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Application of the Routh Method in Computer Simulation of Selected Problems in Collision Theory

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Abstract: In this paper an inelastic collision of two rigid bodies is considered. Friction forces between contacting surfaces of both objects are taken into considerations. The Routh method is applied to obtain the solution and to analyse the collision process. The kinematic state of both bodies after the collision is calculated, and computer simulations of collision are performed.

Key words: Routh method, computer simulation, collision theory

I. INTRODUCTION

We shall consider an inelastic collision between two rigid and finite-sized bodies B_{I} and B_{II} whose surfaces are rough. The effect of the collision is the stepwise but finite change of velocities of the bodies. Time of such a collision is very short, for example the collision between two billard balls lasts for about 10^{-4}-10^{-2} s. The changes of velocities have finite values, therefore the displacements during the collision of all points of the bodies may be neglected. We assume that throughout the collision the bodies are in contact only at one point O. At that point we set up the origin of the rectangular coordinate system $Oxyz$. The reference frame is shown in Fig. 1.

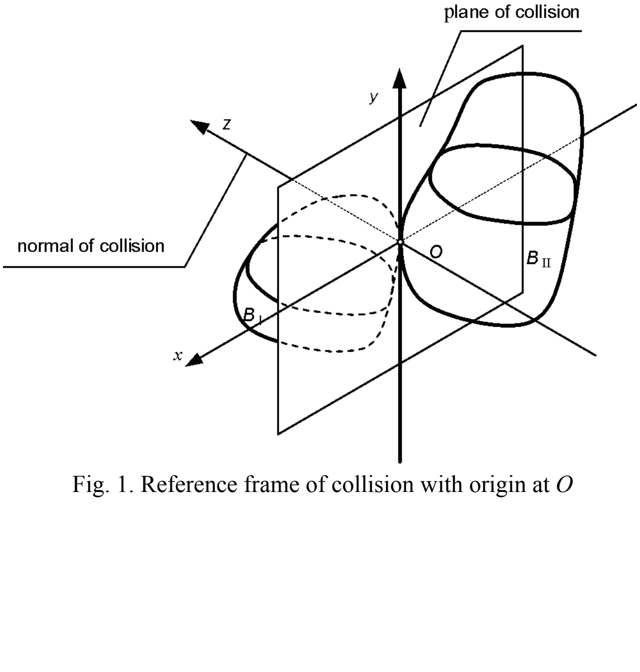

Fig. 1. Reference frame of collision with origin at O

We shall sometimes differentiate point O_{I} which lies on body B_{I} from point O_{II} which lies on body B_{II}.

II. MODELLING OF REACTION FORCES

At the impact the bodies mechanically interact on each other. As is shown in Fig. 2, total reaction force **R** exerted by body B_{II} on body B_{I} is resolved into three orthogonal components: normal force **N** and two components $\mathbf{T_x}$, $\mathbf{T_y}$ of

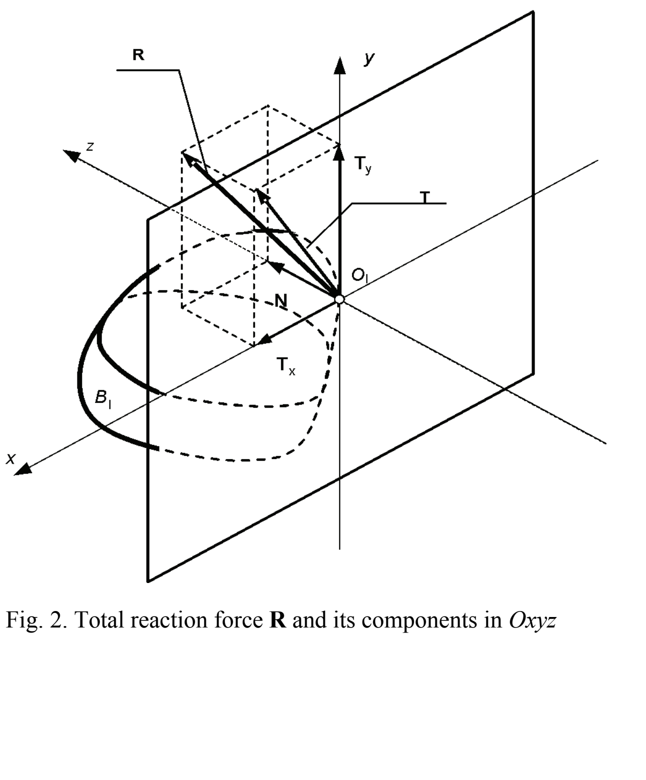

Fig. 2. Total reaction force **R** and its components in $Oxyz$

friction force **T**. If the surfaces of the colliding objects are rough, then the influence of friction force should not be ignored. To describe friction force we assume Coulomb's law.

Since Coulomb's law has the form of an inequality for surfaces at the rest relative to each other, and the form of an equation for surfaces in relative motion, we must distinguish both cases. To this end, we decompose velocity $\mathbf{v}_{O_\mathrm{I}}$ into two orthogonal components: $(\mathbf{v}_{O_\mathrm{I}})_z$ in the direction of axis z and $(\mathbf{v}_{O_\mathrm{I}})_{xy}$ in the plane of collison. Similiarly, velocity $\mathbf{v}_{O_\mathrm{II}}$ is resolved into two components $(\mathbf{v}_{O_{II}})_z$ and $(\mathbf{v}_{O_\mathrm{II}})_{xy}$. Now we can introduce the sliding velocity

$$\mathbf{s} = (\mathbf{v}_{O_\mathrm{I}})_{xy} - (\mathbf{v}_{O_\mathrm{II}})_{xy} \tag{1}$$

and the closing velocity

$$\mathbf{c} = (\mathbf{v}_{O_\mathrm{I}})_z - (\mathbf{v}_{O_\mathrm{II}})_z \,. \tag{2}$$

The magnitude and direction of friction force **T** depend on sliding velocity **s** in the form of the following equations:

$$\begin{array}{llll} T = \mu N & \text{and} \quad \mathbf{T} \parallel \mathbf{s}, & \mathbf{T}\cdot\mathbf{s} < 0 & \text{for} \quad \mathbf{s} \neq \mathbf{0}, \\ T \le \mu N & \text{and} \quad \mathbf{v}^\mathrm{I}_{Os} = \mathbf{v}^{II}_{Os} & & \text{for} \quad \mathbf{s} = \mathbf{0}, \end{array} \tag{3}$$

where μ is the friction coefficient, T and N are the magnitudes of the friction and the normal forces, respectively.

The colliding bodies compress each other by normal force **N**. The compressive strains appear mainly in the region of point O. The collision process can be split into two phases:

– the first phase form t_0 to t_1. In this phase the magnitude of normal force **N** increases. At moment t_1 the magnitude of force **N** achieves the largest value and closing velocity $\mathbf{c} = \mathbf{0}$,

– the second phase form t_1 to t_2 in which the magnitude of normal force **N** decreases to zero.

During the impact, forces **N** and **T** by which the colliding bodies act on each other change extremely qiuckly. Their magnitudes achieve much greater values than the magnitudes of other forces that also act on the bodies at the impact. We say the forces between colliding bodies are the instantaneous forces. For this reason we introduce into our further consideration the following quantities:

– linear impulse $\boldsymbol{\Pi}$ of normal component **N**

$$\boldsymbol{\Pi} = \int_{t_0}^{t} \mathbf{N}(t)\, dt \,, \tag{4}$$

– linear impulse $\boldsymbol{\Upsilon}$ of total friction force **T**

$$\boldsymbol{\Upsilon} = \int_{t_0}^{t} \mathbf{T}(t)\, dt \,, \tag{5}$$

where t_0 – the instant at which the collision begins, t_2 – the instant at which the collision ends, and t – any instant from the interval $\langle t_0, t_2 \rangle$.

During the impact only the impulces of the instantaneous forces have the finite values. The impulses of the other forces exerted on the colliding objects may by neglected because of very short collision time.

III. KINEMATIC QUANTITIES

The motion of each body before the collision is known. In particular, one of the bodies can be at rest. We assume that the initial value of sliding velocity $\mathbf{s}_0 \neq \mathbf{0}$. At any instant t from the interval $\langle t_0, t_2 \rangle$ the kinematic state of the colliding bodies is determined by:

– linear velocities $\mathbf{v}^\mathrm{I}$, $\mathbf{v}^\mathrm{II}$ of their mass centres C_I and C_II, respectively,

– angular velocities $\boldsymbol{\omega}^\mathrm{I}$ and $\boldsymbol{\omega}^\mathrm{II}$.

We have to find the kinematic state of the bodies at instant t_2:

$$\mathbf{v}^\mathrm{I}{}_2,\ \boldsymbol{\omega}^\mathrm{I}{}_2 \quad \text{and} \quad \mathbf{v}^\mathrm{II}{}_2,\ \boldsymbol{\omega}^\mathrm{II}{}_2.$$

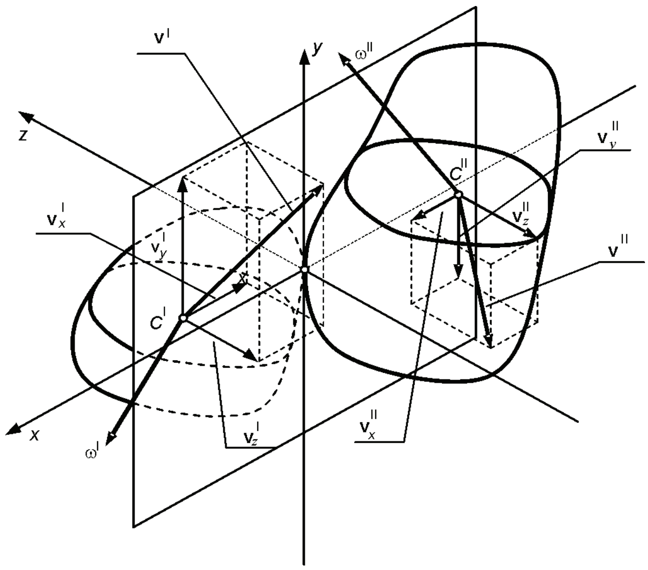

Fig. 3. Kinematic state of colliding bodies at any instant t

Figure 3 shows vectors $\mathbf{v}^\mathrm{I}$, $\boldsymbol{\omega}^\mathrm{I}$ and $\mathbf{v}^\mathrm{II}$, $\boldsymbol{\omega}^\mathrm{II}$ which represent the kinematic state of both bodies at any instant t.

IV. FUNDAMENTAL EQUATIONS

The above problem of collision is described by Euler's laws of motion in the integral form. The law of linear momentum, that is the first Euler's law, has the following form:
– for solid B_{I}

$$
\begin{aligned}
m^{\mathrm{I}}(v^{\mathrm{I}}_x - v^{\mathrm{I}}_{0x}) &= \mathcal{T}_x,\\
m^{\mathrm{I}}(v^{\mathrm{I}}_y - v^{\mathrm{I}}_{0y}) &= \mathcal{T}_y,\\
m^{\mathrm{I}}(v^{\mathrm{I}}_z - v^{\mathrm{I}}_{0z}) &= \mathcal{N},
\end{aligned}
\tag{6}
$$

– for the solid B_{II}

$$
\begin{aligned}
m^{\mathrm{II}}(v^{\mathrm{II}}_x - v^{\mathrm{II}}_{0x}) &= -\mathcal{T}_x,\\
m^{\mathrm{II}}(v^{\mathrm{II}}_y - v^{\mathrm{II}}_{0y}) &= -\mathcal{T}_y,\\
m^{\mathrm{II}}(v^{\mathrm{II}}_z - v^{\mathrm{II}}_{0z}) &= -\mathcal{N},
\end{aligned}
\tag{7}
$$

where
$\mathcal{T}_x$, $\mathcal{T}_y$, $\mathcal{N}$ – the rectangular components of the linear impulses of reaction forces **T** and **N**,
m^{I}, m^{II} – mass of the bodies,
v^{I}_x, v^{I}_y, v^{I}_z – the coordinates of velocity $\mathbf{v}^{\mathrm{I}}$ of mass centre C_{I} at any instant t,
v^{I}_{0x}, v^{I}_{0y}, v^{I}_{0z} – the coordinates of velocity $\mathbf{v}^{\mathrm{I}}_0$ of mass centre C_{I} at instant t_0,
v^{II}_x, v^{II}_y, v^{II}_z – the coordinates of velocity $\mathbf{v}^{\mathrm{II}}$ of mass centre C_{II} at any instant t,
v^{II}_{0x}, v^{II}_{0y}, v^{II}_{0z} – the coordinates of velocity $\mathbf{v}^{\mathrm{II}}_0$ of mass centre C_{II} at instant t_0.

The law of angular momentum (the second Euler's law):
– for solid B_{I} with respect to its mass centre C_{I}

$$
\begin{aligned}
&I^{\mathrm{I}}_{xx}(\omega^{\mathrm{I}}_x - \omega^{\mathrm{I}}_{0x}) - I^{\mathrm{I}}_{xy}(\omega^{\mathrm{I}}_y - \omega^{\mathrm{I}}_{0y}) +\\
&-I^{\mathrm{I}}_{xz}(\omega^{\mathrm{I}}_z - \omega^{\mathrm{I}}_{0z}) = z^{\mathrm{I}}_C \cdot \mathcal{T}_y - y^{\mathrm{I}}_C \cdot \mathcal{N},\\
&-I^{\mathrm{I}}_{yx}(\omega^{\mathrm{I}}_x - \omega^{\mathrm{I}}_{0x}) + I^{\mathrm{I}}_{yy}(\omega^{\mathrm{I}}_y - \omega^{\mathrm{I}}_{0y}) +\\
&-I^{\mathrm{I}}_{yz}(\omega^{\mathrm{I}}_z - \omega^{\mathrm{I}}_{0z}) = -z^{\mathrm{I}}_C \cdot \mathcal{T}_x + x^{\mathrm{I}}_C \cdot \mathcal{N},\\
&-I^{\mathrm{I}}_{zx}(\omega^{\mathrm{I}}_x - \omega^{\mathrm{I}}_{0x}) - I^{\mathrm{I}}_{zy}(\omega^{\mathrm{I}}_y - \omega^{\mathrm{I}}_{0y}) +\\
&+I^{\mathrm{I}}_{zz}(\omega^{\mathrm{I}}_z - \omega^{\mathrm{I}}_{0z}) = -x^{I}_C \cdot \mathcal{T}_y + y^{\mathrm{I}}_C \cdot \mathcal{T}_x
\end{aligned}
\tag{8}
$$

– for solid B_{II} with respect to its mass centre C_{II}

$$
\begin{aligned}
&I^{\mathrm{II}}_{xx}(\omega^{\mathrm{II}}_x - \omega^{\mathrm{II}}_{0x}) - I^{\mathrm{II}}_{xy}(\omega^{\mathrm{II}}_y - \omega^{\mathrm{II}}_{0y}) +\\
&-I^{\mathrm{II}}_{xz}(\omega^{\mathrm{II}}_z - \omega^{\mathrm{II}}_{0z}) = -z^{\mathrm{II}}_C \cdot \mathcal{T}_y + y^{\mathrm{II}}_C \cdot \mathcal{N},\\
&-I^{\mathrm{II}}_{yx}(\omega^{\mathrm{II}}_x - \omega^{\mathrm{II}}_{0x}) + I^{\mathrm{II}}_{yy}(\omega^{\mathrm{II}}_y - \omega^{\mathrm{II}}_{0y}) +\\
&-I^{\mathrm{II}}_{yz}(\omega^{\mathrm{II}}_z - \omega^{\mathrm{II}}_{0z}) = z^{\mathrm{II}}_C \cdot \mathcal{T}_x - x^{\mathrm{II}}_C \cdot \mathcal{N},\\
&-I^{\mathrm{II}}_{zx}(\omega^{\mathrm{II}}_x - \omega^{\mathrm{I}}_{0x}) - I^{\mathrm{II}}_{zy}(\omega^{\mathrm{II}}_y - \omega^{\mathrm{I}}_{0y}) +\\
&+I^{\mathrm{II}}_{zz}(\omega^{\mathrm{II}}_z - \omega^{\mathrm{II}}_{0z}) = x^{\mathrm{II}}_C \cdot \mathcal{T}_y - y^{\mathrm{II}}_C \cdot \mathcal{T}_x,
\end{aligned}
\tag{9}
$$

where
ω^{I}_x, ω^{I}_y, ω^{I}_z and ω^{I}_{0x}, ω^{I}_{0y}, ω^{I}_{0z} – the coordinates of the angular velocity of body B_{I} at any instant t and at instant t_0, respectively,
ω^{II}_x, ω^{II}_y, $\boldsymbol{\omega}^{\mathrm{II}}_z$ and $\omega^{\mathrm{II}}_{0x}$, $\omega^{\mathrm{II}}_{0y}$, $\omega^{\mathrm{II}}_{0z}$ – the coordinates of the angular velocity of body B_{II} at any instant t and at instant t_0, respectively,
I^{I}_{xx}, I^{I}_{yy}, I^{I}_{zz}, I^{I}_{xy}, I^{I}_{xz}, I^{I}_{yz} – the moments and the products of inertia of body B_{I} with respect to the central axes which are parallel to the axes of the *Oxyz* system,
I^{II}_{xx}, I^{II}_{yy}, I^{II}_{zz}, I^{II}_{xy}, I^{II}_{xz}, I^{II}_{yz} – the moments and products of inertia of body B_{II} with respect to the central axes which are parallel to the axes of the *Oxyz* system,

Moreover, in accordance with Poisson's hypothesis we have an additional equation of the form

$$
\mathcal{N}_{12} = k\mathcal{N}_{01},
\tag{10}
$$

where k is the coefficient of restitution and $\mathcal{N}_{12}$, $\mathcal{N}_{01}$ denote linear impulses of normal force **N** at the end of the second and the first phase of the collision, respectively.

In the equations (7)-(11) the quantities

$$
v^{\mathrm{I}}_x,\ v^{\mathrm{I}}_y,\ v^{\mathrm{I}}_z,\ v^{\mathrm{II}}_x,\ v^{\mathrm{II}}_y,\ v^{\mathrm{II}}_z,\ \omega^{\mathrm{I}}_x,\ \omega^{\mathrm{I}}_y,\ \omega^{\mathrm{I}}_z,\ \omega^{\mathrm{II}}_x,\ \omega^{\mathrm{II}}_y,\ \omega^{\mathrm{II}}_z
$$

$$
\text{and } \mathcal{N},\ \mathcal{T}_{\mathrm{x}},\ \mathcal{T}_y
$$

are unknown.

V. THE ROUTH METHOD

All the above equations which govern the collision phenomenon, except the friction law, are of global nature and concern the whole process or its phases. The friction law is formulated for the instantaneous values of the reaction force components and it cannot be generalized for the linear impulses of these forces in a simple way. The values of impulses $\mathcal{T}_x$ and $\mathcal{T}_y$ depend on the existence of a slip between the contacting solids. Furthermore, during the collision the situation may change. So the dual form of the friction laws complicates the solution of the collision problem.

When friction forces play an important role during the collision, we can apply the Routh method [3]. This is an exact method. From the mathematical viewpoint the essence of this method is to eliminate from the governing equations such quantities which deal with neither the sliding conditions nor the conditions of the end of each collision phases.

Using the commonly known velocity relationship for two points of the same rigid body we can write

$$
\begin{aligned}
s\cos\theta - s_0\cos\theta_0 &= v^{\mathrm{I}}_x - v^{\mathrm{I}}_{0x} - z^{\mathrm{I}}(\omega^{\mathrm{I}}_y - \omega^{\mathrm{I}}_{oy}) + y^{\mathrm{I}}(\omega^{\mathrm{I}}_z - \omega^{\mathrm{I}}_{oz}) + \\
&\quad - v^{\mathrm{II}}_x + v^{\mathrm{II}}_{0x} + z^{\mathrm{I}}(\omega^{\mathrm{II}}_y - \omega^{\mathrm{II}}_{oy}) - y^{\mathrm{II}}(\omega^{\mathrm{II}}_z - \omega^{\mathrm{II}}_{oz}), \\
s\sin\theta - s_0\sin\theta_0 &= v^{\mathrm{I}}_y - v^{\mathrm{I}}_{0y} - x^{\mathrm{I}}(\omega^{\mathrm{I}}_z - \omega^{\mathrm{I}}_{oz}) + z^{\mathrm{I}}(\omega^{\mathrm{I}}_x - \omega^{\mathrm{I}}_{ox}) + \\
&\quad - v^{\mathrm{I}}_y + v^{\mathrm{I}}_{0y} + x^{\mathrm{II}}(\omega^{\mathrm{II}}_z - \omega^{\mathrm{II}}_{oz}) - z^{\mathrm{II}}(\omega^{\mathrm{II}}_x - \omega^{\mathrm{II}}_{ox}), \\
c - c_0 &= v^{\mathrm{I}}_z - v^{\mathrm{I}}_{0z} - y^{\mathrm{I}}(\omega^{\mathrm{I}}_x - \omega^{\mathrm{I}}_{ox}) + x^{\mathrm{I}}(\omega^{\mathrm{I}}_y - \omega^{\mathrm{I}}_{oy}) + \\
&\quad - v^{\mathrm{II}}_z + v^{\mathrm{II}}_{0z} + y^{\mathrm{II}}\cdot(\omega^{\mathrm{II}}_x - \omega^{\mathrm{II}}_{ox}) - x^{\mathrm{II}}(\omega^{\mathrm{II}}_y - \omega^{\mathrm{II}}_{oy}),
\end{aligned}
\tag{11}
$$

where θ is the angle between vector **s** and axis $0x$, **s** and $\mathbf{s}_0$ denote magnitudes of sliding velocity **s** at instants t and t_0, respectively. The equations (11) associate vectors **s** and **c** with the vectors $\mathbf{v}^{\mathrm{I}}$, $\boldsymbol{\omega}^{\mathrm{I}}$, $\mathbf{v}^{\mathrm{II}}$ and $\boldsymbol{\omega}^{\mathrm{II}}$. Solving the system (6)-(9) with respect to unknown kinematic quantities:

$$
(v^{\mathrm{I}}_x - v^{\mathrm{I}}_{0x}),\quad (v^{\mathrm{I}}_y - \mathrm{v}^{\mathrm{I}}_{0y}),\quad (v^{\mathrm{I}}_z - v^{\mathrm{I}}_{0z}),\ldots.
$$

$$
(\omega^{\mathrm{II}}_x - \omega^{\mathrm{II}}_{0x}),\quad (\omega^{\mathrm{II}}_y - \omega^{\mathrm{II}}_{0y}),\quad (\omega^{\mathrm{II}}_z - \omega^{\mathrm{II}}_{0z})
$$

and inserting the solutions into the (11) we obtain the following relationships between the sliding and closing velocities and the impulces of the reaction forces

$$
\begin{aligned}
s\cos\theta - s_0\cos\theta_0 &= \alpha\mathbf{T}_x + \gamma\mathbf{T}_y + \varepsilon\mathbf{\Pi}, \\
s\sin\theta - s_0\sin\theta_0 &= \gamma\mathbf{T}_x + \beta\mathbf{T}_y + \delta\mathbf{\Pi}, \\
c - c_0 &= \varepsilon\mathbf{T}_x + \delta\mathbf{T}_y + \chi\mathbf{\Pi},
\end{aligned}
\tag{12}
$$

where α, β, χ, δ, ε, γ are constants which depend on mass, moments and products of inertia of both bodies. They are given by the following formulas:

$$
\alpha = \frac{1}{m^{\mathrm{I}}} + \frac{1}{m^{\mathrm{II}}} + \frac{W^{\mathrm{I}}_{zz}(y^{\mathrm{I}}_C)^2 - 2W^{\mathrm{I}}_{yz}y^{\mathrm{I}}_C z^{\mathrm{I}}_C + W^{\mathrm{I}}_{yy}(z^{\mathrm{I}}_C)^2}{W^{\mathrm{I}}} + \frac{W^{\mathrm{I}}_{zz}(y^{\mathrm{II}}_C)^2 - 2W^{\mathrm{II}}_{yz}y^{\mathrm{II}}_C z^{\mathrm{II}}_C + W^{\mathrm{I}}_{yy}(z^{\mathrm{II}}_C)^2}{W^{\mathrm{II}}},
$$

$$
\beta = \frac{1}{M_{\mathrm{I}}} + \frac{1}{M_{\mathrm{II}}} + \frac{W^{\mathrm{I}}_{zz}(x^{\mathrm{I}}_C)^2 - 2W^{\mathrm{I}}_{xz}x^{\mathrm{I}}_C z^{\mathrm{I}}_C + W^{\mathrm{I}}_{xx}(z^{\mathrm{I}}_C)^2}{W^{\mathrm{I}}} + \frac{W^{\mathrm{II}}_{zz}(x^{\mathrm{II}}_C)^2 - 2W^{\mathrm{II}}_{xz}x^{\mathrm{II}}_C z^{\mathrm{II}}_C + W^{\mathrm{II}}_{xx}(z^{\mathrm{I}}_C)^2(z^{\mathrm{II}}_C)^2}{W^{\mathrm{II}}},
$$

$$
\chi = \frac{1}{M_{\mathrm{I}}} + \frac{1}{M_{\mathrm{II}}} + \frac{W^{\mathrm{I}}_{yy}(x^{\mathrm{I}}_C)^2 - 2W^{\mathrm{I}}_{xy}x^{\mathrm{I}}_C y^{\mathrm{I}}_C + W^{\mathrm{I}}_{xx}(y^{\mathrm{I}}_C)^2}{W^{\mathrm{I}}} + \frac{W^{\mathrm{II}}_{yy}(x^{\mathrm{II}}_C)^2 - 2W^{\mathrm{II}}_{xy}x^{\mathrm{II}}_C y^{\mathrm{II}}_C + W^{\mathrm{II}}_{xx}(y^{\mathrm{II}}_C)^2}{W^{\mathrm{II}}},
$$

$$
\delta = \frac{-W^{\mathrm{I}}_{yz}(x^{\mathrm{I}}_C)^2 + W^{\mathrm{I}}_{xz}x^{\mathrm{I}}_C y^{\mathrm{I}}_C + W^{\mathrm{I}}_{xy}x^{\mathrm{I}}_C z^{\mathrm{I}}_C - W^{\mathrm{I}}_{xx}y^{\mathrm{I}}_C z^{\mathrm{I}}_C}{W^{\mathrm{I}}} + \frac{-W^{\mathrm{II}}_{yz}(x^{\mathrm{II}}_C)^2 + W^{\mathrm{II}}_{xz}x^{\mathrm{II}}_C y^{\mathrm{II}}_C + W^{\mathrm{II}}_{xy}x^{\mathrm{II}}_C z^{\mathrm{II}}_C - W^{\mathrm{II}}_{xx}y^{\mathrm{II}}_C z^{\mathrm{II}}_C}{W^{\mathrm{II}}},
$$

$$
\varepsilon = \frac{-W^{\mathrm{I}}_{xz}(y^{\mathrm{I}}_C)^2 + W^{\mathrm{I}}_{yz}x^{\mathrm{I}}_C y^{\mathrm{I}}_C + W^{\mathrm{I}}_{xy}y^{\mathrm{I}}_C z^{\mathrm{I}}_C - W^{\mathrm{I}}_{yy}x^{\mathrm{I}}_C z^{\mathrm{I}}_C}{W^{\mathrm{I}}} + \frac{-W^{\mathrm{II}}_{xz}(y^{\mathrm{II}}_C)^2 + W^{\mathrm{II}}_{yz}x^{\mathrm{II}}_C y^{\mathrm{II}}_C + W^{\mathrm{II}}_{xy}y^{\mathrm{II}}_C z^{\mathrm{II}}_C - W^{\mathrm{II}}_{yy}x^{\mathrm{II}}_C z^{\mathrm{II}}_C}{W^{\mathrm{II}}},
$$

$$
\gamma = \frac{-W^{\mathrm{I}}_{xy}(z^{\mathrm{I}}_C)^2 + W^{\mathrm{I}}_{yz}x^{\mathrm{I}}_C z^{\mathrm{I}}_C + W^{\mathrm{I}}_{xz}y^{\mathrm{I}}_C z^{\mathrm{I}}_C - W^{\mathrm{I}}_{zz}x^{\mathrm{I}}_C y^{\mathrm{I}}_C}{W^{\mathrm{I}}} + \frac{-W^{\mathrm{II}}_{xy}(z^{\mathrm{II}}_C)^2 + W^{\mathrm{II}}_{yz}x^{\mathrm{II}}_C z^{\mathrm{II}}_C + W^{\mathrm{II}}_{xz}y^{\mathrm{II}}_C z^{\mathrm{II}}_C - W^{\mathrm{II}}_{zz}x^{\mathrm{II}}_C y^{\mathrm{II}}_C}{W^{\mathrm{II}}},
$$

where W^{I} and W^{II} denote the determinants of the tensors of inertia, $W^{\mathrm{I}}_{xx}, W^{\mathrm{I}}_{yy}, W^{\mathrm{I}}_{zz}, W^{\mathrm{I}}_{xy}, \ldots, W^{\mathrm{II}}_{zy}$ are the algebraic complements of elements of these determinants.

The Routh method has a simple geometrical interpretation in a space in which the values of impulses $\mathbf{T}_x$, $\mathbf{T}_y$ and $\mathbf{\Pi}$ are the coordinates of the points. According to the nature of normal force **N**, its impulse satisfies inequality $\mathbf{\Pi} \geq 0$. Therefore we should talk about the semi-space impulse. Any point of this semi-space we denote by $\Gamma = (\mathbf{\Pi}, \mathbf{T}_x, \mathbf{T}_y)$. In this semi-space some geometrical objects are defined:

– the non-sliding line. We can obtain its equation from (12) assuming that $\mathbf{s} = 0$

$$
\begin{aligned}
s_0\cos\theta_0 + \alpha\mathbf{T}_x + \gamma\mathbf{T}_y + \varepsilon\mathbf{\Pi} &= 0, \\
s_0\sin\theta_0 + \gamma\mathbf{T}_x + \beta\mathbf{T}_y + \delta\mathbf{\Pi} &= 0.
\end{aligned}
\tag{13}
$$

– the plane of the greatest compression. We can obtain its equation from (12) assuming that $\mathbf{c} = 0$

$$
\mathrm{c}_0 + \varepsilon\mathbf{T}_x + \delta\mathbf{T}_y + \chi\mathbf{\Pi} = 0. \tag{14}
$$

– the plane of the end of collision. Its equation is

$$
\mathbf{\Pi} = (1+k)\mathbf{\Pi}_{01}. \tag{15}
$$

To find the trajectory representing the relationship between impulses $\mathbf{T}_x$, $\mathbf{T}_y$ and $\mathbf{\Pi}$, we must firstly write the friction law in the differential form

$$d\mathbf{T}_x = -\mu \cos\theta \, d\mathbf{\Pi}, \quad d\mathbf{T}_y = -\mu \sin\theta \, d\mathbf{\Pi} \quad \text{for} \quad \mathbf{s} \neq 0. \tag{16}$$

and then solve a few initial-value problems. The initial problems and their solutions are described in detail in [3]. Finally, we get three functions of the same variable θ

$$\mathbf{\Pi} = \Psi(\theta), \quad \mathbf{T}_x = Z_x(\theta), \quad \mathbf{T}_y = Z_y(\theta). \tag{17}$$

If there is a slip between the surfaces of the bodies, then the functions (17) define the trajectory in the semi-space that represents the relationship between the values of impulses $\mathbf{T}_x$, $\mathbf{T}_y$ and $\mathbf{\Pi}$.

VI. INTERPRETATION OF TWO-DIMENSIONAL COLLISION ON IMPULSE SEMI-PLANE

In the remainder of the paper we shall consider the collision between two bodies which before and also after the impact are in the plane motion parallel to the *Oxz* plane. Therefore we have:

$$\omega^{\mathrm{I}}_x = 0, \quad \omega^{\mathrm{I}}_z = 0, \quad \omega^{\mathrm{II}}_x = 0, \quad \omega^{\mathrm{II}}_z = 0, \quad v^{\mathrm{I}}_y = 0, \quad v^{\mathrm{II}}_y = 0. \tag{18}$$

The constrains are ideal and have no influence on the motion. We assume that the point of contact O and the mass centres C_{I} and C_{II} move in the plane *Oxz*, that is

$$y^{\mathrm{I}}_C = 0, \qquad y^{\mathrm{II}}_C = 0. \tag{19}$$

Axis *Oy* is the principal axis of inertia. Hence the following products of inertia are given as follows:

$$I^{\mathrm{I}}_{xy} = 0, \quad I^{\mathrm{I}}_{yz} = 0, \quad I^{\mathrm{II}}_{xy} = 0, \quad I^{\mathrm{II}}_{yz} = 0\,. \tag{20}$$

In this case, sliding velocity $\mathbf{s}$ lies on axis *Ox* and angle θ can be equal to 0 or π. Therefore component $\mathbf{T}_y$ of the total reaction force $\mathbf{R}$ is equal to zero. Thus the impulse semi-space reduces to the semi-plane and its any point $\Gamma = (\mathbf{\Pi}, \mathbf{T}_x)$. The geometrical objects characteristic for Routh's method are described by the following equations:

– the non-sliding line

$$s_0 \cos\theta_0 + \alpha \mathbf{T}_x + \varepsilon \mathbf{\Pi} = 0, \tag{21}$$

– the plane of the greatest compression which now reduces to the straight line

$$c_0 + \varepsilon \mathbf{T}_x + \chi \mathbf{\Pi} = 0, \tag{22}$$

– the line of the end of collision

$$\mathbf{\Pi} = (1+k)\mathbf{\Pi}_{01} \tag{23}$$

– the trajectory that represents the relationship between impulses $\mathbf{T}_x$ and $\mathbf{\Pi}$ when there is a slip between the surfaces of colliding bodies

$$\mathbf{T}_x = -\mu \mathbf{\Pi} \cos\theta_0. \tag{24}$$

Constants α, χ, δ, ε, γ defined in (12) can be simplified to the form:

$$\alpha = \frac{1}{M_{\mathrm{I}}} + \frac{1}{M_{\mathrm{II}}} + \frac{(z^{\mathrm{I}}_C)^2}{I^{\mathrm{I}}_{yy}} + \frac{(z^{\mathrm{II}}_C)^2}{I^{\mathrm{II}}_{yy}},$$

$$\chi = \frac{1}{M_{\mathrm{I}}} + \frac{1}{M_{\mathrm{II}}} + \frac{(x^{\mathrm{I}}_C)^2}{I^{\mathrm{I}}_{yy}} + \frac{(x^{\mathrm{II}}_C)^2}{I^{\mathrm{II}}_{yy}},$$

$$\varepsilon = -\frac{x^{\mathrm{I}}_C \cdot z^{\mathrm{I}}_C}{I^{\mathrm{I}}_{yy}} - \frac{x^{\mathrm{II}}_C \cdot z^{\mathrm{II}}_C}{I^{\mathrm{II}}_{yy}},$$

$$\delta = 0, \quad \gamma = 0.$$

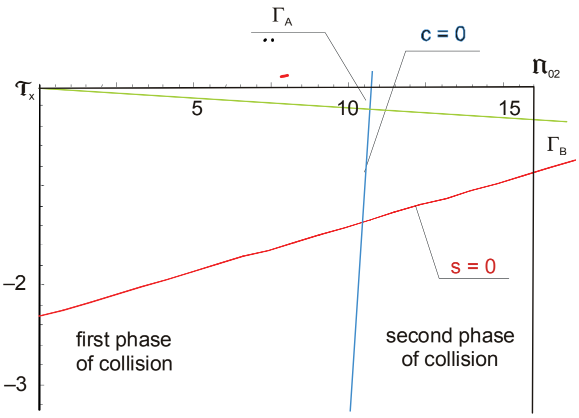

Fig. 4. Interpretation of collision on the impulse semi-plane

In Fig. 4 we can observe the geometrical objects for a collision process on the impulse semi-plane. The trajectory given by (24) is drawn in green. The trajectory representing the relationship between impulses $\mathbf{T}_x$ and $\mathbf{\Pi}$ starts at the origin because the initial value of sliding velocity $\mathbf{s}_0 \neq \mathbf{0}$. The trajectory does not cross the non-sliding line $\mathbf{s} = \mathbf{0}$, so throughout the collision there is a slip between the surfaces of both bodies. At point $\Gamma_A = (\mathbf{\Pi}_{01}, \mathbf{T}_{x\,01})$ the trajectory crosses the line of the greatest compression $\mathbf{c} = \mathbf{0}$. Coordinate $\mathbf{\Pi}_{01}$ of this point is the value of impulse $\mathbf{\Pi}$ at the end of the first phase of the collision. Product $(1+k)\,\mathbf{\Pi}_{01}$ determines the line of the end of collision. This line is drawn in black. The trajectory crosses it at point $\Gamma_B = (\mathbf{\Pi}_{02}, \mathbf{T}_{x\,02})$. The coordinates of Γ_B are the values of the impulses respective of the normal force and the friction force at the instant t_2. Inserting values $\mathbf{\Pi}_{02}$ and $\mathbf{T}_{x\,02}$ into (6)-(9) we get the quantities: $\mathbf{v}^{\mathrm{I}}_2, \boldsymbol{\omega}^{\mathrm{I}}_2$ and $\mathbf{v}^{\mathrm{II}}_2, \boldsymbol{\omega}^{\mathrm{II}}_2$.

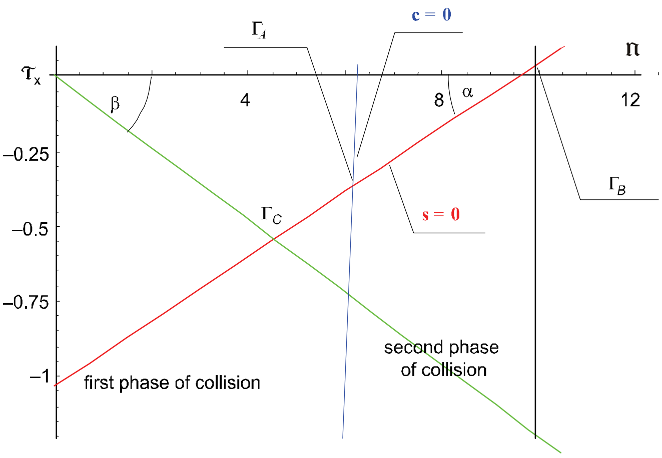

Fig. 5. Cross point Γ_C in first phase of collision

The trajectory shown in Fig. 5 crosses the non-sliding line at point Γ_C in the first phase of the collision. That means that sliding velocity $\mathbf{s}$ is equal to zero at point Γ_C. In other words, the tangent components of velocities of both bodies at point O became equal. The non-sliding line and the trajectory make angles α and β with horizontal axis $\boldsymbol{\mathfrak{n}}$. If angle α is smaller than angle β, the non-sliding contact will continue to hold to the end of the collision. Condition $\alpha < \beta$ is satisfied in the case shown in Fig. 5, so starting at point Γ_C the slip between the surfaces of both bodies vanishes. Point Γ_A at which the non-sliding line crosses the line of the greatest compression $\mathbf{c} = \mathbf{0}$ determines the end of the first phase of the collision. Coordinate $\boldsymbol{\mathfrak{n}}_{01}$ of this point multiplied by $(1+k)$ fixes the line of the end of the collision which is drawn in black in Fig. 5. The coordinates of $\Gamma_B = (\boldsymbol{\mathfrak{n}}_{02}, \boldsymbol{\mathfrak{T}}_{x\,02})$ are these values of impulses of reaction forces which we have to insert into (6)-(9) to find $\mathbf{v}^{\mathrm{I}}_2$, $\boldsymbol{\omega}^{\mathrm{I}}_2$ and $\mathbf{v}^{\mathrm{II}}_2$, $\boldsymbol{\omega}^{\mathrm{II}}_2$.

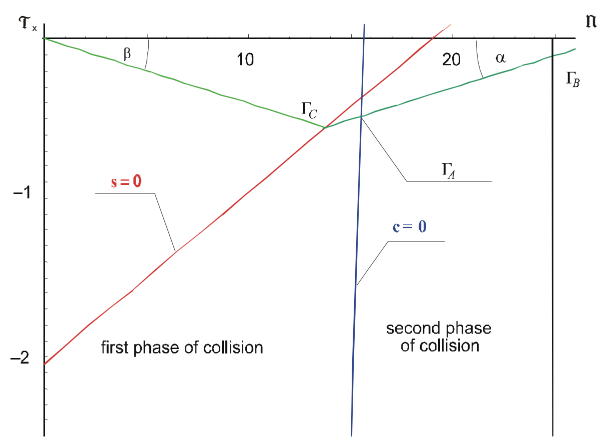

Fig. 6. Discontinuity in the slope of the trajectory at Γ_C

If inequality $\alpha < \beta$ is not satisfied at cross point Γ_C, as is shown in Fig. 6, then the slip between the surfaces of both bodies will hold to the end of the collision. That means the friction force is too small to stop the relative motion of both bodies in the plane of collision. But at point Γ_C the sliding velocity $\mathbf{s}$ and friction force $\mathbf{T}$ change their signs. The effect of these changes is a discontinuity in the slope of the trajectory at Γ_C. At $\Gamma_A = (\boldsymbol{\mathfrak{n}}_{01}, \boldsymbol{\mathfrak{T}}_{x\,01})$ the modified trajectory crosses the line of the greatest compression $\mathbf{c} = \mathbf{0}$. Coordinate $\boldsymbol{\mathfrak{n}}_{01}$ is the value of impulse $\boldsymbol{\mathfrak{n}}$ at the end of the first phase of collision. The line of the end of the collision is determined by product $(1+k)\,\boldsymbol{\mathfrak{n}}_{01}$. Coordinates $\boldsymbol{\mathfrak{n}}_{02}$, $\boldsymbol{\mathfrak{T}}_{x\,02}$ of Γ_B are those we should insert to (6)-(9) to find $\mathbf{v}^{\mathrm{I}}_2$, $\boldsymbol{\omega}^{\mathrm{I}}_2$ and $\mathbf{v}^{\mathrm{II}}_2$, $\boldsymbol{\omega}^{\mathrm{II}}_2$.

If the trajectory and the non-sliding line cross each other, but cross point Γ_C lies in the second phase area, then cross point $\Gamma_A = (\boldsymbol{\mathfrak{n}}_{01}, \boldsymbol{\mathfrak{T}}_{x\,01})$, at which the trajectory crosses the line of the greatest compression $\mathbf{c} = \mathbf{0}$, detrmines the end of the first phase. Coordinate $\boldsymbol{\mathfrak{n}}_{01}$ multiplied by $(1+k)$ fixes the line of end of the collision. Depending on the values of angles α and β at point Γ_C, the slip between the surfaces of both bodies can disappear or not.

VII. EXAMPLES

1. Collision between two discs

Let us consider the central collision between two discs. One disc signified here as B_{I} was at rest before the collision, and the second one denoted by B_{II} moved at constant speed. The following values were set to the calculations:

$$m_{\mathrm{I}} = 8\ \mathrm{kg}, \quad r_{\mathrm{I}} = 0.09\ \mathrm{m}, \quad m_{\mathrm{II}} = 2\ \mathrm{kg}, \quad r = 0.05\ \mathrm{m},$$
$$\mu = 0.25, \quad k = 0.6,$$
$$v^{\mathrm{II}}_{0x} = 0\ \mathrm{m/s}, \quad v^{\mathrm{II}}_{0z} = 1\ \mathrm{m/s}, \quad \omega^{\mathrm{II}}_{0y} = 12\ \mathrm{rad/s}.$$

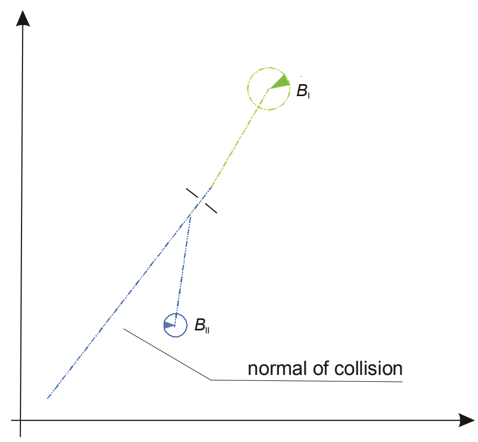

Fig. 7. State after the collision

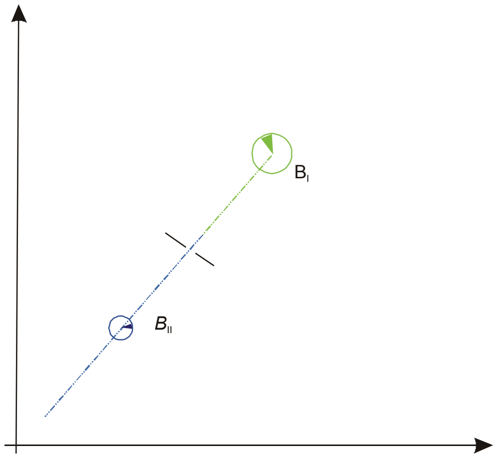

Fig. 8. State after the collision when $\mu = 0$

The effect of this collision is shown in Fig. 7. The angles of reflection for bodies B_I and B_{II} are equal to 7.12° and –150.26°, respectively. For comparison the effect of the collision of the same discs, when the friction coefficient $\mu = 0$, is shown in Fig. 8.

The basic quantities describing the kinematic state after the collision with $\mu = 0.25$ are presented in Table 1.

Table 1. Kinematic state after the collision

$\mathfrak{T}_{x\,02}$ [Ns]	$\mathfrak{n}_{02}$ [Ns]	ω^I_2 [rad/s]	ω^{II}_2 [rad/s]	Components of $\mathbf{v}^I_2$	Components of $\mathbf{v}^{II}_2$
				in frame of collision [m/s]	
0.320	2.560	–0.88889	5.600	[0.04, 0.32]	[–0.16, –0.28]

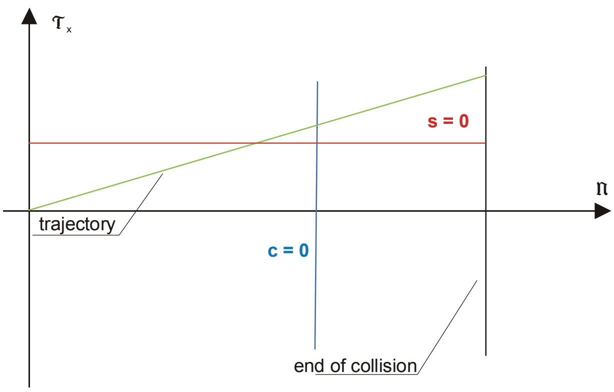

Fig. 9. Interpretation on impulse semi-plane

The analysis of the collision on impulse semi-plane is illustrated in Fig. 9. The slope of the non-sliding line is equal to zero because the constant $\varepsilon = 0$ for the central collision. Moreover, the line of the greatest compression is then perpendicular to the horizontal axis.

The next example concern the central collision between disc B_{II} of mass 2 kg and radius 0.5 m and the motionless disc B_I of mass 2 000 kg and radius 0.9 m. Before the impact, disc B_{II} moved at constant speed 8 m/s and had constant angular velocity equal to 10 rad/s. The influence of coefficient μ on the angle of reflection ϕ_{II} for the disc B_{II} for some values of the coefficient of restitution k is shown in Fig. 10. We can see that for small values of μ all the functions are linear. Starting at a certain value of μ which is dependent on k, angle ϕ_{II} is invariable.

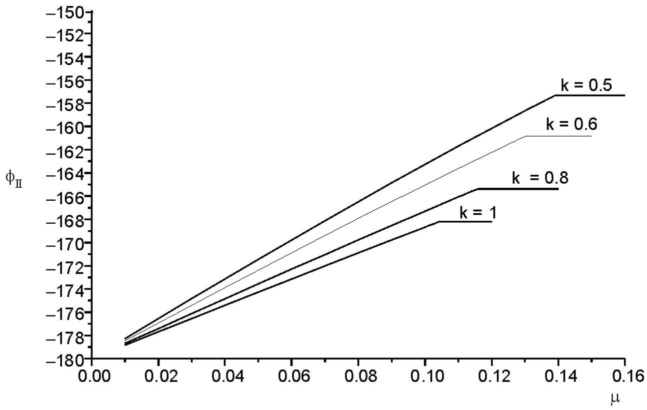

Fig. 10. The reflection angle ϕ_I depending on friction coefficient μ

The analysis of the collision on impulse semi-plane helps us to find an explanation for the above relationships. For small values of μ the trajectory does not cross the non-sliding line (see Fig. 11a). Thus, if coefficient μ rises, then value $\mathfrak{T}_{x\,02}$ of the impulse of friction force at the end of collision increases, too. For a sufficiently large value of μ appears cross point Γ_C (see Fig. 11b), thus starting at Γ_C the slip between the surfaces of both bodies vanishes and

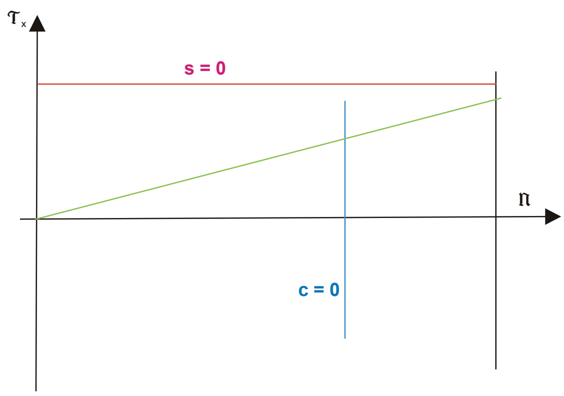

Fig. 11a. Impulse semi-plane for $\mu = 0.1$

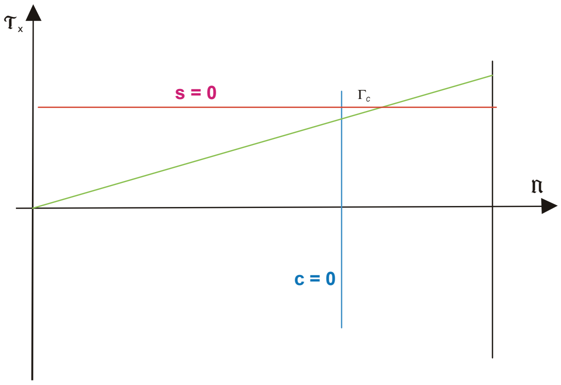

Fig. 11b. Impulse semi-plane for $\mu = 0.15$

$\mathcal{T}_{x02}$ does not change. Value $\mathcal{T}_{x02}$ determines the change of the component of velocity $\mathbf{v}^{\mathrm{II}}$ in the direction of axis x, therefore the reflection angle ϕ_{II} has a constant value for sufficiently large values of coefficient μ.

The changes of the angular velocity of colliding bodies during the collision are another effect of action of friction forces. The influence of coefficient μ on the value of angular velocity $\omega^{\mathrm{II}}{}_{y}$ at the end of the collision for some values of coefficient k is shown in Fig. 12.

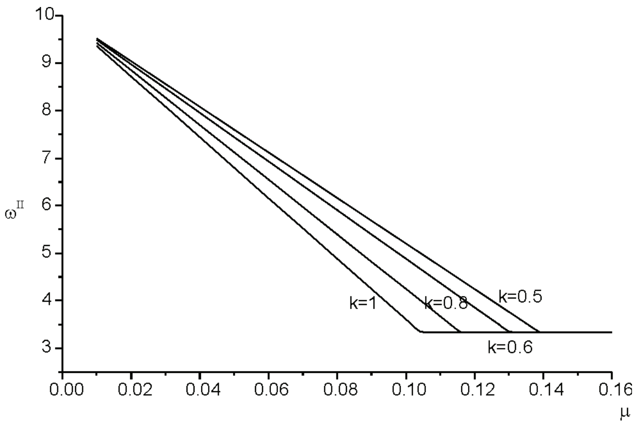

Fig. 12. The angular velocity $\omega^{\mathrm{II}}{}_{y}$ depending on friction coefficient

2. Collision between disc and square plate

Another considered example is the collision between a disc and a square plate which was at rest before the impact. The length of the side of the plate is a. The plate will be denoted by B_{I}, the disc of radius r – by B_{II}. The following values were set to calculations:

$$m_{\mathrm{I}} = 4\ \mathrm{kg},\quad a = 0.2\ \mathrm{m},\quad m_{\mathrm{II}} = 2\ \mathrm{kg},\quad r = 0.05\ \mathrm{m},$$
$$v^{\mathrm{II}}{}_{0x} = 0\ \mathrm{m/s},\quad v^{\mathrm{II}}{}_{0z} = 3\ \mathrm{m/s},\quad \omega^{\mathrm{II}}{}_{0y} = 5\ \mathrm{rad/s},$$
$$\mu = 0.15,\quad k = 0.6.$$

In Fig. 13 the kinematic state of the bodies at the initial instant t_0 is shown. The plate before the impact is at rest, so the initial value of the closing velocity $\mathbf{c}_0 = -v_{\mathrm{II}0z}$. In case of this impact the mass centre of the plate lies beyond the normal of collision. Therefore, constant $\varepsilon \neq 0$ (see (24) and below) and both the nonsliding line and the line of the greatest compression are not parallel to none of the axis. The interpretation of the collision on the impulse semi-plane is illustrated in Fig. 14.

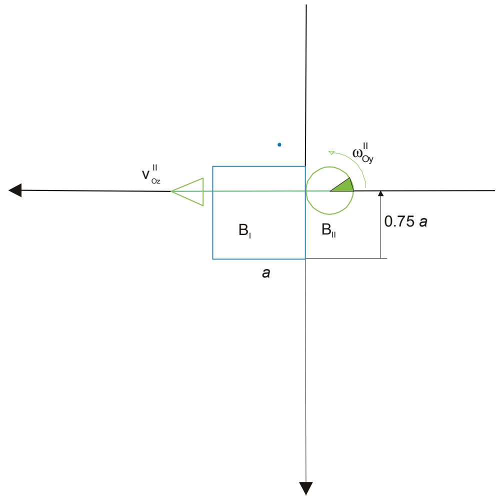

Fig. 13. Kinematic state of both bodies at instant t_0

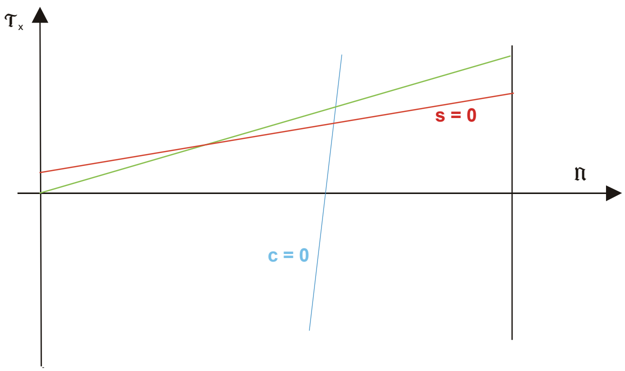

Fig. 14. Interpretation on impulse semi-plane for $v^{\mathrm{II}}{}_{0z} = 3$ m/s

The influence of the coefficient of friction μ on the angles of reflection ϕ_{I} and ϕ_{II} for some chosen magnitudes of the initial closing velocity $\mathbf{c}_0$ is illustrated in Fig. 15 and in Fig. 16, respectively. For small values of μ absolute values of both the angles increase along with coefficient μ. Starting at certain value of μ, which is dependent on the magnitude of $\mathbf{c}_0$, the angles become independent of μ.

In contrast to the case of central collisions, both functions presented in Fig. 15 and 16 are non-linear for small

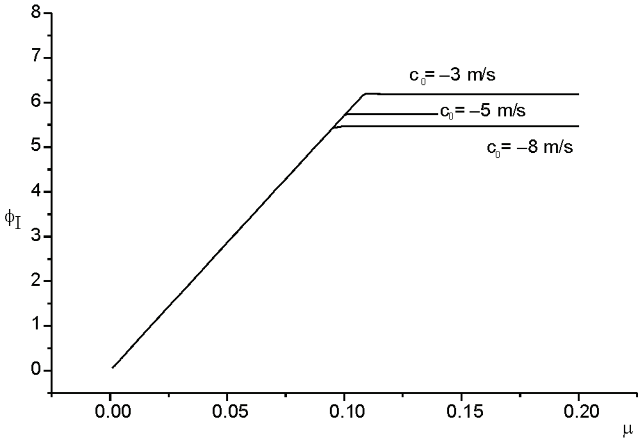

Fig.15. The angle of reflection for the plate in function of μ for some initial closing velocities $\mathbf{c}_0$

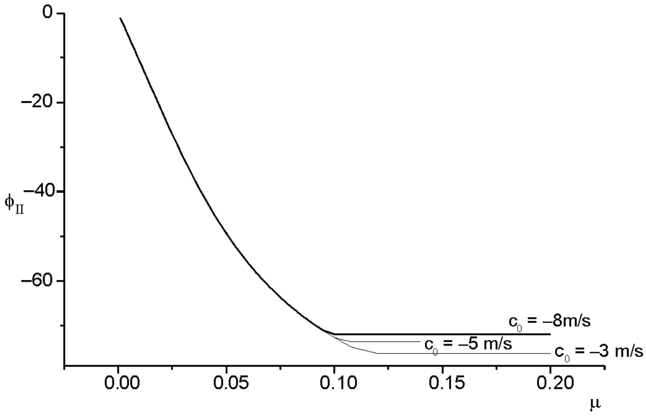

Fig. 16. The angle of reflection for the disc in function of μ for some initial clising velocites $\mathbf{c}_0$

values of coefficient μ. The non-linear character of those relationships is connected with the form of the non-sliding line and the line of the greatest compression. The line of the greatest compression is not parallel to the vertical axis, thus if μ rises, then coordinate $\boldsymbol{\Pi}_{01}$ of the cross point $\Gamma_A = (\boldsymbol{\Pi}_{01}, \boldsymbol{T}_{x\,01})$ varies, too. At that point the trajectory and line $\mathbf{c} = \mathbf{0}$ cross each other, when μ is small enough. Along with $\boldsymbol{\Pi}_{01}$, value $\boldsymbol{\Pi}_{02}$ also increases and the line of the end of collision travels to the right side. In that case the coefficient of friction has an influence on the tangential as well as the normal components of velocities $\mathbf{v}^{\mathrm{I}}$ and $\mathbf{v}^{\mathrm{II}}$.

Acknowledgements

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