} = \mu \cdot \left(u_{i,j} + u_{j,i}\right) + \left(\lambda \cdot u_{k,k} - \gamma \cdot T\right) \cdot \delta_{ij}, \quad (6)$$

and the heat flux vector by the Fourier law

$$q_i = -_o\lambda \cdot T_{,i}, \quad (7)$$

where $_o\lambda = \kappa \cdot _{\varepsilon}c$ thermal conductivity, $_{\varepsilon}c$ – specific heat at constant strain, and δ_{ij} – the Kronecker Delta.

We will call an interval $(0,\infty)$ the time, $(0,t)$ the current time, while $(0,\tilde{t})$ the finite time, where $\tilde{t} > t$ is the final instant and $t > 0$ is the current instant.

The governing equations (1) may describe the phenomena by means of the displacement-temperature pair $\langle u|T\rangle$, provided that the thermoelastic boundary and the initial load is taken into account. This will lead to the initial-boundary value problem as a model of the thermomechanical process. For a fixed current instant, set $\{\langle u|T\rangle(\mathbf{x},t), \mathbf{x} \in D\}$ is called a global thermoelastic state, while value $\langle u|T\rangle(\mathbf{x},t)$ – a local one.

III. INITIAL-SPACE VALUE PROBLEM

We will seek a solution of the thermoelastic problem in an approximate way in the class of twice differentiable functions. Let the fictitious model occupy space Z. For the model of the process taking place in the Cartesian product $Z \times (0,\infty)$, a continuous and bounded in infinity load is defined as a model of interactions on the body

$$\bar{g} : Z \to R, \quad \lim_{\mathbf{z}\to\infty}\left|\bar{g}(\mathbf{z})\right| < \infty,$$

$$\bar{W} : Z \times (0,\infty) \to R, \quad \lim_{\mathbf{z}\to\infty}\left|\bar{W}(\mathbf{z},t)\right| < \infty,$$

$$\bar{X}_i : Z \times (0,\infty) \to R, \quad (8)$$

$$\lim_{\mathbf{z}\to\infty}\left|\bar{X}_i(\mathbf{z},t)\right| < \infty, \quad \mathbf{z} \in Z, t > 0$$

and both temperature and displacement fields are defined as well

$$\bar{T} : Z \times (0,\infty) \to R,$$

$$\bar{u}_i : Z \times (0,\infty) \to R. \quad (9)$$

We assume that a new load does not differ within region D from the thermoelastic load existing there

$$\bar{g}(\mathbf{z}) = g(\mathbf{z}), \quad \bar{W}(\mathbf{z},t) = W(\mathbf{z},t),$$

$$\bar{X}_i(\mathbf{z},t) = X_i(\mathbf{z},t), \quad \mathbf{z} \in D, t > 0. \quad (10)$$

For the thermoelastic process taking place in the Cartesian product $Z \times (0, \infty)$, the initial-space value problem contains improper boundary conditions

$$
\mu \cdot \overline{u}_{i,jj}(\mathbf{z},t) + (\lambda + \mu) \cdot \overline{u}_{j,ij}(\mathbf{z},t) - \gamma \cdot \overline{T}_{,i}(\mathbf{z},t) =
$$
$$
= -\overline{X}_i(\mathbf{z},t),
$$
$$
\overline{T}_{,jj}(\mathbf{z},t) - \frac{\partial}{\partial t}\left[\frac{1}{\kappa} \cdot \overline{T}(\mathbf{z},t) + \eta \cdot \overline{u}_{j,j}(\mathbf{z},t) \right] = \tag{11}
$$
$$
= -\frac{1}{_o\lambda} \cdot \overline{W}(\mathbf{z},t),
$$
$$
\mathbf{z} \in Z, \quad t > 0,
$$

$$
\overline{T}(\mathbf{z},0)/\kappa + \eta \cdot \overline{u}_{j,j}(\mathbf{z},0) = \overline{g}(\mathbf{z}), \qquad \mathbf{z} \in Z
$$
$$
\lim_{\mathbf{z}\to\infty}\left|\overline{T}(\mathbf{z},t)\right| < \infty, \qquad \lim_{\mathbf{z}\to\infty}\left|\overline{u}_i(\mathbf{z},t)\right| < \infty, \quad t > 0. \tag{12}
$$

Suppose $\mathbf{F}$ is a vector point function with $r^2 \cdot |\mathbf{F}|$ bounded at infinity [1, p. 326]. We assume that vectors $\overline{u}_i$ and $\overline{X}_i$ satisfy such a condition and we express them in terms of Helmholtz potentials

$$
\overline{u}_i = \tilde{\lambda} \cdot \phi_{,i} + \tilde{\mu} \cdot \varepsilon_{ijk} \cdot \psi_{k,j}, \quad \psi_{k,k} = 0,
$$
$$
\tilde{\lambda} = (\lambda + 2 \cdot \mu)^{-1}, \ \tilde{\mu} = (\mu)^{-1} \tag{13}
$$
$$
\overline{X}_i = \beta_{,i} + \varepsilon_{ijk} \cdot \chi_{k,j}, \qquad \chi_{k,k} = 0.
$$

ε_{ijk} denotes here the permutation symbol, and zeroing of the divergence is called a marking condition. In an infinite thermoelastic space the Helmholtz potentials can be represented by the body force [2, p. 182]

$$
\beta(\mathbf{z},t) = -\frac{1}{4\cdot\pi} \cdot \int_Z \frac{\overline{X}_{i,i}(\mathbf{z},t)}{|\mathbf{z}-\mathbf{y}|} \cdot d\mathbf{y} + {}_o\beta(t),
$$
$$
\chi_i(\mathbf{z},t) = -\frac{1}{4\cdot\pi} \cdot \int_Z \frac{\varepsilon_{ijk} \cdot \overline{X}_{k,j}(\mathbf{y},t)}{|\mathbf{z}-\mathbf{y}|} \cdot d\mathbf{y} + {}_o\chi_i(t), \tag{14}
$$
$$
\mathbf{z} \in Z, \ t > 0.
$$

For the problem (11-12) such temperatures and potentials that satisfy the following Helmholtz space problem are required:

$$
\left[\phi_{,jj}(\mathbf{z},t) + \beta_{,i}(\mathbf{z},t) - \gamma \cdot \overline{T}(\mathbf{z},t) \right]_{,i} +
$$
$$
+ \varepsilon_{ijk} \cdot \left[\psi_{k,llj}(\mathbf{z},t) + \chi_{k,j}(\mathbf{z},t) \right] = 0,
$$
$$
\overline{T}_{,jj}(\mathbf{z},t) - \frac{\partial}{\partial t}\left[\frac{1}{\kappa} \cdot \overline{T}(\mathbf{z},t) + \eta \cdot \tilde{\lambda} \cdot \phi_{,jj}(\mathbf{z},t) \right] = \tag{15}
$$
$$
= -\frac{1}{_o\lambda} \cdot \overline{W}(\mathbf{z},t), \quad \mathbf{z} \in Z, \quad t > 0,
$$

$$
\overline{T}(\mathbf{z},0)/\kappa + \eta \cdot \overline{u}_{j,j}(\mathbf{z},0) = \overline{g}(\mathbf{z}), \qquad \mathbf{z} \in Z
$$
$$
\lim_{\mathbf{z}\to\infty}\left|\overline{T}(\mathbf{z},t)\right| < \infty, \qquad \lim_{\mathbf{z}\to\infty}\left|\overline{u}_i(\mathbf{z},t)\right| < \infty, \quad t > 0. \tag{16}
$$

For the Helmholtz potentials we have

$$
\phi(\mathbf{z},t) = \left[\gamma \cdot \overline{T}(Z,t) - \beta(Z,t) \right] * E(\mathbf{z}),
$$
$$
\psi_i(\mathbf{z},t) = -\chi_i(Z,t) * E(\mathbf{z}). \tag{17}
$$

We can check that the functions satisfying the problem (11-12) have the form [5]:

$$
\overline{T}(\mathbf{z},t) =
$$
$$
= \tilde{\kappa} \cdot \left[\eta \cdot \tilde{\lambda} \cdot \frac{\partial}{\partial t}\beta(Z \times (0,t)) + \overline{W}(Z \times (0,t))/{}_o\lambda \right] * F(\mathbf{z},t) + \tag{18}
$$
$$
+ \tilde{\kappa} \cdot \left[\overline{g}(Z) + \eta \cdot \tilde{\lambda} \cdot \beta(Z,0) \right] * F(\mathbf{z},t),
$$
$$
\mathbf{z} \in Z, t > 0,
$$

$$
\overline{u}_i(\mathbf{z},t) =
$$
$$
= \tilde{\lambda} \cdot \left[\gamma \cdot \overline{T}(Z,t) - \beta(Z,t) \right] * E_{,i}(\mathbf{z}) + \tag{19}
$$
$$
- \tilde{\mu} \cdot \varepsilon_{ijk} \cdot \chi_k(Z,t) * E_{,j}(\mathbf{z}).
$$

Here $\tilde{\kappa}$ plays the role of the substitute thermal diffusivity

$$
\frac{1}{\tilde{\kappa}} = \frac{1}{\kappa} + \eta \cdot \gamma \cdot {}_o\lambda. \tag{20}
$$

The denotation $(Z, \cdot) *$ will be called the Laplace convolution multiplication on set Z, and its result – the Laplace convolution product (of two functions)

$$
f(Z,t) * E(\mathbf{z}) \equiv \int_Z f(\mathbf{y},t) \cdot E(\mathbf{z}-\mathbf{y}) \cdot d\mathbf{y}, \tag{21}
$$
$$
\mathbf{z} \in Z.
$$

Function E is the fundamental solution of the elliptic operator

$$
E(\mathbf{z}) \equiv -(4 \cdot \pi \cdot |\mathbf{z}|)^{-1}, \quad \mathbf{z} \neq 0. \tag{22}
$$

The denotation $(Z) *$ will be called the space convolution multiplication on set Z, and its result the space convolution product

$$
f(Z) * F(\mathbf{z},t) \equiv \int_Z f(\mathbf{y}) \cdot F(\mathbf{z}-\mathbf{y},t) \cdot d\mathbf{y}, \tag{23}
$$
$$
\mathbf{z} \in Z, \quad t > 0.
$$

Function F is the fundamental solution of the parabolic operator

$$F(\mathbf{z},t) \equiv \begin{cases} 0, & t<0, \quad \mathbf{z}\in Z \\ 0, & t=0, \quad \mathbf{z}\neq 0 \quad (24) \\ \left(2\cdot\sqrt{\pi\cdot\tilde{\kappa}\cdot t}\right)^{-3}\cdot\exp\left(-\dfrac{|\mathbf{z}|^2}{4\cdot\tilde{\kappa}\cdot t}\right), & t>0, \quad \mathbf{z}\in Z. \end{cases}$$

The denotation $(Z\times(0,t))*$ will be called the space-time convolution multiplication on set $Z\times(0,t)$, and its result – the space-time convolution product

$$f\left(Z\times(0,t)\right)*F(\mathbf{z},t) \equiv$$
$$\equiv \int_0^t\int_Z f(\mathbf{y},s)\cdot F(\mathbf{z}-\mathbf{y},t-s)\cdot d\mathbf{y}ds, \qquad (25)$$
$$\mathbf{z}\in Z, \ t>0.$$

The space-time convolution satisfies an important identity for any integrable function f [3, p. 230]

$$\frac{\partial}{\partial t}\left[f(Z\times(0,t))*F(\mathbf{z},t)\right]-\tilde{\kappa}\cdot\left[f(Z\times(0,t))*F(\mathbf{z},t)\right]_{,jj} = \qquad (26)$$
$$= f(\mathbf{z},t), \quad \mathbf{z}\in Z, \ t>0.$$

An improper boundary condition for function $\bar{T}$ is met due to the properties of the F function and boundedness (8) in infinity. Functions convolutionally multiplied by F are assumed in a continuous and bounded manner, hence the formula (18) presents a continuous and bounded function in $Z\times(0,\tau)$, where $0<\tau<\infty$ [4, pp. 361-363]. So function $\bar{T}$ is bounded in infinity and from relation (14) it appears that the Laplace convolutions in formula (19) gives a continuously differentiable function on a bounded and closed domain, if the convolution is defined on this domain [4, p. 191]. Therefore, improper boundary conditions for functions u_i are also satisfied.

IV. TEMPERATURE-DISPLACEMENT FICTITIOUS COMPONENTS

Let us make use of the law of additivity of the integral as the set function, and make use of the formula (10)

$$\bar{T}(\mathbf{z},t) = \tilde{\kappa}\cdot\left[\eta\cdot\tilde{\lambda}\cdot\frac{\partial}{\partial t}\beta\left(D\times(0,t)\right)+\right.$$
$$\left.+\frac{1}{_o\lambda}\cdot W\left(D\times(0,t)\right)\right]*F(\mathbf{z},t)+ \qquad (27)$$
$$+\tilde{\kappa}\cdot\left[g(D)+\eta\cdot\tilde{\lambda}\cdot\beta(D,0)\right]*F(\mathbf{z},t)+$$
$$+{_f}T(\mathbf{z},t), \quad \mathbf{z}\in Z, \quad t>0,$$

$$\bar{u}_i(\mathbf{z},t) = \tilde{\lambda}\cdot\left[\gamma\cdot\bar{T}(D,t)-\beta(D,t)\right]*E_{,i}(\mathbf{z})+$$
$$-\tilde{\mu}\cdot\varepsilon_{ijk}\cdot\chi_k(D,t)*E_{,j}(\mathbf{z})+ \qquad (28)$$
$$+{_f}u_i(\mathbf{z},t), \qquad \mathbf{z}\in Z, \ t>0$$

where the following denotations have been used

$${_f}u_i(\mathbf{z},t) \equiv \tilde{\lambda}\cdot\left[\gamma\cdot\bar{T}(Z\setminus D,t)-\beta(Z\setminus D,t)\right]*E_{,i}(\mathbf{z})+$$
$$-\varepsilon_{ijk}\cdot\chi_k(Z\setminus D,t)*E_{,j}(\mathbf{z})/\mu,$$
$${_f}T(\mathbf{z},t) \equiv \tilde{\kappa}\cdot\left[\eta\cdot\tilde{\lambda}\cdot\frac{\partial}{\partial t}\beta\left((Z\setminus D)\times(0,t)\right)+\right.$$
$$\left.+\frac{1}{_o\lambda}\cdot\bar{W}\left((Z\setminus D)\times(0,t)\right)\right]*F(\mathbf{z},t)+ \qquad (29)$$
$$+\tilde{\kappa}\cdot\left[\eta\cdot\tilde{\lambda}\cdot\beta(Z\setminus D,0)+\bar{g}(Z\setminus D)\right]*F(\mathbf{z},t),$$
$$\mathbf{z}\in Z, \ t>0.$$

Functions ${_f}T$ and ${_f}u_i$ are the temperature and displacement fictitious components. They are defined in time $(0,\infty)$ on the whole space Z. The functions (27-28) are called the component integral because they satisfy strictly (11-12) and contain the fictitious components. The fictitious components have to satisfy the component admissibility conditions

$$\mu\cdot{_f}u_{i,jj}(\mathbf{x},t)+(\lambda+\mu)\cdot{_f}u_{j,ij}(\mathbf{x},t) = 0,$$
$${_f}T_{,jj}(\mathbf{x},t) = \frac{1}{\tilde{\kappa}}\cdot\frac{\partial}{\partial t}{_f}T(\mathbf{x},t), \qquad (30)$$
$$\mathbf{x}\in D, \quad t>0,$$
$$T(\mathbf{x},0) = 0, \qquad \mathbf{x}\in D. \qquad (31)$$

It is easy to notice that assuming the displacement fictitious component in the form of a gradient of the elliptic fundamental solution and the temperature fictitious component in the form of the parabolic fundamental solution, the component admissibility conditions are fulfilled. Therefore, the approximate fictitious components will be searched for in the form:

$${_f^a}T(\mathbf{z},t) \equiv \sum_{l=1}^p\sum_{m=1}^n T_l^m\cdot F\left(\mathbf{z}-{_m}\mathbf{y},t-{^l}s\right),$$
$$t>0, \mathbf{z}\in Z, \quad T_l^m, {^l}s\in R, \quad {_m}\mathbf{y}\in Z\setminus D, \qquad (32)$$

$${_f^a}u_i(\mathbf{z},t) \equiv \sum_{m=1}^n u_j^m(t)\cdot E_{,ij}(\mathbf{z}-{_m}\mathbf{y}),$$
$$t>0, \ \mathbf{z}\in Z, \ u_i^m(t)\in R, \qquad (33)$$
$$u_i^m(0) = 0, \qquad {_m}\mathbf{y}\in Z\setminus D.$$

The functions (32-33) are called approximate fictitious components and the coefficients accompanying the fundamental solutions – component capacities: $_f^a T$ – the approximate temperature component, and $_f^a u_k$ – the approximate displacement component. Here $_m\mathbf{y}$ is the fictitious place, $^l s$ – the fictitious instant, w_l^m, $u_k^m(t)$ – the temperature-displacement capacity in the $_m\mathbf{y}$ place, the sets $\{_m\mathbf{y} : m = \overline{1,n}\}$. $\{^l s : l = \overline{1,p}\}$ fictitious location and time. The approximate fictitious components defined this way satisfy the component admissibility conditions.

Let us replace the fictitious components in the component integral with their approximate form (32-33)

$$\hat{T}(\mathbf{z},t) \equiv$$
$$\equiv \tilde{\kappa}\cdot\left[\eta\cdot\tilde{\lambda}\cdot\frac{\partial}{\partial t}\beta(D\times(0,t)) + \frac{1}{_o\lambda}\cdot W(D\times(0,t))\right]*F(\mathbf{z},t) \tag{34}$$
$$+\tilde{\kappa}\cdot\left[g(D) + \eta\cdot\tilde{\lambda}\cdot\beta(D,0)\right]*F(\mathbf{z},t) + _f^a W(\mathbf{z},t),$$
$$\mathbf{z} \in Z, \quad t > 0,$$

$$\hat{u}_i(\mathbf{z},t) \equiv \tilde{\lambda}\cdot\left[\gamma\cdot\hat{T}(D,t) - \beta(D,t)\right]*E_{,i}(\mathbf{z}) +$$
$$-\tilde{\mu}\cdot\varepsilon_{ijk}\cdot\chi_k(D,t)*E_{,j}(\mathbf{z}) + \tag{35}$$
$$+ _f^a u_i(\mathbf{z},t),$$
$$\mathbf{z} \in Z, \quad t > 0.$$

These functions are the approximate integral. The capacities of the approximate components should be derived to satisfy the boundary condition (3) in the best possible way. Therefore, the time $(0,\infty)$ will be contracted to the finite time $(0,\tilde{t})$. The temperature capacity values can be determined independently. For the thermoelastic stress tensor (6) in the finite time the normal to the boundary surface will be assumed as independent of the current instant

$$2\cdot\mu\cdot v_j(_1\mathbf{r})\cdot\sum_{m=1}^{n}u_k^m(t)\cdot E_{,ijk}(_1\mathbf{r},t) = \hat{P}_i(_1\mathbf{r},t), \tag{36}$$
$$_1\mathbf{r}\in\partial_1 D, \quad t\in(0,\tilde{t}),$$

$$\sum_{m=1}^{n}u_j^m(t)\cdot E_{,ij}\left(_2\mathbf{r} - _m\mathbf{y}\right) = \hat{U}_i\left(_2\mathbf{r},t\right), \tag{37}$$
$$_2\mathbf{r}\in\partial_2 D, \quad t\in(0,\tilde{t}),$$

$$\sum_{l=1}^{p}\sum_{m=1}^{n}w_l^m\cdot\left[_o\lambda\cdot v_i(\mathbf{r})\cdot F_{,i} + \alpha\cdot F\right](\mathbf{r} - _m\mathbf{y}, t - ^l s) = \hat{V}(\mathbf{r},t), \tag{38}$$
$$\mathbf{r}\in\partial D, \quad t\in(0,\tilde{t})$$

where the following denotations have been used

$$\hat{P}_i(\mathbf{r},t) \equiv {}^v P_i(\mathbf{r},t) +$$
$$+\tilde{\lambda}\cdot\left[\lambda\cdot\beta(\mathbf{r},t) + 2\cdot\mu\cdot\gamma\cdot\hat{T}(\mathbf{r},t)\right]\cdot v_i(\mathbf{r}) + v_j(\mathbf{r})$$
$$\cdot\left\{2\cdot\mu\cdot\tilde{\lambda}\cdot\left[\beta(D,t) - \gamma\cdot\hat{T}(D,t)\right]*E_{,ij}(\mathbf{r}) + \tag{39}\right.$$
$$\left. +\chi_l(D,t)*\left[\varepsilon_{ikl}\cdot E_{,kj}(\mathbf{r}) + \varepsilon_{jkl}\cdot E_{,ki}(\mathbf{r})\right]\right\},$$

$$\hat{U}_i(\mathbf{r},t) \equiv U_i(\mathbf{r},t) + \tilde{\lambda}\cdot\left[\beta(D,t) - \gamma\cdot\hat{T}(D,t)\right]*E_{,i}(\mathbf{r}) + \tag{40}$$
$$+\varepsilon_{ijk}\cdot\tilde{\mu}\cdot\chi_k(D,t)*E_{,j}(\mathbf{r}),$$

$$\hat{V}(\mathbf{r},t) \equiv V(\mathbf{r},t) + \alpha\cdot _b T(\mathbf{r},t) +$$
$$-\tilde{\kappa}\cdot\left[\eta\cdot\tilde{\lambda}\cdot\frac{\partial}{\partial t}\beta(D\times(0,t)) +\right.$$
$$\left. +\frac{1}{_o\lambda}\cdot W(D\times(0,t))\right]*[v_j(\mathbf{r})\cdot _o\lambda\cdot F_{,j}(\mathbf{r},t) + \alpha\cdot F(\mathbf{r},t)] + \tag{41}$$
$$-\tilde{\kappa}\left[g(D) + \eta\cdot\tilde{\lambda}\beta(D,0)\right]*\left[v_j(\mathbf{r})\cdot _o\lambda\cdot F_{,j}(\mathbf{r},t) + \alpha\cdot F(\mathbf{r},t)\right].$$

We shall cover boundary ∂D with mesh $_h\mathbf{r}$, $h = \overline{1,N}$, where $N > n$. Let us call set $\{^l t : l = \overline{1,p}\}$ of test instants $^k t$ the test time, where $0 < {}^1 t < {}^2 t < ... < {}^p t < \tilde{t}$. We shall write the formula (38) at the nodes $_h\mathbf{r}\in\partial D$ and the test instants $^k t\in(0,\tilde{t})$

$$\sum_{l=1}^{p}\sum_{m=1}^{n}T_l^m\cdot B_{hm}^{kl} = \hat{V}(_h\mathbf{r}, {}^k t), \quad _h\mathbf{r}\in\partial D,$$
$$h = \overline{1,N}, \quad {}^k t\in(0,\tilde{t}), \quad k = \overline{1,p}. \tag{42}$$

$$B_{hm}^{kl} \equiv {}_o\lambda\cdot F_{,j}(_h\mathbf{r} - _m\mathbf{y}, {}^k t - {}^l s)\cdot v_j(_h\mathbf{r}) +$$
$$+\alpha\cdot F(_h\mathbf{r} - _m\mathbf{y}, {}^k t - {}^l s).$$

The coupled system of the $3\cdot N$ equations with the $3\cdot n$ unknown values $u_l^m(t)$, $l = \overline{1,3}, m = \overline{1,n}$ can be written in a condensed form for adequately numbered load values and coefficients of the unknown

$$\sum_{l=1}^{3}\sum_{m=1}^{n}A_{hm}^{il}\cdot u_l^m(t) = p_h^i(t), \quad h = \overline{1,N}, \quad i = \overline{1,3}. \tag{43}$$

The following denotations have been used

$$p_h^i(t) \equiv \begin{cases} \tilde{P}_i(_h\mathbf{r},t), & h = \overline{1, {}^1 N}, \\ \tilde{U}_i(_h\mathbf{r},t), & h = \overline{{}^1 N + 1, N}, \end{cases} \quad i = \overline{1,3}, \tag{44}$$

$$A_{hm}^{il} \equiv \begin{cases} 2 \cdot \mu \cdot E_{,ijl}(_h\mathbf{r} - {_m}\mathbf{y}) \cdot v_j(_h\mathbf{r}), & h = \overline{1, {}^1N}, \\ \mu \cdot E_{,il}(_h\mathbf{r} - {_m}\mathbf{y}), & h = \overline{{}^1N+1, N}, \end{cases} \quad (45)$$

$$i, l = \overline{1,3}, \ m = \overline{1,n}.$$

New temperature and displacement distributions may be defined in region D. For this purpose a contraction to the region D of the functions $\hat{T}, \hat{u}_i$, defined by the formulae (34-35), will be carried out

$$^aT(\mathbf{x},t) = -\tilde{\kappa} \cdot \left[\eta \cdot \tilde{\lambda} \cdot \frac{\partial}{\partial t} \beta(D \times (0,t)) + \right.$$
$$\left. + \frac{1}{_o\lambda} \cdot W(D \times (0,t)) \right] * F(\mathbf{x},t) +$$
$$- \tilde{\kappa} \cdot \left[g(D) + \eta \cdot \tilde{\lambda} \cdot \beta(D,0) \right] * F(\mathbf{x},t) +$$
$$+ \sum_{l=1}^{p} \sum_{m=1}^{n} \tilde{T}_l^m \cdot F(\mathbf{x} - {_m}\mathbf{y}, t - {}^ls), \ (\mathbf{x},t) \in D \times (0,\infty), \quad (46)$$

$$^au_i(\mathbf{x},t) \equiv \tilde{\lambda} \cdot \left[\gamma \cdot {}^aT(D,t) - \beta(D,t) \right] * E_{,i}(\mathbf{z}) +$$
$$- \varepsilon_{ijk} \cdot \chi_k(D,t) * E_{,j}(\mathbf{z}) / \mu +$$
$$+ \sum_{m=1}^{n} \tilde{u}_j^m(t) \cdot E_{,ij}(\mathbf{x} - {_m}\mathbf{y}), \quad (47)$$

$$\mathbf{x} \in D, \ t > 0.$$

The expressions (46-47) create an approximate solution to the thermoelastic problem. The Duhamel-Neumann relation enables determination of the thermoelastic stress tensor

$$^a\sigma_{ij}(\mathbf{x},t) = 2 \cdot \mu \cdot \left\{ \tilde{\lambda} \cdot \left[\gamma \cdot {}^aT(D,t) - \beta(D,t) \right] * E_{,ij}(\mathbf{x}) + \right.$$
$$\left. + \sum_{m=1}^{n} u_k^m(t) \cdot E_{,ijk}(\mathbf{x} - {_m}\mathbf{y}) \right\} +$$
$$- \chi_l(D,t) * \left[\varepsilon_{ikl} \cdot E_{,kj}(\mathbf{x}) + \varepsilon_{jkl} \cdot E_{,ki}(\mathbf{x}) \right] +$$
$$- \delta_{ij} \cdot \tilde{\lambda} \cdot \left[2 \cdot \mu \cdot \gamma \cdot {}^aT(\mathbf{x},t) + \lambda \cdot \beta(\mathbf{x},t) \right]. \quad (48)$$

The Fourier law (7) allows deriving the thermoelastic heat flux vector

$$^aq_i(\mathbf{x},t) = -\tilde{\kappa} \cdot \left[{_o}\lambda \cdot \eta \cdot \tilde{\lambda} \cdot \frac{\partial}{\partial t} \beta(D \times (0,t)) + \right.$$
$$\left. + W(D \times (0,t)) \right] * F_{,i}(\mathbf{x},t) +$$
$$- \tilde{\kappa} \cdot {_o}\lambda \cdot \left[g(D) + \eta \cdot \tilde{\lambda} \cdot \beta(D,0) \right] * F_{,i}(\mathbf{x},t) +$$
$$- {_o}\lambda \cdot \sum_{l=1}^{p} \sum_{m=1}^{n} T_l^m \cdot F_{,i}(\mathbf{x} - {_m}\mathbf{y}, t - {}^ls). \quad (49)$$

The approximate thermoelastic flow through the surface with normal v_i will be derived

$$^vP_i(\mathbf{x},t) = 2 \cdot \mu \cdot \left\{ \tilde{\lambda} \cdot \left[\gamma \cdot {}^aT(D,t) - \beta(D,t) \right] * \frac{\partial}{\partial v} E_{,i}(\mathbf{x}) + \right.$$
$$\left. + \sum_{m=1}^{n} u_j^m(t) \cdot \frac{\partial}{\partial v} E_{,ij}(\mathbf{x} - {_m}\mathbf{y}) \right\} +$$
$$- \chi_l(D,t) * \left[\varepsilon_{ikl} \cdot \frac{\partial}{\partial v} E_{,k}(\mathbf{x}) + \varepsilon_{jkl} \cdot v_j \cdot E_{,ki}(\mathbf{x}) \right] +$$
$$- v_i \cdot \tilde{\lambda} \cdot \left[2 \cdot \mu \cdot \gamma \cdot {}^aT(\mathbf{x},t) + \lambda \cdot \beta(\mathbf{x},t) \right], \quad (50)$$

$$^vq(\mathbf{x},t) = -\tilde{\kappa} \cdot \left[{_o}\lambda \cdot \eta \cdot \tilde{\lambda} \cdot \frac{\partial}{\partial t} \right] \beta(D \times (0,t)) +$$
$$+ W(D \times (0,t)) \right] * \frac{\partial}{\partial v} F(\mathbf{x},t) +$$
$$- \tilde{\kappa} \cdot {_o}\lambda \cdot [g(D) + \eta \cdot \tilde{\lambda} \cdot \beta(D,0)] * \frac{\partial}{\partial v} F(\mathbf{x},t) +$$
$$- {_o}\lambda \cdot \sum_{l=1}^{p} \sum_{m=1}^{n} T_l^m \cdot \frac{\partial}{\partial v} F(\mathbf{x} - {_m}\mathbf{y}, t - {}^ls). \quad (51)$$

It gives a full description of thermomechanical phenomena with the use of the displacement-temperature pair.

V. CONCLUSIONS

The paper presents the influence of fictitious load on the solution of the quasi-static problem of coupled thermoelasticity. A coupled thermoelastic initial condition, as a logical consequence of the coupled governing equations, significantly simplified the considerations. The Helmholtz representations of the displacement and body force vectors enabled the solution to the initial-space value problem in a fictitious unbounded medium. Convolution products of fictitious load and the fundamental solutions of differential operators of both elliptic and parabolic type have been obtained as a result. Their components, called fictitious components, have been used for verification of the boundary condition in the finite time. The solution is continuously differentiable, thus enabling the determination of stress and heat flux distributions.

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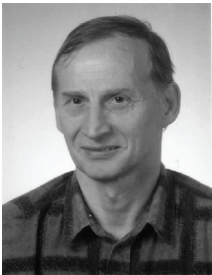
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