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NEGATIVE POISSON'S RATIOS FROM ROTATING RECTANGLES

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Abstract: Materials with a negative Poisson's ratio exhibit the unexpected property of becoming fatter when stretched and narrower when compressed. This counter-intuitive behaviour is known as 'auxetic behaviour' and imparts many beneficial effects on the material's macroscopic properties. This paper discusses the potential of systems composed of rigid rectangles connected together through flexible hinges at their vertices. It will be shown that, on application of uniaxial loads, these rigid rectangles will rotate with respect to each other to form, in some cases, a more open structure hence giving rise to a negative Poisson's ratio.

Key words: Negative Poisson's ratio, auxetic, mechanical properties

1. INTRODUCTION

Materials with a negative Poisson's ratio (auxetic) exhibit the very unusual property of becoming wider when stretched and thinner when compressed [1]. This unusual behaviour can normally be described in terms of models based on the geometry of the system (in the case of materials, the geometry of material's internal structure) and the way this geometry changes as a result of applied loads (deformation mechanism).

In recent years, various two and three dimensional theoretical models and structures which can lead to negative Poisson's ratio have been proposed including, two and three-dimensional 're-entrant' systems [1-6], models based on rigid 'free' molecules [7-10], chiral structures [11-13], systems made from rotating squares and triangles [14-19], composites [20], and fractal structures [21]. In all of these systems, the Poisson's ratio does not depend on scale and the systems may be constructed at the nano- (molecular), micro- or at the macro- level - the only requirement is the right combination of the geometry and the deformation mechanism.

In fact, some of these two and three dimensional theoretical models are thought to be responsible for negative Poisson's ratios in various classes of nano- and microstructured materials. For example the negative Poisson's ratio in open cell foams has been explained in terms of re-entrant models [22-25] and a chiral model [13], whilst the auxetic behaviour

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predicted in some zeolites has been attributed to rotation of squares or triangles [14, 17, 26]. In this respect, one should mention that the scale independence of the Poisson's ratio has been exploited by various researchers in their quest for designing new auxetic materials by first proposing auxetic macrostructures which are then downscaled to the molecular level to produce nanostructured materials that mimic the auxetic macrostructures (the 'downscaling technique'). For example, the downscaling technique has been used to propose theoretical nanostructured auxetic polymers which mimic re-entrant honeycomb systems [1] and the 'rotating triangles' model [15].

A particular 'geometry/deformation mechanism' which has attracted a considerable amount of interest can be constructed using rigid squares hinged at their vertices [16-18]. It has been shown that the Poisson's ratios of such systems where the squares are perfectly rigid will exhibit constant Poisson's ratios equal to -1 [17, 26] whilst if the squares are allowed to deform, the Poisson's ratio will be dependent on the relative rigidity of the squares with respect to the rigidity of the hinges and on the direction of loading [14, 17, 19]. It has also been shown that this geometry/mechanism is likely to be responsible for negative Poisson's ratios in various classes of nanostructured materials including materials belonging to the KH_2PO_4 family of the D_{2d} point group [18], silicates and zeolites [14, 17, 26].

In this work we will present an extension to the rotating rigid squares model, namely one where the squares are replaced rigid rectangles.

2. THE ROTATING RECTANGLES MODEL

Consider a two dimensional tessellation built with rigid rectangles of side lengths *a*, *b* hinged at their corners and aligned in the Ox_{12} plane as shown in Fig. 1. Let the angle between

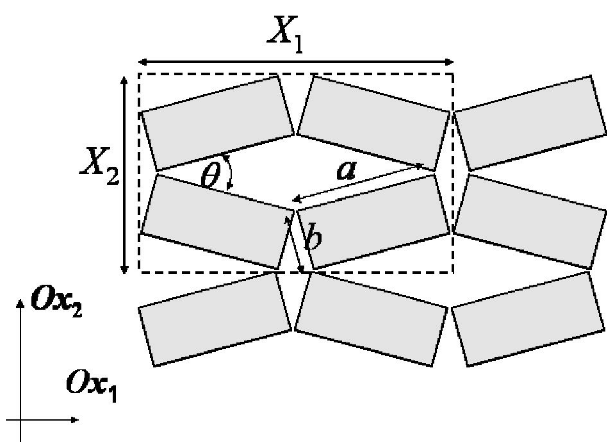

Fig. 1. The system composed of hinged 'rotating rectangles' of size $(a \times b)$ with a rectangular unit cell of dimensions $(X_1 \times X_2)$

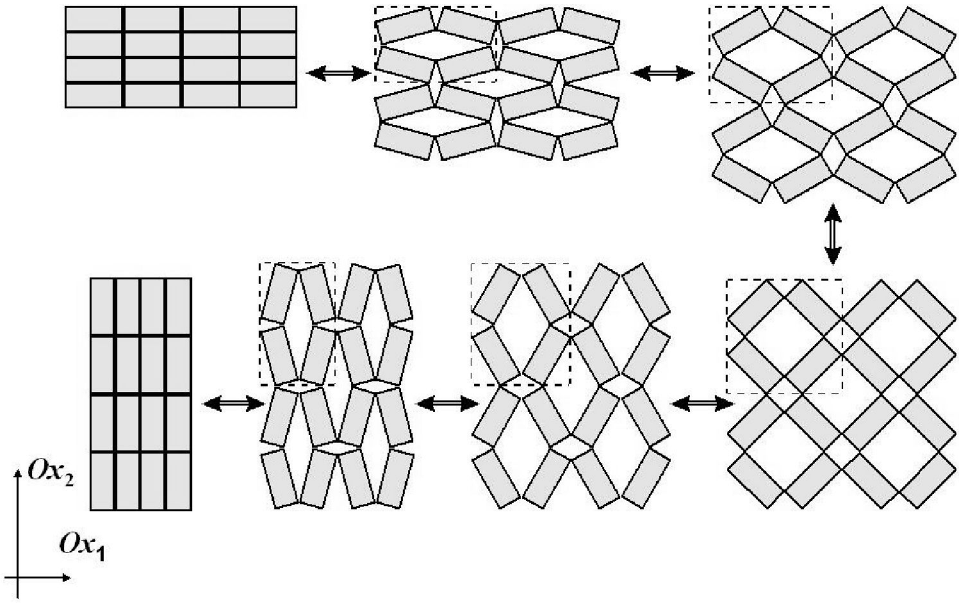

Fig. 2. The 'rotating rectangles' structure composed of rectangles of size ($a \times b$) at various values of 'q'
In this system, $a \approx 2b$

two such rectangles be θ. The shape of the structure for various values of θ is shown in Fig. 2. These different configurations may be obtained from one another through loading in an Ox_i direction. Fig. 2 visually suggests that the structure is auxetic in the Ox_{12} plane for some, but not all, values of θ.

As illustrated in Fig. 1, a rectangular unit cell may be used to describe this tessellation where the cell sides are parallel to the Ox_1 and Ox_2 axis. This unit cell contains four a-b rectangles (*i.e.* rectangles of side lengths a and b) with projections in the Ox_i directions are given by:

$$X_1 = 2\left[a\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\theta}{2}\right)\right] \quad \text{and} \quad X_2 = 2\left[a\sin\left(\frac{\theta}{2}\right)+b\cos\left(\frac{\theta}{2}\right)\right]. \tag{1}$$

In this model we shall assume that the structure deforms solely by relative rotation of the rectangles, then a and b are constants and hence X_i are functions of the single variable θ *i.e.* $X_i = X_i(\theta)$. We shall also assume that the stiffness of the structure (and hence the Young's moduli) is a result of the stiffness of the hinges, that is, a stiffness which opposes changes in the angles θ. In particular, we shall assume that the hinges satisfy the equation:

$$M = K_h\left(\delta\theta\right), \tag{2}$$

where M is the moment applied to the rectangles, $\delta\theta$ is the angular displacement due to M, and K_h is the spring constant for the hinge.

The rigidity of the rectangles result in a structure which is geometrically constrained not to shear. This results in a value of zero for the five elements of compliance matrix which are associated with shearing a hence the compliance matrix for this system is hence of the form:

$$\mathbf{S} = \left[s_{ij}\right] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

where ν_{ij} represent the Poisson's ratios (or more precisely the Poisson's functions) in the Ox_{ij} plane for loading in the Ox_i direction, defined by:

$$\nu_{ij} = \left(\nu_{ji}\right)^{-1} = -\frac{d\varepsilon_j}{d\varepsilon_i} \qquad i, j = 1, 2\,, \tag{4}$$

whilst E_i are the Young's moduli for loading in the Ox_i directions given by:

$$E_i = \frac{d\sigma_i}{d\varepsilon_i} \text{i} \qquad = \qquad 1,2, \tag{5}$$

where $d\sigma_i$ and $d\varepsilon_i$ are infinitesimally small stresses and strains for loading in the Ox_i directions respectively.

2.1. The Poisson's ratios

The infinitesimally small strains $d\varepsilon_i$ in the Ox_i directions may be defined by:

$$d\varepsilon_i = \frac{dX_i}{X_i} \tag{6}$$

and since $X_i = X_i\ (\theta)$,

$$\nu_{21} = \left(\nu_{12}\right)^{-1} = -\frac{d\varepsilon_1}{d\varepsilon_2} = -\frac{dX_1/X_1}{dX_2/X_2} = -\frac{dX_1/d\theta}{dX_2/d\theta}.\frac{X_2}{X_1}\ . \tag{7}$$

Differentiating Eq. 1 we obtain:

$$\frac{dX_1}{d\theta} = -a\sin\left(\frac{\theta}{2}\right) + b\cos\left(\frac{\theta}{2}\right) \qquad \text{and} \qquad \frac{dX_2}{d\theta} = a\cos\left(\frac{\theta}{2}\right) - b\sin\left(\frac{\theta}{2}\right), \tag{8}$$

i.e. from Eq. 1, Eq. 7 and Eq. 8 the on-axis Poisson's ratios:

$$\nu_{21} = (\nu_{12})^{-1} = \frac{a^2 \sin^2\left(\frac{\theta}{2}\right) - b^2 \cos^2\left(\frac{\theta}{2}\right)}{a^2 \cos^2\left(\frac{\theta}{2}\right) - b^2 \sin^2\left(\frac{\theta}{2}\right)}, \tag{9}$$

which for particular case when $a = b$ *(i. e.* the rectangle is a square) are given by:

$$\nu_{21} = \nu_{12} = -1. \tag{10}$$

2.2. The Young's moduli

The work done by each unit cell due to the changes in the inter-rectangle angles from θ to $\theta + d\theta$ that accompany a small strain is given by:

$$W = N\left[\frac{1}{2}K_h (d\theta)^2\right] = 8\left[\frac{1}{2}K_h (d\theta)^2\right], \tag{11}$$

where N is the number of hinges per unit cell, which in this case is equal to eight. (One unit cell contains four rectangles, each rectangle has four vertices, and two vertices contribute to one hinge) and K_h is the stiffness constant of the hinges as defined through Eq. 2.

Also, since $X_i = X_i(\theta)$, the work done per unit volume due to an infinitesimally small strain $d\varepsilon_i$ for loading in the Ox_i direction ($i = 1, 2$) is given by:

$$U = \frac{1}{2}E_i (d\varepsilon_i)^2 = \frac{1}{2}E_i\left(\frac{dX_i}{X_i}\right)^2 = \frac{1}{2}E_i\left(\frac{1}{X_i}\frac{dX_i}{d\theta}\right)^2 (d\theta)^2. \tag{12}$$

Form the principle of conservation of energy:

$$U = \frac{1}{V}W, \tag{13}$$

where V is the volume of the unit cell given by (assuming a unit thickness in the third dimension):

$$V = X_1 X_2. \tag{14}$$

Thus from Eq. 11 to Eq. 14 we have:

$$\frac{1}{2}E_i\left(\frac{1}{X_i}\frac{dX_i}{d\theta}\right)^2 (d\theta)^2 = \frac{1}{X_1 X_2} 8\left[\frac{1}{2}K_h (d\theta)^2\right] \tag{15}$$

and hence the on-axis Young's moduli E_i ($i = 1, 2$) are given by:

$$E_i = 8K_h \frac{X_i^2}{X_1 X_2}\left(\frac{dX_i}{d\theta}\right)^{-2} \qquad i = 1,2, \tag{16}$$

i.e.:

$$E_1 = 8K_h \frac{X_1}{X_2}\left(\frac{dX_1}{d\theta}\right)^{-2} = 8K_h\left(\frac{\left[a\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\theta}{2}\right)\right]}{\left[a\sin\left(\frac{\theta}{2}\right)+b\cos\left(\frac{\theta}{2}\right)\right]\left[-a\sin\left(\frac{\theta}{2}\right)+b\cos\left(\frac{\theta}{2}\right)\right]^2}\right),$$

(17)

$$E_2 = 8K_h \frac{X_2}{X_1}\left(\frac{dX_2}{d\theta}\right)^{-2} = 8K_h\left(\frac{\left[a\sin\left(\frac{\theta}{2}\right)+b\cos\left(\frac{\theta}{2}\right)\right]}{\left[a\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\theta}{2}\right)\right]\left[a\cos\left(\frac{\theta}{2}\right)-b\sin\left(\frac{\theta}{2}\right)\right]^2}\right).$$

In the particular case when $a = b$ (*i.e.* the rectangle is a square), the on-axis Young's moduli simplify to:

$$E = E_1 = E_2 = K_h \frac{8}{a^2}\left[\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right]^{-2} = K_h \frac{8}{a^2}\frac{1}{\left[1-\sin(\theta)\right]}. \tag{18}$$

These equations for the Poisson's ratios and Young's moduli for both the general case of the rectangles and the particular case of the squares satisfy the thermodynamic requirements given by:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \qquad \left|\nu_{ij}\right| \le \sqrt{\frac{E_i}{E_j}}. \tag{19}$$

Plots of E_i and ν_{ij} *vs* θ for a rectangle measuring $(a \times b)$ = (1.5 × 1.0) with K_h = 1 are given in Fig. 3.

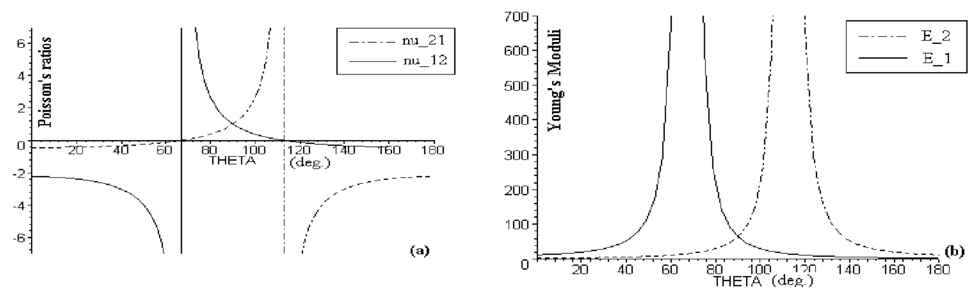

Fig. 3. Plots of (a) the Poisson's ratios and (b) Young's moduli of the rotating rectangles structure where $(a \times b)$ = (1.5 × l) and K_h= 1

3. DISCUSSION

The expressions derived above suggest that for systems composed of hinged rigid rectangles connected together as shown in Fig. 1 and Fig. 2, the Poisson's ratios are dependent on the geometry of the system and that the same structure can exhibit both positive and negative Poisson's ratios. Also, since a finite strain in any of the two Ox_i directions has to be accompanied by a change in the angle θ, and since Eq. 9 suggests that for a given system of $a \times b$ rectangles where $a \neq b$, the Poisson's ratio is dependent on θ, then the Poisson's ratio will also be strain dependent and will change when uniaxial loads are applied. In particular, it is also possible that the Poisson's ratio will change sign as the system is being loaded. For example, referring to Fig. 3a, loading in the Ox_2 direction of a system made from rectangles of size $(a \times b) = (1.5 \times 1)$ with $\theta_{\text{initial}} = 40°$ will initially exhibit auxetic behaviour but then start exhibiting conventional after the θ will become greater than 67.4°. In fact, for systems such as the one discussed here where the geometry is dependent on a single variable (=θ), the Poisson's ratios v_{ij} are given by:

$$v_{ij} = -\frac{d\varepsilon_j}{d\varepsilon_i} = -\frac{dX_j/d\theta}{dX_i/d\theta} \cdot \frac{X_i}{X_j} \qquad i,j = 1,2 \tag{20}$$

and for negative Poisson's ratios, we require:

$$\frac{dX_j/d\theta}{dX_i/d\theta} \cdot \frac{X_i}{X_j} > 0\,.$$

Since the unit cells are always positive (*i.e.* $\frac{X_i}{X_j} > 0$ to all values of θ), this requirement reduces to the requirement that the two derivatives $\frac{dX_1}{d\theta}$ and $\frac{dX_2}{d\theta}$ have the same sign (both positive or both negative), *i.e.* for this particular case, for negative Poisson's ratios it is required that either:

$$-a\sin\left(\frac{\theta}{2}\right) + b\cos\left(\frac{\theta}{2}\right) > 0 \quad \text{and} \quad a\cos\left(\frac{\theta}{2}\right) - b\sin\left(\frac{\theta}{2}\right) > 0$$

or: (21)

$$-a\sin\left(\frac{\theta}{2}\right) + b\cos\left(\frac{\theta}{2}\right) < 0 \quad \text{and} \quad a\cos\left(\frac{\theta}{2}\right) - b\sin\left(\frac{\theta}{2}\right) < 0\,,$$

i.e. for $\theta \in (0,\pi)$:

$$\frac{b}{a} > \tan\left(\frac{\theta}{2}\right) \quad \text{and} \quad \frac{a}{b} > \tan\left(\frac{\theta}{2}\right) \quad \Rightarrow \quad 0 < \theta < 2\tan^{-1}\left[\min\left(\frac{a}{b}, \frac{b}{a}\right)\right]$$

or:

$$\frac{b}{a} < \tan\left(\frac{\theta}{2}\right) \quad \text{and} \quad \frac{a}{b} < \tan\left(\frac{\theta}{2}\right) \quad \Rightarrow \quad 2\tan^{-1}\left[\max\left(\frac{a}{b},\frac{b}{a}\right)\right] < \theta < \pi \,. \tag{22}$$

The Poisson's ratios are positive for the other values of $\theta \in (0,\pi)$, *i.e.:*

$$2\tan^{-1}\left[\min\left(\frac{a}{b},\frac{b}{a}\right)\right] < \theta < 2\tan^{-1}\left[\max\left(\frac{a}{b},\frac{b}{a}\right)\right] \tag{23}$$

This means that irrespective of the size of the rectangle, if we load the fully closed structure ($\theta = 0$) the structure is initially auxetic until θ reaches **the value** of $2\tan^{-1}[\min(a/b, b/a)]$ where the Poisson's ratio becomes positive and remains positive until θ is $2\tan^{-1}[\max(a/b, b/a)]$ when it becomes negative again and remains negative until $\theta = \pi$, a conformation were the structure is once again fully closed.

All this, is in sharp contrast to the system constructed from rigid squares where the Poisson's ratios were strain independent and always equal to -1 [16]. In this respect it should be noted that the region of negative Poisson's ratio for the 'rotating rectangles model' may be increased by decreasing the difference between a and b, and in the limit when $a = b$ (*i.e.* the rectangle is a square), the Poisson's ratio will be negative for all values of θ and equal to -1. However, this increase in the range will occur on the expense of a decrease in the magnitude of the negative Poisson's ratios (the rotating rectangles can exhibit Poisson's ratios which are less than -1). Also, it is important to realise that unless $a = b$, the extent of auxeticity will be dependent on the direction of loading. (The off-axis mechanical properties may be obtained may be obtained by using standard axis transformation techniques [28].)

Finally, it should also be noted that in this discussion we have assumed that the rectangles in the systems are perfectly rigid. As for in the case of the rotating squares, the values of the Poisson's ratios for real systems are likely to be effected by the relative rigidity of the rectangles with respect to the rigidity of the hinges [9, 13, 19] and the auxetic behaviour may become less pronounced as the rectangles loose their rigidity.

4. CONCLUSION

In this paper we have shown that auxetic behaviour may be achieved from rigid rectangles which rotate relative to each other. This new model offers the advantage over the earlier model constructed using squares since:

- This model is more general (all squares are special rectangles, but not *vice-versa)* and may hence it be applied to a wider range of systems.
- Systems constructed from rectangles can exhibit both positive and negative Poisson's ratios (depending on the angle between the rectangles). Rotating squares can only exhibit negative Poisson's ratios.

- Systems constructed from rectangles can exhibit Poisson's ratios which more negative than -1 that can be obtained from rotating squares.

We have shown through analytical modelling that in this case of 'rotating rigid rectangles', the extent of auxeticity depends, amongst other things, on the actual geometry of the system and the strains the system is being subjected to. It also argued that for real systems, the extent of auxeticity will also be dependent on the direction of loading and relative rigidity of rectangles when compared to the hinges. All this is very significant and different from more particular case when the system is constructed from rigid squares [16] and can provide us with a clearer understanding of the potential of rotating rigid units for generating auxetic behaviour.

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