I. INTRODUCTION

The wave equation is crucial to many domains of technology. We want to get a solution of the wave equation expressed in terms of some functions which satisfy this equation. We present three methods to obtain the Trefftz functions (wave polynomials) for the wave equation. The first method is connected with a generating function, and leads to recurrent formulas for polynomials and their derivatives, while the second method is based on a Taylor series expansion. Both methods lead to the same form of the Trefftz functions \[1, 2\]. The third method presented here, apart from the Taylor series expansion, additionally uses the inverse Laplace operator, leading to another form of the Trefftz wave functions which are still polynomials.

II. SOLUTION OF THE WAVE EQUATION BY THE WAVE POLYNOMIALS METHOD

Let us consider a non-homogeneous wave equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Q(x, y, t) = \frac{\partial^2 u}{\partial t^2}
\]

with the general solution given by

\[
u(x, y, t) = L^{-1}(0) + L^{-1}(Q),
\]

where \(L^{-1}(0)\) is the general solution of a homogeneous equation and \(L^{-1}(Q)\) is the particular solution of the non-homogeneous one. A general solution is approximated by the linear combination of the Trefftz functions \(v_n(x, y, t)\) (wave polynomials)

\[
\Theta(x, y, t) = \sum_{n=1}^{N} a_n v_n(x, y, t),
\]

satisfying the homogenous wave equation. To find unknown coefficients \(a_n\), we minimize the functional describing the adjustment of the approximation (in the mean square sense) to the boundary and initial conditions. The formula for the particular solution \(L^{-1}(Q)\) is given in \[3\]. Here we present the method of forming an approximate solution based on the Trefftz functions for the non-homogenous wave equation which seems to be interesting in generating different forms of these functions.

III. GENERATING THE TREFFTZ FUNCTIONS

Let us consider the two-dimensional homogenous wave equation
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2} \quad (4) \]

We expand the solution \( u = u(x, y, t) \) of (3) into a Taylor series in the neighbourhood of an arbitrary point \( (x_0, y_0, t_0) \)
\[ u(x, y, t) = u(x_0, y_0, t_0) + \sum_{n=1}^{\infty} \frac{d^n u(x_0, y_0, t_0)}{n!} (x-x_0, y-y_0, t-t_0) \]

where
\[ d^n u(x_0, y_0, t_0)(x-x_0, y-y_0, t-t_0) = \left( \frac{\partial}{\partial x}(x-x_0) + \frac{\partial}{\partial y}(y-y_0) + \frac{\partial}{\partial t}(t-t_0) \right)^n u(x_0, y_0, t_0). \]

Using (4) in order to eliminate \( \partial^2 u/\partial t^2 \), we transform the right hand side of (5), and then group other partial derivatives in a specific way (it is essential to separate \( \Delta u \)). This leads to the following form of the expansion
\[ u(x, y, t) = u_0 + \frac{\partial u}{\partial x} x + \frac{\partial u}{\partial y} y + \frac{\partial u}{\partial t} t + \frac{\partial^2 u}{\partial x^2} \left( \frac{x^2}{2!} - \frac{y^2}{2!} \right) + \]
\[ + \frac{\partial^2 u}{\partial x \partial y} x y + \frac{\partial^2 u}{\partial y \partial t} y t + \frac{\partial^2 u}{\partial t^2} \left( \frac{t^2}{2!} - \frac{y^2}{2!} \right) + \]
\[ + \frac{\partial^3 u}{\partial x^2 \partial t} \left( \frac{t^2}{2!} - \frac{y^2}{2!} \right) + \frac{\partial^3 u}{\partial x \partial y \partial t} x y t + \frac{\partial^3 u}{\partial y \partial t^2} y t^2 + \Delta u \left( \frac{t^2}{2!} - \frac{y^2}{2!} \right) + \]
\[ + \frac{\partial^3 u}{\partial x^2 \partial t^2} \left( \frac{t^2}{2!} - \frac{y^2}{2!} \right) + \frac{\partial^3 u}{\partial x \partial y \partial t^2} x y t^2 + \frac{\partial^3 u}{\partial y^2 \partial t^2} y t^2 + \Delta u \left( \frac{t^2}{2!} - \frac{y^2}{2!} \right) + \]
\[ + \frac{\partial \Delta u}{\partial x} \left( \frac{t^2}{2!} - \frac{y^2}{2!} \right) + \frac{\partial \Delta u}{\partial y} \left( \frac{t^2}{2!} - \frac{y^2}{2!} \right) + \frac{\partial \Delta u}{\partial t} \left( \frac{t^2}{2!} - \frac{y^2}{2!} \right) + ... \]

where \( u_0 = u(x_0, y_0, t_0), \quad \bar{x} = x-x_0, \quad \bar{y} = y-y_0, \quad \bar{t} = t-t_0. \)

In the above-presented expansion partial derivatives \( \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x \partial y \partial t}, \frac{\partial^3 u}{\partial y \partial t^2} \) at \( (x_0, y_0, t_0) \) are multiplied by well known harmonic functions \((7, 8)\) whereas other derivatives are multiplied by the polynomials satisfying Eq. (4).

\[ F_r(x, y) = Re \left( \frac{(x+i y)^r}{n!} \right) = \sum_{k=0}^{n} \left( \frac{(r)!}{k!(n-k)!} \right) \frac{\bar{x}^k \bar{y}^r}{k!} \quad n \geq 0 \quad (7) \]

\[ G_r(x, y) = Im \left( \frac{(x+i y)^r}{n!} \right) = \sum_{k=0}^{n} \left( \frac{(r-1)!}{k!(n-k)!} \right) \frac{\bar{x}^k \bar{y}^r}{k!} \quad n \geq 1 \quad (8) \]

All coefficients in (6) are Trefftz functions (wave polynomials) for Eq. (4). Moreover, they are different from wave polynomials described in \([3]\). In the next part we introduce the inverse Laplace operator for harmonic functions. It will be used to get wave polynomials appearing in (6).

### IV. AN INVERSE LAPLACE OPERATOR FOR HARMONIC FUNCTIONS

On the analogy of \([1, 7]\), we take into account a differential operator \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \)

Basis harmonic functions \( F_r(x, y), G_r(x, y) \) are linear combinations of monomials \( (x^n /m!)(y^k /k!) \); to calculate their inverse Laplace operators it is enough to know \( \Delta^{-1} (x^n /m!)(y^k /k!) \).

Since
\[ \Delta \left( \frac{x^n y^k}{m! \ k!} \right) = \frac{x^{n-2} y^k}{(m-2)! \ k!} + \frac{x^n y^{k-2}}{m! \ (k-2)!} \]

we may write
\[ \Delta^{-1} \left( \frac{x^n y^{k-2}}{m! \ (k-2)!} \right) = \frac{x^n y^k}{m! \ k!} = \Delta^{-1} \left( \frac{x^{n-2} y^k}{(m-2)! \ k!} \right) \]

\[ m \geq 2, k \geq 2. \]

After substituting \( k+2 \) in place of \( k \) we obtain the relation
\[ \Delta^{-1} \left( \frac{x^n y^k}{m! \ (k+2)!} \right) = \frac{x^n y^{k+2}}{m! \ (k+2)!} = \Delta^{-1} \left( \frac{x^{n-2} y^{k+2}}{(m-2)! \ (k+2)!} \right) \]

\[ m \geq 2, \quad k \geq 0. \]

If we put
\[ \Delta^{-1} \left( \frac{y^k}{k!} \right) = \frac{y^{k+2}}{(k+2)!}, \quad \Delta^{-1} \left( \frac{y^k}{k!} \right) = \frac{y^{k+1}}{(k+2)!} \]

then from (11), (12) we compute the inverse Laplace operator for an arbitrary monomial (13)
Solutions of Two-Dimensional Wave Equation using some Form of the Trefftz Functions

\[ \Delta^{-1}\left( \frac{x^m y^k}{m! \cdot k!} \right) = \begin{cases} x^n y^{k+2} & m = 0, m = 1, k \geq 0 \\ x^n y^{k+2} & m = 2, k \geq 0 \\ \frac{x^{m-2} y^{k+2}}{(m-2)! \cdot (k+2)!} & m \geq 2, k \geq 0 \end{cases} \]

(13)

The formula (13) may be written in a closed form

\[ \Delta^{-1}\left( \frac{x^m y^k}{m! \cdot k!} \right) = \sum_{j=0}^{\pi} (-1)^j \frac{x^{m-2j} y^{k+2j}}{(m-2j)! \cdot (k+2j)!} \]

(14)

Harmonic functions \( F_n, G_n \) are symmetrical with respect to variables \( x \) and \( y \), whereas calculations of successive inverse operations in accordance with the formula (11) distinguish variable \( y \) (observe that \( \Delta^{-1}(1) = y^2/2! \)). By symmetry of \( \Delta \) with respect to variables \( x \) and \( y \), it is possible to define \( \Delta^{-1} \) which distinguishes variable \( x \). Both possibilities are shown in Table 1.

<table>
<thead>
<tr>
<th>Distinguished variable</th>
<th>Inverse operation ( \Delta^{-1}\left( \frac{x^m y^k}{m! \cdot k!} \right) )</th>
</tr>
</thead>
</table>
| \( x \) \( \Delta^{-1}(1) = \frac{x^2}{2!} \) | \[ \Delta^{-1}\left( \frac{x^m y^k}{m! \cdot k!} \right) = \begin{cases} x^{m-2} y^k & k = 0, k = 1, m \geq 0 \\ -\Delta^{-1}\left( \frac{x^{m+2} y^{k-2}}{(m+2)! \cdot (k-2)!} \right) & k \geq 2, m \geq 0 \end{cases} \]
| \( y \) \( \Delta^{-1}(1) = \frac{y^2}{2!} \) | \[ \Delta^{-1}\left( \frac{x^m y^k}{m! \cdot k!} \right) = \begin{cases} \frac{x^m y^{k+2}}{m! \cdot (k+2)!} & m = 0, m = 1, k \geq 0 \\ -\Delta^{-1}\left( \frac{x^{m+2} y^{k+2}}{(m+2)! \cdot (k+2)!} \right) & m \geq 2, k \geq 0 \end{cases} \]

Similarly, the following inverse operations are calculated

\[ \Delta^{-n}(1) = \Delta^{-1}(\Delta^{-n}). \]

The inverse operations for a few successive harmonic functions \( F_n(x,y) \) and \( G_n(x,y) \) are shown below.

\[ \Delta^{-1}(F_0) = \Delta^{-1}(1) = \frac{y^2}{2!} \]

\[ \Delta^{-2}(F_0) = \Delta^{-1}\left( \frac{y^5}{4!} \right) = \frac{y^4}{4!} \]

\[ \Delta^{-1}(G_1) = \Delta^{-1}(x) = \frac{xy^2}{2!} \]

\[ \Delta^{-2}(G_1) = \Delta^{-1}\left( \frac{xy^2}{2!} \right) = \frac{xy^4}{4!} \]

\[ \Delta^{-1}(F_1) = \Delta^{-1}(y) = \frac{y^3}{3!} \]

\[ \Delta^{-2}(F_1) = \Delta^{-1}\left( \frac{y^3}{3!} \right) = \frac{y^5}{5!} \]

\[ \Delta^{-1}(G_2) = \Delta^{-1}(xy) = \frac{xy^3}{3!} \]

\[ \Delta^{-2}(G_2) = \Delta^{-1}\left( \frac{xy^3}{3!} \right) = \frac{xy^5}{5!} \]

(16)
\[ \Delta^{-1}(F_i) = \Delta^{-1}\left(\frac{x^2}{2!} - \frac{y^2}{2!}\right) = \frac{x^2 y^2}{2!} - \frac{2 y^4}{4!}, \quad \Delta^{-2}(F_i) = \Delta^{-1}\left(\frac{x^2 y^2}{2!} - \frac{2 y^4}{4!}\right) = \frac{x^2 y^4}{2!} - \frac{3 y^6}{6!} \]

V. REPRESENTATION OF WAVE FUNCTIONS IN TERMS OF INVERSE LAPLACE OPERATIONS

Let us divide the Taylor series expansion (6) into two parts (17). The first one includes harmonic functions \( F_n \) and functions connected with them by inverse operations, while the second one contains their Laplace inverse apart from harmonic functions \( G_n \)

\[ u(x, y, t) = u_{1, i} + \frac{\partial u}{\partial x} \bigg|_{x=x_0} + \frac{\partial u}{\partial t} \bigg|_{t=t_0} + \frac{\partial^2 u}{\partial x^2} \left(\frac{x^2}{2!} - \frac{y^2}{2!}\right) + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t}\right) + \Delta u \left(\frac{\partial^2 F_i}{2!} + \frac{\partial \Delta u}{\partial t}\right) + \frac{\partial^2 u}{\partial x^2} \left(\frac{x^2 y^2}{2!} - \frac{2 y^4}{4!}\right) + \Delta u \left(\frac{\partial^2 F_i}{2!} + \frac{\partial \Delta u}{\partial t}\right) + \ldots \]

Using \( \Delta^{-1}(F_i), \Delta^{-i}(G_k), i = 1, 2, \ldots, N, k = 0, 1, 2, \ldots \) allows us to rewrite formula (17)

\[ u(x, y, t) = u_{1, i} + \frac{\partial u}{\partial x} F_i + \frac{\partial u}{\partial t} F_i + \frac{\partial^2 u}{\partial x^2} F_i + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t}\right) F_i + \Delta u \left(\frac{\partial^2 F_i}{2!} + \frac{\partial \Delta u}{\partial t}\right) + \frac{\partial^2 u}{\partial x^2} \left(\frac{x^2 y^2}{2!} - \frac{2 y^4}{4!}\right) + \Delta u \left(\frac{\partial^2 F_i}{2!} + \frac{\partial \Delta u}{\partial t}\right) + \ldots \]

(18)
Solutions of Two-Dimensional Wave Equation using some Form of the Trefftz Functions

\[ u(x,y,t) = \sum_{n=0}^{N} \sum_{k=0}^{N} \Delta u_{nk} (x,y) \]  

\[ \sum_{j=0}^{n-1} \sum_{k=0}^{n-k-1} \frac{\Delta^j u}{\partial x^j \partial y^{n-j-1}} \left( \frac{x}{2} \right)^{n-j-1} \gamma_j^{n-j-1} F_{x-n+1} (x,y) \]  

\[ + \sum_{j=0}^{n-k} \sum_{k=0}^{n-k} \frac{\Delta^j u}{\partial x^j \partial y^{n-k}} \left( \frac{x}{2} \right)^{n-k} \gamma_j^{n-k} G_{x-n+1} (x,y) \]  

\[ + R_{N+1}, \]

where \([x]\) gives the greatest integer less than or equal to \(x\).

The obtained form of \(u(x,y,t)\) consists of two independent sequences of wave polynomials given by inverse operations of harmonic functions.

We can repeat all the previous considerations to eliminate the partial derivatives from the Taylor series expansion with the help of the wave equation. Then we obtain a similar form of function \(u\), but now the inverse operations are calculated in a different way – with distinguished different variables. The results of all possible eliminations and consecutive expansions are presented in Table 2.

<table>
<thead>
<tr>
<th>Eliminated derivatives</th>
<th>Distinguished variable in inverse operations</th>
<th>Expanded form of (u = u(x, y, t))</th>
</tr>
</thead>
</table>
| \( \frac{\partial^2 u}{\partial x^2} \) | \( x \) \( \left( \Delta^j (t) = \frac{x^j}{2!} \right) \) | \( u(x,y,t) = \sum_{n=0}^{N} \sum_{k=0}^{N} \frac{\partial^2 u}{\partial x^2 \partial y^{n-k}} \left[ \frac{n-k}{2} \right] F_{x-n+1} (x,y) \) \]  

\[ + \sum_{n=1}^{N} \sum_{k=0}^{n-k-1} \frac{\partial^2 u}{\partial x^2 \partial y^{n-k-1}} \left[ \frac{n-k-1}{2} \right] G_{x-n+1} (x,y) \]  

\[ + R_{N+1}, \]

| \( \frac{\partial^2 u}{\partial y^2} \) | \( y \) \( \left( \Delta^j (t) = \frac{y^j}{2!} \right) \) | \( u(x,y,t) = \sum_{n=0}^{N} \sum_{k=0}^{N} \frac{\partial^2 u}{\partial y^2 \partial x^{n-k}} \left[ \frac{n-k}{2} \right] F_{y-n+1} (x,y) \) \]  

\[ + \sum_{n=1}^{N} \sum_{k=0}^{n-k-1} \frac{\partial^2 u}{\partial y^2 \partial x^{n-k-1}} \left[ \frac{n-k-1}{2} \right] G_{y-n+1} (x,y) \]  

\[ + R_{N+1}, \]
VI. STRUCTURE OF AN APPROXIMATE SOLUTION OF THE WAVE EQUATION

The solution of the wave equation is approximated by the sum of two linear combinations of wave polynomials

\[ \Theta(x, y, t) = \sum_{n=0}^{N} a_{n} \cdot \varphi_{n}(x, y, t) + \sum_{k=1}^{N} a_{k} \psi_{k}(x, y, t) \]  

(19)

where \( \varphi_{n}(x, y, t) \), \( \psi_{k}(x, y, t) \) denote functions given by inverse operations of harmonic functions \( F_{n} \), \( G_{k} \) respectively, \( N \) – degree of harmonic functions.

Denote \( h_{2n}(x, y) = F_{n}(x, y) \), \( h_{2n+1}(x, y) = G_{n+1}(x, y) \), \( n \geq 0 \) and arrange wave polynomials \( \varphi_{n}(x, y, t) \), \( \psi_{k}(x, y, t) \) into a triangle matrix (20). In every row there are polynomials of the fixed degree with respect to variables \( x, y \). Each column contains polynomials of the same degree with respect to variable \( t \).

\[
\begin{align*}
\varphi_{0} &= h_{0} \\
\varphi_{1} &= h_{1} \\
\varphi_{2} &= h_{2} \quad \varphi_{2} = th_{0} \\
\varphi_{3} &= h_{3} \quad \varphi_{3} = th_{1} \\
\varphi_{4} &= h_{4} \quad \varphi_{4} = th_{2} \quad \varphi_{5} = \frac{t^{2}}{2!} h_{0} + \Delta^{-1} h_{0} \\
\varphi_{5} &= h_{5} \quad \varphi_{5} = th_{3} \quad \varphi_{6} = \frac{t^{2}}{2!} h_{1} + \Delta^{-1} h_{1} \\
\varphi_{6} &= h_{6} \quad \varphi_{7} = th_{4} \quad \varphi_{8} = \frac{t^{2}}{2!} h_{2} + \Delta^{-1} h_{2} \quad \varphi_{9} = \frac{t^{3}}{3!} h_{0} + t \Delta^{-1} h_{0} \\
\varphi_{7} &= h_{7} \quad \varphi_{8} = th_{5} \quad \varphi_{9} = \frac{t^{2}}{2!} h_{3} + \Delta^{-1} h_{3} \quad \varphi_{10} = \frac{t^{3}}{3!} h_{1} + t \Delta^{-1} h_{1} \\
\varphi_{9} &= h_{9} \quad \varphi_{10} = th_{6} \quad \varphi_{11} = \frac{t^{2}}{2!} h_{4} + \Delta^{-1} h_{4} \quad \varphi_{12} = \frac{t^{3}}{3!} h_{2} + t \Delta^{-1} h_{2} \quad \varphi_{13} = \frac{t^{4}}{4!} h_{0} + \frac{t^{2}}{2!} - \Delta^{-1} h_{0} + \Delta^{-2} h_{0}.
\end{align*}
\]

(20)

For an approximation purpose we always require a finite number of functions, so the question is in which way and how many functions we should choose. Both the degree of harmonic functions and the assumption of accuracy with respect to the time variable determine the unambiguous choice of functions presented in Table 3.
Table 3. Elements of linear combinations of base functions depending on order approximations with respect to time

<table>
<thead>
<tr>
<th>Order of polynomials with respect to space variables</th>
<th>$\sim t^0$</th>
<th>$\sim t^1$</th>
<th>$\sim t^2$</th>
<th>$\sim t^3$</th>
<th>$\sim t^4$</th>
<th>$\sim t^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harmonic base function $h_i$</td>
<td>$h_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Harmonic base function $h_i$</td>
<td>$h_1$</td>
<td>$h_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Biharmonic base function $\Delta^1 h_i$</td>
<td>$h_2$</td>
<td></td>
<td>$h_0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Biharmonic base function $\Delta^3 h_i$</td>
<td></td>
<td>$h_1$</td>
<td>$h_0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base function $\Delta^2 h_i$</td>
<td></td>
<td></td>
<td>$h_2$</td>
<td>$h_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base function $\Delta^4 h_i$</td>
<td></td>
<td></td>
<td></td>
<td>$h_1$</td>
<td>$h_0$</td>
<td></td>
</tr>
</tbody>
</table>

The hypothetical system of approximating functions is shown below.

The linear combination of base functions, which gives polynomials up to degree 1 with respect to time and up to degree 3 with respect to space variables consists of 12 functions

$$h_0, h_1, h_2, h_3, h_4, h_5, h_6,$$

If we require the quadratic approximation with respect to $t$ and the approximation up to degree 4 with respect to $x$ and $y$, we should take 25 functions

$$h_{00}, h_{11}, h_{22}, h_{33}, h_{44}, h_{55}, h_{66}, h_{01}, h_{12}, h_{23}, h_{34}, h_{45}, h_{56}, h_{02}, h_{13}, h_{24}, h_{35}, h_{46}.$$
with number \( j (1 \leq j \leq m - 1) \) in Table 3, \( 2(n - j + 1) \) consecutive functions should be taken. Notice that column 1 includes only harmonic functions \( h_i \), whereas column \( j \) includes components \( \Delta^{-[j-1]} h_i \). When we omit a restriction on the order of time, the approximation is exactly the truncated Taylor expansion of \( u = u(x, y, t) \) at \((x_0, y_0, t_0)\) up to the order \( n \).

VII. POLAR WAVE FUNCTIONS

In polar coordinates the two-dimensional wave equation has the following form

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2}.
\]  

(21)

Like in Cartesian coordinates, the solution of a problem given by equation (21) with appropriate initial and boundary conditions can be approximated by a linear combination of polar wave functions (23) satisfying the governing equation. These functions are obtained by substituting

\[
x = r \cos \varphi, \quad y = r \sin \varphi
\]

(22)
to into (20).

A few of such functions are presented below:

\[
h_0 = 1, \quad h_1 = r \sin \varphi, \quad h_2 = r \cos \varphi, \quad t h_3 = t, \quad h_4 = r^2 \sin \varphi \cos \varphi, \quad h_5 = tr \sin \varphi, \quad h_6 = r^2 \cos \varphi, \quad h_7 = tr \cos \varphi, \quad h_8 = r^2 \sin \varphi, \quad h_9 = tr \sin \varphi, \quad h_{10} = r^2 \cos \varphi
\]

(23)

An approximate solution is formed exactly in the same way as in Cartesian coordinates.

VIII. NUMERICAL EXAMPLES

The authors of papers [1] and [3] have shown how the wave-polynomial method works for polynomials generated for one and two-dimensional Cartesian coordinates. Numerical examples presented here demonstrate how this method works for wave polynomials given by inverse Laplace operators.

- Vibrations of a square membrane

The two-dimensional wave equation

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (x, y) \in (0,1) \times (0,1), \quad t \geq 0
\]

(24)
together with conditions:

\[
u(0, y, t) = u(l, y, t) = u(x, 0, t) = u(x, l, t) = 0 \\
u(x, y, 0) = xy(x-1)(y-1) \quad \frac{\partial u}{\partial t}(x, y, 0) = 0
\]

(25)
has the exact solution given by [5]

\[
u(x, y, t) = \frac{64}{\pi^6} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin(\pi x (2n+1)) \sin(\pi y (2m+1)) \cos \left( \frac{\pi t \sqrt{(2n+1)^2 + (2m+1)^2}}{} \right)}{(2n+1)^3 (2m+1)^3}.
\]

(26)
An approximate solution of this problem \( \Theta(x, y, t) \) is determined in a successive manner in the \( n \)-th time interval \( \langle (n-1)\delta t, n\delta t \rangle \) as a linear combination of the Trefftz functions

\[
\Theta_n(x, y, t) = \sum_{k=0}^{\infty} a_k h_k(x, y, t).
\]  

(27)

The unknown coefficients of this approximation \( a_k \) are obtained from the minimization of the following functional

\[
J_n = \int_0^{\delta t} \int_0^{\delta t} \left[ \Theta_n(x, y, (n-1)\delta t) - \Theta_{n-1}(x, y, (n-1)\delta t) \right]^2 \, dx \, dy + \int_0^{\delta t} \int_0^{\delta t} \left[ \frac{\partial \Theta_n(x, y, (n-1)\delta t)}{\partial t} - \frac{\partial \Theta_{n-1}(x, y, (n-1)\delta t)}{\partial t} \right]^2 \, dx \, dy + \int_0^{\delta t} \int_0^{\delta t} \left( \Theta_n(0, 0, t) \right)^2 \, dx \, dy + \int_0^{\delta t} \int_0^{\delta t} \left( \Theta_n(1, 1, t) \right)^2 \, dx \, dy + \int_0^{\delta t} \int_0^{\delta t} \left( \Theta_n(0, y, t) \right)^2 \, dy \, dx + \int_0^{\delta t} \int_0^{\delta t} \left( \Theta_n(x, 1, t) \right)^2 \, dx \, dy,
\]  

(28)

which leads to a linear system of equations on \( a_k \). We assume that in the first time interval, \( \Theta_0(x, y, 0) \) is known from the initial conditions. In the \( n \)-th time interval \( \langle (n-1)\delta t, n\delta t \rangle \) is given by the approximate solution at the end of the previous time interval.

To solve the considered problem we take an approximation by the Trefftz polynomials whose degrees are bounded by 10 with respect to space variables and 9 with respect to time.

Figure 1 shows the results obtained at time \( t = 0.5 \) for \( \delta t = 1 \).

Because the greatest deflection takes place in the middle of the membrane, Fig. 3 shows the results obtained in the whole time interval at this point.

The approximate solution obtained for a given time interval can be extended over this interval. Figure 4 shows that the extrapolation in time of the approximate solution is close to the exact one.

The extrapolation can obviously be made not only in time but also in space, so the method should give good results in solving inverse problems.

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Fig. 1. Approximate solution (a), difference between exact and approximate solution (b) at \( t = 0.5 \)

Fig. 2. Approximate and exact solution (a), difference between exact and approximate solution (b) at point (0.5, 0.5) for time interval \( [0, 1] \)
Vibrations of a square membrane with a constant source

The next example shows the vibration of a membrane with the constant source $Q(x, y, t) = 0.1$. The equation of motion is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Q(x, y, t), \quad (x, y) \in (0,1) \times (0,1), \quad t > 0$$

(29)

with initial and boundary conditions:

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$$
$$u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = 0$$

(30)

The exact solution for this problem is known [5]

$$u(x, y, t) = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{4 \sin(k \pi x) \sin(l \pi y)}{10(k \pi)^2 + (l \pi)^2} \times \sum_{m=0}^\infty \sum_{n=0}^\infty 10(k \pi)^2 + (l \pi)^2 (t - \tau) dy dx$$

(31)

As in the previous example the approximate solution to this problem $\Theta(x, y, t)$ is determined in a successive manner in the $n$-th time interval as a sum of linear combination of the Trefftz functions (27) and additionally function $g_p(x, y, t)$ which is the particular solution to the non-homogenous equation (29),

$$\Theta_n(x, y, t) = \sum_{k=0}^N a_k h_k(x, y, t) + g_p(x, y, t).$$

(32)

The unknown coefficients of this approximation $a_k$ are obtained from minimization of the functional (28), whereas function $g_p(x, y, t)$ is calculated as [3]

$$g_p = L^{-1}(Q) = L^{-1}\left(\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{Q(x_n, y_n, t_0)}{k!l!} \frac{x_n^2 y_n^2}{k!l!} \right) =$$

$$= \sum_{n=0}^\infty \sum_{m=0}^\infty c_{mn} L^{-1}(x_n^2 y_n^2),$$

$$L^{-1}(x_n^2 y_n^2) = \frac{(w_{p1} + w_{p2} + w_{p3})}{3},$$

(33)

$$w_{p1} = L^{-1}(x_1^2 y_1^2) = \frac{1}{(k+2)(k+1)} \times$$

$$\times (-x_1^4 y_1^4 + m(m-1)L^{-1}(x_1^4 y_1^4) +$$

$$-l(l-1)L^{-1}(x_1^4 y_1^4)),$$

$$w_{p2} = L^{-1}(x_2^2 y_2^2) = \frac{1}{(l+2)(l+1)} \times$$

$$\times (-x_2^4 y_2^4 + m(m-1)L^{-1}(x_2^4 y_2^4) +$$

$$-k(k-1)L^{-1}(x_2^4 y_2^4)),$$

$$w_{p3} = L^{-1}(x_3^2 y_3^2) = \frac{1}{(m+2)(m+1)} \times$$

$$\times (-x_3^4 y_3^4 + k(k-1)L^{-1}(x_3^4 y_3^4) +$$

$$-l(l-1)L^{-1}(x_3^4 y_3^4)),$$

(34)

where $x = x - x_0, y = y - y_0, t = t - t_0$.

The same polynomials as in the previous example are used for the solution; function $g_p(x, y, t)$ is taken as

$$g_p(x, y, t) = \frac{-x^2 - y^2 + t^2}{60}.$$

(35)

Figure 5 shows the results obtained at time $t = 0.5$ for $\delta t = 1$.

The approximate solution in the middle of the membrane and its extrapolation in time are shown in Fig. 6.

The error of approximation in the middle of the membrane defined as

$$\delta_{x, y, t} = \frac{\int_0^{\delta t} \left(\Theta(0.5, 0.5, t) - u(0.5, 0.5, t)\right)^2 dt}{\int_0^{\delta t} \left(u(0.5, 0.5, t)\right)^2 dt}.$$ 

(36)

is calculated. The results are shown in Table 4.
Table 4. Error dependence on the number of polynomials and length of time interval

<table>
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<tr>
<th>Number of polynomials</th>
<th>Length of time interval</th>
</tr>
</thead>
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<tr>
<td></td>
<td>$\delta t = 0.5$</td>
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<tr>
<td>$N = 49$</td>
<td>0.0070</td>
</tr>
<tr>
<td>$N = 85$</td>
<td>0.0565</td>
</tr>
<tr>
<td>$N = 121$</td>
<td>0.0141</td>
</tr>
<tr>
<td>$N = 153$</td>
<td>0.0200</td>
</tr>
</tbody>
</table>

The presented test example shows good accuracy of the approximate solution with wave polynomials as base functions. Moreover, wave polynomials do not yield any restrictions on the shape of the area, so the method can be used for more complicated shapes of the considered areas. In these cases wave polynomials may also be used as base functions in different variants of FEM, which leads to better accuracy of approximation in each element.

The method of generating wave functions presented here can also be easily extended to a three-dimensional equation.

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References


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