THREE- AND FOUR-STAGE IMPLICIT INTERVAL METHODS OF RUNGE-KUTTA TYPE

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Abstract: The paper presents three- and four-stage implicit interval methods of Runge-Kutta type and is a continuation of our previous paper [6] dealing with one- and two-stage methods of this kind. It is shown that the exact solution of the initial value problem belongs to interval-solutions obtained by both kinds of these methods (three- and four-stage). We also present some approximations of the widths of interval-solutions.

1. INTRODUCTION

Interval methods for solving the initial value problem are interesting due to interval-solutions obtained by such methods which contain their errors. Computer implementations of interval methods in floating-point interval arithmetic together with the representation of initial data in the form of minimal machine intervals, i.e. by intervals which ends are equal or neighboring machine numbers, yield interval solutions which contain all possible numerical errors.

In this paper we consider three- and four-stage implicit interval methods of Runge-Kutta type, which are presented in Sections 3 and 4. As we proved in [6], the exact solution of the initial value problem belongs to interval-solutions obtained by both kinds of these methods. In Sect. 5 some approximations of the widths of interval-solution are given.

2. THE INITIAL VALUE PROBLEM AND CONVENTIONAL RUNGE-KUTTA METHODS

As is well-known (see e.g. [5]), the initial value problem consists in finding the function $y - y(x)$, such that

$$y' = f(t, y(t)), \quad y(0) = y_0,$$

(1)

where $t \in [0, T]$, $y \in \mathbf{R}^N$ and $f : [0, T] \times \mathbf{R}^N \to \mathbf{R}^N$. We will assume that the solution of (1) exists and is unique.

To carry out a single step by a conventional, m-stage Runge-Kutta method we apply the formula (see e.g. [1])

$$y_{k+1} = y_k + h \sum_{i=1}^{m} w_i \kappa_i, \quad k = 0, 1, \ldots,$$

(2)

where

$$\kappa_i = f(t_k + c_i h, y_k + h \sum_{j=1}^{s} a_{ij} \kappa_j), \quad i = 1, 2, \ldots, m, \tag{3}$$

$$\sum_{i=1}^{m} w_i = 1, \quad c_i = \sum_{j=1}^{s} a_{ij}, \tag{4}$$

and where $s = i - 1$ for an explicit method, and $s = m$ for an implicit one. The set of numbers w_i, c_i, a_{ij} are constants which characterize a particular method.

The local truncation error of step $k + 1$ for a Runge-Kutta method (explicit and implicit) of order p can be written in the form (see e.g. [1], [3] or [5])

$$r_{k+1}(h) = \psi(t_k, y(t_k))h^{p+1} + O(h^{p+2})$$
$$= r_{k+1}^{(p+1)}(0)\frac{h^{p+1}}{(p+1)!} + r_{k+1}^{(p+2)}(\theta h)\frac{h^{p+2}}{(p+2)!}, \tag{5}$$

where

$$0 < \theta < 1, \quad \left| \frac{r_{k+1}^{(p+2)}(\theta h)}{(p+2)!} \right| \le M. \tag{6}$$

This error is equal to the difference between the exact value $y(t_k + h)$ and its approximation evaluated on the basis of the exact value $y(t_k)$. The function $\psi(t, y(t))$ depends on coefficients w_i, c_i, a_{ij}, and on partial derivatives of the function $f(t, y)$ occuring in (1). The form of $\psi(t, y(t))$ is very complicated and cannot be written in a general form for an arbitrary p (see e.g. [1], [5] or [8]).

3. THREE-STAGE IMPLICIT INTERVAL METHODS

Let us denote:
A, and A   - sets in which the function $f(t, y)$ is defined, i.e.

$$\Delta_t = \{ t \in \mathbf{R}: 0 \le t \le a \},$$
$$\Delta_y = \{ y = (y_1, y_2, \ldots, y_N)^T \in \mathbf{R}^N : \underline{b}_i \le y_i \le \overline{b}_i, \quad i = 1, 2, \ldots, N \},$$

$F(T,Y)$ - an interval extension of $f(t, y)$,
    $\psi(T, Y)$ - an interval extension of $\psi(t, y)$ (see (5)),
and let us assume that:
· the function $F(T,Y)$ is defined and continuous for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
· the function $F(T,Y)$ is monotonic with respect to inclusion, i.e.

$$T_1 \subset T_2 \wedge Y_1 \subset Y_2 \Rightarrow F(T_1, Y_1) \subset F(T_2, Y_2),$$

- for each $T \subset \Delta_t$ and for each $Y \subset \Delta_y$ there exists a constant $L > 0$ such that

$$d(F(T, Y)) \leq L(d(T) + d(Y)),$$

where $d(A)$ denotes the width of A (if $A_1, A_2, ... A_N$ $)^T$, then the number $d(A)$ is defined by $d(A) = \max\limits_{i = 1, 2, ..., N} d(A_i)$),

- the function $\Psi(T, Y)$ is defined for all $T \subset \Delta_t$ and $Y \subset \Delta_y$,
- the function $\Psi(T, Y)$ is monotonie with respect to inclusion.

From the theory of conventional Runge-Kutta methods it is known that two-stage implicit methods, which are characterized by the set of coefficients w_i, c_i, a_{ij}, where $i, j = 1, 2, 3$, can have order up to six, and the maximum order condition $(p = 6)$ is achieved if (see e.g. [1] or [3])

$$w_1 = w_3 = \frac{5}{18}, \quad w_2 = \frac{4}{9}, \quad c_1 = \frac{1}{2} - \frac{\sqrt{15}}{10}, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{2} + \frac{\sqrt{15}}{10},$$

$$a_{11} = a_{33} = \frac{5}{36}, \quad a_{12} = \frac{2}{9} - \frac{\sqrt{15}}{15}, \quad a_{13} = \frac{5}{36} - \frac{\sqrt{15}}{30},$$

$$a_{21} = \frac{5}{36} + \frac{\sqrt{15}}{24}, \quad a_{22} = \frac{2}{9}, \quad a_{23} = \frac{5}{36} - \frac{\sqrt{15}}{24}, \quad a_{31} = \frac{5}{36} + \frac{\sqrt{15}}{30}, \quad a_{32} = \frac{2}{9} + \frac{\sqrt{15}}{15}.$$

In general, for $t_o = 0$ and $y_0 \in Y_0$ three-stage implicit interval method of Runge-Kutta type we define as follows:

$$Y_n(t_0) = Y_n(0) = Y_0,$$
$$Y_n(t_{k+1}) = Y_n(t_k) + h\left(w_1 K_{1,k}(h) + w_2 K_{2,k}(h) + w_3 K_{1,k}(h)\right)$$
$$\qquad + \left(\Psi(T_k, Y_n(t_k)) + [-\alpha, \alpha]\right) h^{p+1}, \tag{7}$$
$$k = 0, 1, ..., n - 1,$$

where $p \leq 6$, and

$$K_{i,k}(h) = F\left(T_k + c_i h, Y_n(t_k) + h\left(a_{i1} K_{1,k}(h) + a_{i2} K_{2,k}(h) + a_{i3} K_{3,k}(h)\right)\right),$$
$$i = 1, 2, 3, \quad \alpha = M h_0. \tag{8}$$

The step-size h of the method (7), where $0 < h \leq h_0$, h_0 denotes a given number, is calculated from the formula

$$h = \frac{\eta_3^*}{n}, \tag{9}$$

where

$$\eta_3^* = \min\left\{\eta_0, \eta_1, \eta_2, \eta_3\right\}, \tag{10}$$

The numbers $\eta_i > 0$ $(i = 1, 2, 3)$ are evaluated in such a way that

$$Y_0 + \eta_i \, c_i \, F(\Delta_t, \Delta_y) \subset \Delta_y, \quad i = 1, 2, 3, \tag{11}$$

and the number η_0 — from the relation

$$\begin{aligned} Y_0 + \eta_0 &\left(w_1 F(\Delta_t, \Delta_y) + w_2 F(\Delta_t, \Delta_y) + w_3 F(\Delta_t, \Delta_y) \right) \\ &+ \left(\Psi(\Delta_t, \Delta_y) + [-\alpha, \alpha] \right) h_0^p \subset \Delta_y, \end{aligned} \tag{12}$$

for $Y_0 \subset \Delta_y$ and $y_0 \in Y_0$.

The interval $[0, \eta_3^*]$ we divide into n parts by the points $t_k = kh$ $(k = 0, 1, \ldots, n)$, and the intervals T_k, which appear in the method (7) - (8), we choose in such a way that

$$t_k = kh \in T_k \subset [0, \eta_3^*].$$

From (8) it follows that in each step of the method we have to solve a (vector) interval equation of the form

$$X = G(T, X),$$

where

$$T \in \Delta_t \subset I(\mathbf{R}), \; X = (X_1, X_2, \ldots, X_N)^T \in I(\Delta_y) \subset I(\mathbf{R}^N),$$

$$G : I(\Delta_t) \times I(\Delta_y) \to I(\mathbf{R}^N).$$

If we assume that the function G is a contraction mapping, then the corresponding iteration process follows from the fixed-point theorem.

4. FOUR-STAGE IMPLICIT INTERVAL METHODS

Four-stage implicit methods, which are characterized by the set of coefficients w_i, c_i, a_{ij}, $(i, j - 1, 2, 3, 4)$, can have order up to eight, and if (see e.g. [3])

$$w_1 = w_4 = \frac{1}{4} - \frac{\sqrt{30}}{72}, \quad w_2 = w_3 = \frac{1}{4} + \frac{\sqrt{30}}{72},$$

$$c_1 = \frac{1}{2} - A_1, \quad c_2 = \frac{1}{2} - B_1, \quad c_3 = \frac{1}{2} + B_1, \quad c_4 = \frac{1}{2} + A_1,$$

$$a_{11} = a_{44} = \frac{w_1}{2}, \quad a_{12} = \frac{w_2}{2} - A_2 + B_3, \quad a_{13} = \frac{w_2}{2} - A_2 - B_3, \quad a_{14} = \frac{w_1}{2} - A_4,$$

$$a_{21} = \frac{w_1}{2} - B_2 + A_3, \quad a_{22} = a_{33} = \frac{w_2}{2}, \quad a_{23} = \frac{w_2}{2} - B_4, \quad a_{24} = \frac{w_1}{2} - B_2 - A_3,$$

$$a_{31} = \frac{w_1}{2} + B_2 + A_3, \quad a_{32} = \frac{w_2}{2} + B_4, \quad a_{34} = \frac{w_1}{2} + B_2 - A_3,$$

$$a_{41} = \frac{w_1}{2} + A_4, \quad a_{42} = \frac{w_2}{2} + A_2 + B_3, \quad a_{43} = \frac{w_2}{2} + A_2 - B_3,$$

where

$$A_1 = \frac{1}{2}\sqrt{\frac{15+2\sqrt{30}}{35}}, \quad A_2 = A_1\left(\frac{1}{6}+\frac{\sqrt{30}}{24}\right), \quad A_3 = A_1\left(\frac{1}{21}+\frac{5\sqrt{30}}{168}\right), \quad A_4 = A_1 - 2A_2,$$

$$B_1 = \frac{1}{2}\sqrt{\frac{15-2\sqrt{30}}{35}}, \quad B_2 = A_1\left(\frac{1}{6}-\frac{\sqrt{30}}{24}\right), \quad B_3 = B_1\left(\frac{1}{21}-\frac{5\sqrt{30}}{168}\right), \quad B_4 = B_1 - 2B_2,$$

then the method is of the maximum order ($p = 8$).

For $t_0=0$ and $y_0 \in Y_0$ four-stage implicit interval methods are defined by the following set of formulas:

$$Y_n(t_0) = Y_n(0) = Y_0,$$
$$Y_n(t_{k+1}) = Y_n(t_k) + h\left(w_1 K_{1,k}(h) + w_2 K_{2,k}(h) + w_3 K_{3,k}(h) + w_4 K_{4,k}(h)\right)$$
$$\hspace{2cm} + \left(\Psi(T_k, Y_n(t_k)) + [-\alpha, \alpha]\right)h^{p+1}, \tag{13}$$
$$k = 0, 1, \ldots, n-1,$$

where $p \le 8$, and

$$K_{i,k}(h) = F\left(T_k + c_i h,\right.$$
$$\left. Y_n(t_k) + h\left(a_{i1} K_{1,k}(h) + a_{i2} K_{2,k}(h) + a_{i3} K_{3,k}(h) + a_{i4} K_{4,k}(h)\right)\right), \tag{14}$$
$$i = 1, 2, 3, 4, \quad \alpha = M h_0.$$

The step-size h, $0 < h \le h_0$, where h_0 is given, can be found from the formula

$$h = \frac{\eta_4^*}{n}, \tag{15}$$

where

$$\eta_4^* = \min\left\{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4\right\}. \tag{16}$$

The numbers $\eta_i > 0$ $(i = 1, 2, 3, 4)$ should fulfilled the conditions

$$Y_0 + \eta_i c_i F(\Delta_t, \Delta_y) \subset \Delta_y, \quad i = 1, 2, 3, 4, \tag{17}$$

and the number η_0 should be choosen in such a way that

$$Y_0 + \eta_0\left(w_1 F(\Delta_t, \Delta_y) + w_2 F(\Delta_t, \Delta_y) + w_3 F(\Delta_t, \Delta_y) + w_4 F(\Delta_t, \Delta_y)\right)$$
$$+ \left(\Psi(\Delta_t, \Delta_y) + [-\alpha, \alpha]\right)h_0^p \subset \Delta_y, \tag{18}$$

for $Y_0 \subset \Delta_y$ and $y_0 \in Y_0$.

In [6] we proved (Theorem 1 in Sect. 5) that for one- and two-stage implicit interval methods the exact solution of the initial value problem (1) belonged to the intervals obtained by these methods. Since in that theorem there are no restrictions to one- or two-stage methods, the same conclusion we get for the methods (7) - (8) and (13) - (14).

5. WIDTHS OF INTERVAL SOLUTIONS

First, let us consider the widths of intervals $K_{i,k}(h)$ given by (8) and (14). From these formulas and properties of the function F (see Sect. 3) it follows that

$$d(K_{i,k}(h)) \le L\{d(T_k) + d(Y_n(t_k))\} + hL \sum_{j=1}^{m} |a_{ij}| d(K_{j,k}(h)), \qquad (19)$$

where $m = 3$, $i = 1, 2, 3$ for the method (7) - (8), and $m = 4$, $i = 1, 2, 3, 4$ for the method (13) - (14). The inequalities (19) are of the form

$$(1 - \alpha_{ii}) x_i - \sum_{\substack{j=1 \\ j \ne i}}^{m} \alpha_{ij} x_j \le \beta, \quad i = 1, 2, \dots, m. \qquad (20)$$

For $m = 3$ we get the following solution of (20):

$$x_1 \le \frac{\beta}{W_3}[(1 - \alpha_{22})(1 - \alpha_{33}) + (1 + \alpha_{23} - \alpha_{33})\alpha_{12} + (1 + \alpha_{32} - \alpha_{22})\alpha_{13} - \alpha_{23}\alpha_{32}],$$

$$x_2 \le \frac{\beta}{W_3}[(1 - \alpha_{11})(1 - \alpha_{33}) + (1 + \alpha_{13} - \alpha_{33})\alpha_{21} + (1 + \alpha_{31} - \alpha_{11})\alpha_{23} - \alpha_{13}\alpha_{31}],$$

$$x_3 \le \frac{\beta}{W_3}[(1 - \alpha_{11})(1 - \alpha_{22}) + (1 + \alpha_{12} - \alpha_{22})\alpha_{31} + (1 + \alpha_{21} - \alpha_{11})\alpha_{32} - \alpha_{12}\alpha_{21}],$$

where
$$W_3 = (1 - \alpha_{11})(1 - \alpha_{22})(1 - \alpha_{33}) - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32}$$
$$-(1 - \alpha_{11})\alpha_{23}\alpha_{32} - (1 - \alpha_{22})\alpha_{13}\alpha_{31} - (1 - \alpha_{33})\alpha_{12}\alpha_{21},$$

and where
$$1 - \alpha_{11} > 0, \quad 1 - \alpha_{22} > 0, \quad 1 - \alpha_{33} > 0,$$
$$(1 - \alpha_{11})(1 - \alpha_{22}) - \alpha_{12}\alpha_{21} > 0, \quad (1 - \alpha_{11})(1 - \alpha_{33}) - \alpha_{13}\alpha_{31} > 0,$$
$$(1 - \alpha_{22})(1 - \alpha_{33}) - \alpha_{23}\alpha_{32} > 0, \quad W_3 > 0.$$

For $m = 4$ the solution of (20) is of the form

$$x_1 \le \frac{\beta}{W_4}\{(1 - \alpha_{22})(1 - \alpha_{33})(1 - \alpha_{44})$$

$$+[1 + (1 + \alpha_{34} - \alpha_{44})\alpha_{23} + (1 + \alpha_{43} - \alpha_{33})\alpha_{24} - (1 - \alpha_{44})\alpha_{33} - \alpha_{34}\alpha_{43} - \alpha_{44}]\alpha_{12}$$

$$+[1+(1+\alpha_{24}-\alpha_{44})\alpha_{32}+(1+\alpha_{42}-\alpha_{22})\alpha_{34}-(1-\alpha_{44})\alpha_{22}-\alpha_{24}\alpha_{42}-\alpha_{44}]\alpha_{13}$$
$$+[1+(1+\alpha_{23}-\alpha_{33})\alpha_{42}+(1+\alpha_{32}-\alpha_{22})\alpha_{43}-(1-\alpha_{33})\alpha_{22}-\alpha_{23}\alpha_{32}-\alpha_{33}]\alpha_{14}$$
$$-(1-\alpha_{44})\alpha_{23}\alpha_{32}-(1-\alpha_{33})\alpha_{24}\alpha_{42}-(1-\alpha_{22})\alpha_{34}\alpha_{43}$$
$$-\alpha_{23}\alpha_{34}\alpha_{42}-\alpha_{24}\alpha_{32}\alpha_{43}\},$$

$$x_2\le\frac{\beta}{W_4}\{(1-\alpha_{11})(1-\alpha_{33})(1-\alpha_{44})$$
$$+[1+(1+\alpha_{34}-\alpha_{44})\alpha_{13}+(1+\alpha_{43}-\alpha_{33})\alpha_{14}-(1-\alpha_{44})\alpha_{33}-\alpha_{34}\alpha_{43}-\alpha_{44}]\alpha_{21}$$
$$+[1+(1+\alpha_{14}-\alpha_{44})\alpha_{31}+(1+\alpha_{41}-\alpha_{11})\alpha_{34}-(1-\alpha_{44})\alpha_{11}-\alpha_{14}\alpha_{41}-\alpha_{44}]\alpha_{23}$$
$$+[1+(1+\alpha_{13}-\alpha_{33})\alpha_{41}+(1+\alpha_{31}-\alpha_{11})\alpha_{43}-(1-\alpha_{33})\alpha_{11}-\alpha_{13}\alpha_{31}-\alpha_{33}]\alpha_{24}$$
$$-(1-\alpha_{44})\alpha_{13}\alpha_{31}-(1-\alpha_{33})\alpha_{14}\alpha_{41}-(1-\alpha_{11})\alpha_{34}\alpha_{43}$$
$$-\alpha_{13}\alpha_{34}\alpha_{41}-\alpha_{14}\alpha_{31}\alpha_{43}\},$$

$$x_3\le\frac{\beta}{W_4}\{(1-\alpha_{11})(1-\alpha_{22})(1-\alpha_{44}) \tag{22}$$
$$+[1+(1+\alpha_{24}-\alpha_{44})\alpha_{12}+(1+\alpha_{42}-\alpha_{22})\alpha_{14}-(1-\alpha_{44})\alpha_{22}-\alpha_{24}\alpha_{42}-\alpha_{44}]\alpha_{31}$$
$$+[1+(1+\alpha_{14}-\alpha_{44})\alpha_{21}+(1+\alpha_{41}-\alpha_{11})\alpha_{24}-(1-\alpha_{44})\alpha_{11}-\alpha_{14}\alpha_{41}-\alpha_{44}]\alpha_{32}$$
$$+[1+(1+\alpha_{12}-\alpha_{22})\alpha_{41}+(1+\alpha_{21}-\alpha_{11})\alpha_{42}-(1-\alpha_{11})\alpha_{22}-\alpha_{12}\alpha_{21}-\alpha_{11}]\alpha_{34}$$
$$-(1-\alpha_{44})\alpha_{12}\alpha_{21}-(1-\alpha_{11})\alpha_{24}\alpha_{42}-(1-\alpha_{22})\alpha_{14}\alpha_{41}$$
$$-\alpha_{12}\alpha_{24}\alpha_{41}-\alpha_{14}\alpha_{21}\alpha_{42}\},$$

$$x_4\le\frac{\beta}{W_4}\{(1-\alpha_{11})(1-\alpha_{22})(1-\alpha_{33})$$
$$+[1+(1+\alpha_{23}-\alpha_{33})\alpha_{12}+(1+\alpha_{32}-\alpha_{22})\alpha_{13}-(1-\alpha_{33})\alpha_{22}-\alpha_{23}\alpha_{32}-\alpha_{33}]\alpha_{41}$$
$$+[1+(1+\alpha_{13}-\alpha_{33})\alpha_{21}+(1+\alpha_{31}-\alpha_{11})\alpha_{23}-(1-\alpha_{33})\alpha_{11}-\alpha_{13}\alpha_{31}-\alpha_{33}]\alpha_{42}$$
$$+[1+(1+\alpha_{12}-\alpha_{22})\alpha_{31}+(1+\alpha_{21}-\alpha_{11})\alpha_{32}-(1-\alpha_{11})\alpha_{22}-\alpha_{12}\alpha_{21}-\alpha_{11}]\alpha_{43}$$
$$-(1-\alpha_{11})\alpha_{23}\alpha_{32}-(1-\alpha_{22})\alpha_{13}\alpha_{31}-(1-\alpha_{33})\alpha_{12}\alpha_{21}$$
$$-\alpha_{12}\alpha_{23}\alpha_{31}-\alpha_{13}\alpha_{21}\alpha_{32}\},$$

where

$$W_4=(1-\alpha_{11})(1-\alpha_{22})(1-\alpha_{33})(1-\alpha_{44})$$
$$-(1-\alpha_{44})[(1-\alpha_{11})\alpha_{23}\alpha_{32}+(1-\alpha_{22})\alpha_{13}\alpha_{31}+(1-\alpha_{33})\alpha_{12}\alpha_{21}]$$
$$-(1-\alpha_{11})(\alpha_{23}\alpha_{34}\alpha_{42}+\alpha_{24}\alpha_{32}\alpha_{43})-(1-\alpha_{22})(\alpha_{13}\alpha_{34}\alpha_{41}+\alpha_{14}\alpha_{31}\alpha_{43})$$
$$-(1-\alpha_{33})(\alpha_{12}\alpha_{24}\alpha_{41}+\alpha_{14}\alpha_{21}\alpha_{42})-(1-\alpha_{44})(\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{13}\alpha_{21}\alpha_{32})$$
$$-\alpha_{14}\alpha_{41}[(1-\alpha_{22})(1-\alpha_{33})-\alpha_{23}\alpha_{32}]-\alpha_{24}\alpha_{42}[(1-\alpha_{11})(1-\alpha_{33})-\alpha_{13}\alpha_{31})]$$
$$-\alpha_{34}\alpha_{43}[(1-\alpha_{11})(1-\alpha_{22})-\alpha_{12}\alpha_{21})]$$
$$-\alpha_{12}(\alpha_{23}\alpha_{34}\alpha_{41}+\alpha_{24}\alpha_{31}\alpha_{43})-\alpha_{13}(\alpha_{21}\alpha_{34}\alpha_{42}+\alpha_{24}\alpha_{32}\alpha_{41})$$
$$-\alpha_{14}(\alpha_{21}\alpha_{32}\alpha_{43}+\alpha_{23}\alpha_{31}\alpha_{42}),$$

and where

$$1 - \alpha_{11} > 0, \quad 1 - \alpha_{22} > 0, \quad 1 - \alpha_{33} > 0, \quad 1 - \alpha_{44} > 0,$$

$$(1 - \alpha_{11})(1 - \alpha_{22}) - \alpha_{12}\alpha_{21} > 0, \quad (1 - \alpha_{11})(1 - \alpha_{33}) - \alpha_{13}\alpha_{31} > 0,$$

$$(1 - \alpha_{11})(1 - \alpha_{44}) - \alpha_{14}\alpha_{41} > 0, \quad (1 - \alpha_{22})(1 - \alpha_{33}) - \alpha_{23}\alpha_{32} > 0,$$

$$(1 - \alpha_{22})(1 - \alpha_{44}) - \alpha_{24}\alpha_{42} > 0, \quad (1 - \alpha_{33})(1 - \alpha_{44}) - \alpha_{34}\alpha_{43} > 0,$$

$$(1 - \alpha_{11})(1 - \alpha_{22})(1 - \alpha_{33}) - (1 - \alpha_{11})\alpha_{23}\alpha_{32} - (1 - \alpha_{22})\alpha_{13}\alpha_{31}$$
$$\quad - (1 - \alpha_{33})\alpha_{12}\alpha_{21} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{13}\alpha_{21}\alpha_{32} > 0,$$

$$(1 - \alpha_{11})(1 - \alpha_{22})(1 - \alpha_{44}) - (1 - \alpha_{11})\alpha_{24}\alpha_{42} - (1 - \alpha_{22})\alpha_{14}\alpha_{41}$$
$$\quad - (1 - \alpha_{44})\alpha_{12}\alpha_{21} - \alpha_{12}\alpha_{24}\alpha_{41} - \alpha_{14}\alpha_{21}\alpha_{42} > 0,$$

$$(1 - \alpha_{11})(1 - \alpha_{33})(1 - \alpha_{44}) - (1 - \alpha_{11})\alpha_{34}\alpha_{43} - (1 - \alpha_{33})\alpha_{14}\alpha_{41}$$
$$\quad - (1 - \alpha_{44})\alpha_{13}\alpha_{31} - \alpha_{13}\alpha_{34}\alpha_{41} - \alpha_{14}\alpha_{31}\alpha_{43} > 0,$$

$$(1 - \alpha_{22})(1 - \alpha_{33})(1 - \alpha_{44}) - (1 - \alpha_{22})\alpha_{34}\alpha_{43} - (1 - \alpha_{33})\alpha_{24}\alpha_{42}$$
$$\quad - (1 - \alpha_{44})\alpha_{23}\alpha_{32} - \alpha_{23}\alpha_{34}\alpha_{42} - \alpha_{24}\alpha_{32}\alpha_{43} > 0,$$

$$W_4 > 0.$$

Using the above results we can estimate the widths of interval solutions for implict three- and four-stage methods presented in the previous sections.

Theorem 1. *If Y_n (t_k) $(k = 0, 1, \ldots, n)$ are obtained on the basis of the method (7) - (8), then for h_0 such that*

$$h_0 < \min \left\{ 1, \frac{1}{L|a_{11}|}, \frac{1}{L|a_{22}|}, \frac{1}{L|a_{33}|}, \frac{1}{L(|a_{11}| + |a_{22}|) + L^2|a_{12}||a_{21}|}, \right.$$

$$\frac{1}{L(|a_{11}| + |a_{33}|) + L^2|a_{13}||a_{31}|}, \tag{23}$$

$$\left. \frac{1}{L(|a_{22}| + |a_{33}|) + L^2|a_{23}||a_{32}|}, \frac{1}{L\Omega_1 + L^2\Omega_2 + L^3\Omega_3} \right\},$$

where

$$\Omega_1 = |a_{11}| + |a_{22}| + |a_{33}|,$$
$$\Omega_2 = |a_{12}||a_{21}| + |a_{13}||a_{31}| + |a_{23}||a_{32}|,$$
$$\Omega_3 = |a_{11}||a_{22}||a_{33}| + |a_{12}||a_{23}||a_{31}| + |a_{13}||a_{21}||a_{32}|,$$

we have

$$d(Y_n(t_k)) \le Qh^p + Rd(Y_0) + S \cdot \max_{l=1, 2, \ldots, n} d(T_l), \tag{24}$$

where Q, R and S denote some nonnegative constants, and $p \le 6$.

Proof. From (7) we get

$$
\begin{aligned}
d(Y_n(t_{k+1})) \le\ & d(Y_n(t_k)) \\
& + h\Big(\big|w_1\big| d(K_{1k}(h)) + \big|w_2\big| d(K_{2k}(h)) + \big|w_3\big| d(K_{3k}(h))\Big) \\
& + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha] h^{p+1},
\end{aligned}
\tag{25}
$$

where $p \le 6$, $k = 0, 1, \ldots, n-1$. On the basis of (8) we have

$$
\begin{aligned}
d(K_{i,k}(h)) \le\ & L[d(T_k) + d(Y_n(t_k))] \\
& + hL\Big(\big|a_{i1}\big| d(K_{1k}(h)) + hL\big|a_{i2}\big| d(K_{2k}(h)) + hL\big|a_{i3}\big| d(K_{3k}(h))\Big), \\
& i = 1, 2, 3.
\end{aligned}
\tag{26}
$$

From (21) it follows that if

$$
\begin{aligned}
& 1 - hL\big|a_{ii}\big| > 0 \quad i = 1, 2, 3, \\
& \Big(1 - hL\big|a_{ii}\big|\Big)\Big(1 - hL\big|a_{jj}\big|\Big) - h^2 L^2 \big|a_{ij}\big|\,\big|a_{ji}\big| > 0, \quad i, j = 1, 2, 3, \quad i \ne j, \\
& \Big(1 - hL\big|a_{11}\big|\Big)\Big(1 - hL\big|a_{22}\big|\Big)\Big(1 - hL\big|a_{33}\big|\Big) \\
& \quad - h^2 L^2 \Big(\big|a_{12}\big|\,\big|a_{21}\big| + \big|a_{13}\big|\,\big|a_{31}\big| + \big|a_{23}\big|\,\big|a_{32}\big|\Big) \\
& \quad - h^3 L^3 \Big(\big|a_{12}\big|\,\big|a_{23}\big|\,\big|a_{31}\big| + \big|a_{13}\big|\,\big|a_{21}\big|\,\big|a_{32}\big| \\
& \qquad\qquad - \big|a_{11}\big|\,\big|a_{23}\big|\,\big|a_{32}\big| - \big|a_{13}\big|\,\big|a_{22}\big|\,\big|a_{31}\big| - \big|a_{12}\big|\,\big|a_{21}\big|\,\big|a_{33}\big|\Big) > 0,
\end{aligned}
\tag{27}
$$

then the solution of inequalities (26) is of the form

$$
d(K_{i,k}(h)) \le \frac{U_i}{U} L[d(T_k) + d(Y_n(t_k))], \quad i = 1, 2, 3,
\tag{28}
$$

where

$$
\begin{aligned}
U_i =\ & \Big(1 - hL\big|a_{jj}\big|\Big)\Big(1 - hL\big|a_{kk}\big|\Big) + \Big(1 + hL\big|a_{jk}\big| - hL\big|a_{kk}\big|\Big) hL\big|a_{ij}\big| \\
& + \Big(1 + hL\big|a_{kj}\big| - hL\big|a_{jj}\big|\Big) hL\big|a_{ik}\big| - h^2 L^2 \big|a_{jk}\big|\,\big|a_{kj}\big|, \\
& i, k, j = 1, 2, 3, \quad i \ne k, \quad k \ne j, \quad j \ne i, \\
U =\ & \Big(1 - hL\big|a_{11}\big|\Big)\Big(1 - hL\big|a_{22}\big|\Big)\Big(1 - hL\big|a_{33}\big|\Big) \\
& - h^2 L^2 \Big[\Big(1 - hL\big|a_{11}\big|\Big)\big|a_{23}\big|\,\big|a_{32}\big| + \Big(1 - hL\big|a_{22}\big|\Big)\big|a_{13}\big|\,\big|a_{31}\big| + \Big(1 - hL\big|a_{33}\big|\Big)\big|a_{12}\big|\,\big|a_{21}\big|\Big] \\
& - h^3 L^3 \Big(\big|a_{12}\big|\,\big|a_{23}\big|\,\big|a_{31}\big| + \big|a_{13}\big|\,\big|a_{21}\big|\,\big|a_{32}\big|\Big).
\end{aligned}
$$

Taking into account the assumption (23), the first six inequalities in (27) are self-evident. The last inequality in (27) is also an implication of this assumption, because from the inequality

$$h_0 < \frac{1}{L\Omega_1 + L^2\Omega_2 + L^3\Omega_3}$$

it follows that for $h \leq h_0$ we have

$$1 - hL\left(\left|a_{11}\right| + \left|a_{22}\right| + \left|a_{33}\right|\right) - hL^2\left(\left|a_{12}\right|\left|a_{21}\right| + \left|a_{13}\right|\left|a_{31}\right| + \left|a_{23}\right|\left|a_{32}\right|\right)$$
$$- hL^3\left(\left|a_{11}\right|\left|a_{22}\right|\left|a_{33}\right| + \left|a_{13}\right|\left|a_{23}\right|\left|a_{31}\right| + \left|a_{13}\right|\left|a_{21}\right|\left|a_{32}\right|\right) > 0.$$

Since $h < 1$ (as a consequence of $/_{i0} < 1$), then $h^7 < h$ and $h^4 < h$. Thus, from the last inequality it follows that

$$1 - hL\left(\left|a_{11}\right| + \left|a_{22}\right| + \left|a_{33}\right|\right) - h^2L^2\left(\left|a_{12}\right|\left|a_{21}\right| + \left|a_{13}\right|\left|a_{31}\right| + \left|a_{23}\right|\left|a_{32}\right|\right)$$
$$- h^3L^3\left(\left|a_{11}\right|\left|a_{22}\right|\left|a_{33}\right| + \left|a_{13}\right|\left|a_{23}\right|\left|a_{31}\right| + \left|a_{13}\right|\left|a_{21}\right|\left|a_{32}\right|\right) > 0$$

and hence — all the more —

$$1 - hL\left(\left|a_{11}\right| + \left|a_{22}\right| + \left|a_{33}\right|\right) - h^2L^2\left(\left|a_{12}\right|\left|a_{21}\right| + \left|a_{13}\right|\left|a_{31}\right| + \left|a_{23}\right|\left|a_{32}\right|\right)$$
$$+ h^2L^2\left(\left|a_{11}\right|\left|a_{22}\right| + \left|a_{11}\right|\left|a_{33}\right| + \left|a_{22}\right|\left|a_{33}\right|\right)$$
$$- h^3L^3\left(\left|a_{11}\right|\left|a_{22}\right|\left|a_{33}\right| + \left|a_{13}\right|\left|a_{23}\right|\left|a_{31}\right| + \left|a_{13}\right|\left|a_{21}\right|\left|a_{32}\right|\right)$$
$$+ h^3L^3\left(\left|a_{11}\right|\left|a_{23}\right|\left|a_{32}\right| + \left|a_{22}\right|\left|a_{13}\right|\left|a_{31}\right| + \left|a_{33}\right|\left|a_{12}\right|\left|a_{21}\right|\right) > 0.$$

But the left-hand side of this inequality is equal to the left-hand side of the last inequality in (27). Taking into account that $h \leq h_0$, from (28) we get

$$d(K_{1,k}(h)) \leq \frac{1 + h_0L\left[\left|a_{12}\right|\left(1 + h_0L\left|a_{23}\right|\right) + \left|a_{13}\right|\left(1 + h_0L\left|a_{32}\right|\right)\right] + h_0^2L^2\left|a_{22}\right|\left|a_{33}\right|}{\overline{U}}$$
$$\cdot L[d(T_k) + d(Y_n(t_k))],$$

$$d(K_{2,k}(h)) \leq \frac{1 + h_0L\left[\left|a_{21}\right|\left(1 + h_0L\left|a_{13}\right|\right) + \left|a_{23}\right|\left(1 + h_0L\left|a_{31}\right|\right)\right] + h_0^2L^2\left|a_{11}\right|\left|a_{33}\right|}{\overline{U}}$$
$$\cdot L[d(T_k) + d(Y_n(t_k))],$$

$$d(K_{3,k}(h)) \leq \frac{1 + h_0L\left[\left|a_{31}\right|\left(1 + h_0L\left|a_{12}\right|\right) + \left|a_{32}\right|\left(1 + h_0L\left|a_{21}\right|\right)\right] + h_0^2L^2\left|a_{11}\right|\left|a_{22}\right|}{\overline{U}}$$
$$\cdot L[d(T_k) + d(Y_n(t_k))],$$

where

$$\overline{U} = \left(1 - h_0L\left|a_{11}\right|\right)\left(1 - h_0L\left|a_{22}\right|\right)\left(1 - h_0L\left|a_{33}\right|\right)$$
$$- h_0^2L^2\left(\left|a_{12}\right|\left|a_{21}\right| + \left|a_{13}\right|\left|a_{31}\right| + \left|a_{23}\right|\left|a_{32}\right|\right) - h_0^3L^3\left(\left|a_{12}\right|\left|a_{23}\right|\left|a_{31}\right| + \left|a_{13}\right|\left|a_{21}\right|\left|a_{32}\right|\right).$$

Using these estimates, from the inequality (25) we obtain

$$d(Y_n(t_{k+1})) \le d(Y_n(t_k))(1 + \nu_3 hL) + \nu_3 hLd(T_k) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1},$$
$$k = 0, 1, \ldots, n-1,$$

(29)

where

$$\nu_3 = \frac{1}{U}\Big\{ |w_1|\Big(1 + h_0 L\big[|a_{12}|\big(1 + h_0 L|a_{23}|\big) + |a_{13}|\big(1 + h_0 L|a_{32}|\big)\big] + h_0^2 L^2 |a_{22}\|a_{33}|\Big)$$
$$+ |w_2|\Big(1 + h_0 L\big[|a_{21}|\big(1 + h_0 L|a_{13}|\big) + |a_{23}|\big(1 + h_0 L|a_{31}|\big)\big] + h_0^2 L^2 |a_{11}\|a_{33}|\Big)$$
$$+ |w_3|\Big(1 + h_0 L\big[|a_{31}|\big(1 + h_0 L|a_{12}|\big) + |a_{32}|\big(1 + h_0 L|a_{21}|\big)\big] + h_0^2 L^2 |a_{11}\|a_{22}|\Big)\Big\}.$$

From (29) it follows that

$$d(Y_n(t_1)) \le d(Y_n(t_0))(1 + \nu_3 hL) + \nu_3 hLd(T_0) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1},$$

$$d(Y_n(t_2)) \le d(Y_n(t_1))(1 + \nu_3 hL) + \nu_3 hLd(T_1) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1}$$
$$\le \Big(d(Y_n(t_0))(1 + \nu_3 hL) + \nu_3 hLd(T_0) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1}\Big)$$
$$\cdot (1 + \nu_3 hL) + \nu_3 hLd(T_1) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1}$$
$$= d(Y_n(t_0))(1 + \nu_3 hL)^2$$
$$+ \Big(\nu_3 hL \cdot \max_{l=0,1} d(T_l) + d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1}\Big)[1 + (1 + \nu_3 hL)],$$

$$d(Y_n(t_3)) \le d(Y_n(t_2))(1 + \nu_3 hL) + \nu_3 hLd(T_2) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1}$$
$$\le \Big[d(Y_n(t_0))(1 + \nu_3 hL)^2$$
$$+ \Big(\nu_3 hL \cdot \max_{l=0,1} d(T_l) + d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1}\Big)(1 + (1 + \nu_3 hL))\Big]$$
$$\cdot (1 + \nu_3 hL) + \nu_3 hLd(T_2) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1}$$
$$\le d(Y_n(t_0))(1 + \nu_3 hL)^3$$
$$+ \Big(\nu_3 hL \cdot \max_{l=0,1,2} d(T_l) + d(\Psi(\Delta_t, \Delta_y)) + 2\alpha]h^{p+1}\Big)$$
$$\cdot \Big(1 + (1 + \nu_3 hL) + (1 + \nu_3 hL)^2\Big),$$

....

Thus, for each $k = 1, 2, \ldots, n$ we have

$$d(Y_n(t_k)) \le d(Y_n(t_0))(1 + \nu_3 hL)^k$$

$$+ \left(\nu_3 hL \cdot \max_{l=0,1,\ldots,k-1} d(T_l) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha] h^{p+1} \right) \sum_{i=0}^{k-1} (1 + \nu_3 hL)^i.$$

But

$$\sum_{i=0}^{k-1} (1 + \nu_3 hL)^i = \frac{(1 + \nu_3 hL)^k - 1}{\nu_3 hL}$$

$$\le \frac{\exp(\nu_3 khL) - 1}{\nu_3 hL} \le \frac{\exp(\nu_3 nhL) - 1}{\nu_3 hL} = \frac{\exp(\nu_3 \eta_3^* L) - 1}{\nu_3 hL}.$$

Hence

$$d(Y_n(t_k)) \le R d(Y_n(t_0)) + S \cdot \max_{l=0,1,\ldots,k} d(T_l) + Q h^{p+1}, \tag{30}$$

where

$$R = \exp(\nu_3 \eta_3^* L), \quad S = R - 1, \quad Q = \frac{\exp(\nu_3 \eta_3^* L) - 1}{\nu_3 L}[d(\Psi(\Delta_t, \Delta_y)) + 2\alpha].$$

Taking into account that $d(T_0) = 0$, from (30) the inequality (24) follows immediately.

Theorem 2. $If\ Y_n\{t_k\}$ $(k = 0,1,\ldots,n)$ are obtained on the basis of the method (13)-(14), then for h_0 such that

$$h_0 < \min \left\{ 1, \frac{1}{L|a_{11}|}, \frac{1}{L|a_{22}|}, \frac{1}{L|a_{33}|}, \frac{1}{L|a_{44}|}, \right.$$

$$\frac{1}{L(|a_{11}| + |a_{22}|) + L^2|a_{12}||a_{21}|}, \frac{1}{L(|a_{11}| + |a_{33}|) + L^2|a_{13}||a_{31}|},$$

$$\frac{1}{L(|a_{11}| + |a_{44}|) + L^2|a_{14}||a_{41}|}, \frac{1}{L(|a_{22}| + |a_{33}|) + L^2|a_{23}||a_{32}|},$$

$$\frac{1}{L(|a_{22}| + |a_{44}|) + L^2|a_{24}||a_{42}|}, \frac{1}{L(|a_{33}| + |a_{44}|) + L^2|a_{34}||a_{43}|}, \tag{31}$$

$$\frac{1}{L\Theta_{11} + L^2\Theta_{12} + L^3\Theta_{13}}, \frac{1}{L\Theta_{21} + L^2\Theta_{22} + L^3\Theta_{23}},$$

$$\frac{1}{L\Theta_{31} + L^2\Theta_{32} + L^3\Theta_{33}}, \frac{1}{L\Theta_{41} + L^2\Theta_{42} + L^3\Theta_{43}},$$

$$\left. \frac{1}{L\Xi_1 + L^2\Xi_2 + L^3\Xi_3 + L^4\Xi_4} \right\},$$

where

$$\Theta_{11} = \left| a_{11} \right| + \left| a_{22} \right| + \left| a_{33} \right|,$$

$$\Theta_{12} = \left| a_{12} \right| \left| a_{21} \right| + \left| a_{13} \right| \left| a_{31} \right| + \left| a_{23} \right| \left| a_{32} \right|,$$

$$\Theta_{13} = \left| a_{11} \right| \left| a_{22} \right| \left| a_{33} \right| + \left| a_{12} \right| \left| a_{23} \right| \left| a_{31} \right| + \left| a_{13} \right| \left| a_{21} \right| \left| a_{32} \right|,$$

$$\Theta_{21} = \left| a_{11} \right| + \left| a_{22} \right| + \left| a_{44} \right|,$$

$$\Theta_{22} = \left| a_{12} \right| \left| a_{21} \right| + \left| a_{14} \right| \left| a_{41} \right| + \left| a_{24} \right| \left| a_{42} \right|,$$

$$\Theta_{23} = \left| a_{11} \right| \left| a_{22} \right| \left| a_{44} \right| + \left| a_{12} \right| \left| a_{24} \right| \left| a_{41} \right| + \left| a_{14} \right| \left| a_{21} \right| \left| a_{42} \right|,$$

$$\Theta_{31} = \left| a_{11} \right| + \left| a_{33} \right| + \left| a_{44} \right|,$$

$$\Theta_{32} = \left| a_{13} \right| \left| a_{31} \right| + \left| a_{14} \right| \left| a_{41} \right| + \left| a_{34} \right| \left| a_{43} \right|,$$

$$\Theta_{33} = \left| a_{11} \right| \left| a_{33} \right| \left| a_{44} \right| + \left| a_{13} \right| \left| a_{34} \right| \left| a_{41} \right| + \left| a_{14} \right| \left| a_{31} \right| \left| a_{43} \right|,$$

$$\Theta_{41} = \left| a_{22} \right| + \left| a_{33} \right| + \left| a_{44} \right|,$$

$$\Theta_{42} = \left| a_{23} \right| \left| a_{32} \right| + \left| a_{24} \right| \left| a_{42} \right| + \left| a_{34} \right| \left| a_{43} \right|,$$

$$\Theta_{43} = \left| a_{22} \right| \left| a_{33} \right| \left| a_{44} \right| + \left| a_{23} \right| \left| a_{34} \right| \left| a_{42} \right| + \left| a_{24} \right| \left| a_{32} \right| \left| a_{43} \right|,$$

$$\Xi_1 = \left| a_{11} \right| + \left| a_{22} \right| + \left| a_{33} \right| + \left| a_{44} \right|,$$

$$\Xi_2 = \left| a_{12} \right| \left| a_{21} \right| + \left| a_{13} \right| \left| a_{31} \right| + \left| a_{14} \right| \left| a_{41} \right|$$
$$+ \left| a_{23} \right| \left| a_{32} \right| + \left| a_{24} \right| \left| a_{42} \right| + \left| a_{34} \right| \left| a_{43} \right|,$$

$$\Xi_3 = \left| a_{11} \right| \left| a_{22} \right| \left| a_{33} \right| + \left| a_{11} \right| \left| a_{22} \right| \left| a_{44} \right| + \left| a_{11} \right| \left| a_{33} \right| \left| a_{44} \right|$$
$$+ \left| a_{12} \right| \left| a_{23} \right| \left| a_{31} \right| + \left| a_{12} \right| \left| a_{24} \right| \left| a_{41} \right| + \left| a_{13} \right| \left| a_{21} \right| \left| a_{32} \right|$$
$$+ \left| a_{13} \right| \left| a_{34} \right| \left| a_{41} \right| + \left| a_{14} \right| \left| a_{21} \right| \left| a_{42} \right| + \left| a_{14} \right| \left| a_{31} \right| \left| a_{43} \right|$$
$$+ \left| a_{22} \right| \left| a_{33} \right| \left| a_{44} \right| + \left| a_{23} \right| \left| a_{34} \right| \left| a_{42} \right| + \left| a_{24} \right| \left| a_{32} \right| \left| a_{43} \right|,$$

$$\Xi_4 = \left| a_{11} \right| \left| a_{22} \right| \left| a_{34} \right| \left| a_{43} \right| + \left| a_{11} \right| \left| a_{23} \right| \left| a_{32} \right| \left| a_{44} \right|$$
$$+ \left| a_{11} \right| \left| a_{24} \right| \left| a_{33} \right| \left| a_{42} \right| + \left| a_{12} \right| \left| a_{21} \right| \left| a_{33} \right| \left| a_{44} \right|$$
$$+ \left| a_{12} \right| \left| a_{23} \right| \left| a_{34} \right| \left| a_{41} \right| + \left| a_{12} \right| \left| a_{24} \right| \left| a_{31} \right| \left| a_{43} \right|$$
$$+ \left| a_{13} \right| \left| a_{21} \right| \left| a_{34} \right| \left| a_{42} \right| + \left| a_{13} \right| \left| a_{22} \right| \left| a_{31} \right| \left| a_{44} \right|$$
$$+ \left| a_{13} \right| \left| a_{24} \right| \left| a_{32} \right| \left| a_{41} \right| + \left| a_{14} \right| \left| a_{21} \right| \left| a_{32} \right| \left| a_{43} \right|$$
$$+ \left| a_{14} \right| \left| a_{22} \right| \left| a_{33} \right| \left| a_{41} \right| + \left| a_{14} \right| \left| a_{23} \right| \left| a_{31} \right| \left| a_{42} \right|,$$

we have

$$d(Y_n(t_k)) \leq Q h^p + R d(Y_0) + S \cdot \max_{l=1,2,\ldots,n} d(T_l), \tag{32}$$

where Q, R and S   denote some nonnegative constants, and $p \leq 8$.

Proof. The formulas (13) yield

$$d(Y_n(t_{k+1})) \le d(Y_n(t_k)) + h \sum_{i=1}^{4} |w_i| d(K_{ik}(h)) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha] h^{p+1}, \quad (33)$$

where $p \le 8$, $k = 0, 1, \dots, n-1$. On the basis of (14) we have

$$d(K_{i,k}(h)) \le L[d(T_k) + d(Y_n(t_k))] + hL \sum_{j=1}^{4} |a_{ij}| d(K_{j,k}(h)), \quad (34)$$

$$i = 1, 2, 3, 4.$$

From (22) it follows that if

$$1 - hL|a_{ii}| > 0, \quad i = 1, 2, 3, 4,$$

$$\left(1 - hL|a_{ii}|\right)\left(1 - hL|a_{jj}|\right) - h^2 L^2 |a_{ij}||a_{ji}| > 0, \quad i, j = 1, 2, 3, 4, \quad i \ne j,$$

$$\left(1 - hL|a_{ii}|\right)\left(1 - hL|a_{jj}|\right)\left(1 - hL|a_{kk}|\right)$$

$$- h^2 L^2 \left[\left(1 - hL|a_{ii}|\right)|a_{jk}||a_{kj}| + \left(1 - hL|a_{jj}|\right)|a_{ik}||a_{ki}|\right.$$

$$\left. + \left(1 - hL|a_{kk}|\right)|a_{ij}||a_{kj}|\right]$$

$$- h^3 L^3 \left(|a_{ij}||a_{jk}||a_{ki}| + |a_{ik}||a_{ji}||a_{kj}|\right) > 0,$$

$$i, j, k = 1, 2, 3, 4, \quad i \ne j, \quad j \ne k, \quad k \ne i,$$

$$V \equiv \left(1 - hL|a_{11}|\right)\left(1 - hL|a_{22}|\right)\left(1 - hL|a_{33}|\right)\left(1 - hL|a_{44}|\right)$$

$$- h^2 L^2 \left[\left(1 - hL|a_{11}|\right)\left(1 - hL|a_{22}|\right)|a_{34}||a_{43}|\right.$$

$$+ \left(1 - hL|a_{11}|\right)\left(1 - hL|a_{33}|\right)|a_{24}||a_{42}|$$

$$+ \left(1 - hL|a_{11}|\right)\left(1 - hL|a_{44}|\right)|a_{23}||a_{32}|$$

$$+ \left(1 - hL|a_{22}|\right)\left(1 - hL|a_{33}|\right)|a_{14}||a_{41}|$$

$$+ \left(1 - hL|a_{22}|\right)\left(1 - hL|a_{44}|\right)|a_{13}||a_{31}|$$

$$\left. + \left(1 - hL|a_{33}|\right)\left(1 - hL|a_{44}|\right)|a_{12}||a_{21}|\right]$$

$$- h^3 L^3 \left[\left(1 - hL|a_{11}|\right)\left(|a_{23}||a_{34}||a_{41}| + |a_{24}||a_{32}||a_{43}|\right)\right.$$

$$+ \left(1 - hL|a_{22}|\right)\left(|a_{13}||a_{34}||a_{41}| + |a_{14}||a_{31}||a_{43}|\right)$$

$$+ \left(1 - hL|a_{33}|\right)\left(|a_{12}||a_{24}||a_{41}| + |a_{14}||a_{21}||a_{42}|\right)$$

$$\left. + \left(1 - hL|a_{44}|\right)\left(|a_{12}||a_{23}||a_{31}| + |a_{13}||a_{21}||a_{32}|\right)\right]$$

$$(35)$$

$$-h^4 L^4 \Big[\big|a_{12}\big| \big(\big|a_{23}\big| \big\|a_{34}\big\| a_{41}\big| + \big|a_{24}\big| \big\|a_{31}\big\| a_{43}\big| - \big|a_{21}\big| \big\|a_{34}\big\| a_{43}\big| \big)$$

$$+ \big|a_{13}\big| \big(\big|a_{21}\big| \big\|a_{34}\big\| a_{42}\big| + \big|a_{24}\big| \big\|a_{32}\big\| a_{41}\big| - \big|a_{24}\big| \big\|a_{31}\big\| a_{42}\big| \big)$$

$$+ \big|a_{14}\big| \big(\big|a_{21}\big| \big\|a_{32}\big\| a_{43}\big| + \big|a_{23}\big| \big\|a_{31}\big\| a_{42}\big| - \big|a_{23}\big| \big\|a_{32}\big\| a_{41}\big| \big) \Big] > 0,$$

then

$$d(K_{i,k}(h)) \le \frac{V_i}{V} L[d(T_k) + d(Y_n(t_k))], \quad i = 1, 2, 3, 4, \tag{36}$$

where

$$V_i = \Big(1 - hL\big|a_{jj}\big|\Big)\Big(1 - hL\big|a_{kk}\big|\Big)\Big(1 - hL\big|a_{ll}\big|\Big)$$

$$+ \Big[1 + hL\big(\big|a_{jk}\big| + \big|a_{jl}\big|\big) - hL\big(\big|a_{kk}\big| + \big|a_{ll}\big|\big)\Big] hL\big|a_{ij}\big|$$

$$+ \Big[1 + hL\big(\big|a_{kj}\big| + \big|a_{kl}\big|\big) - hL\big(\big|a_{jj}\big| + \big|a_{ll}\big|\big)\Big] hL\big|a_{ik}\big|$$

$$+ \Big[1 + hL\big(\big|a_{lj}\big| + \big|a_{lk}\big|\big) - hL\big(\big|a_{jj}\big| + \big|a_{kk}\big|\big)\Big] hL\big|a_{il}\big|$$

$$- \Big(1 + hL\big|a_{ij}\big| - hL\big|a_{jj}\big|\Big) h^2 L^2 \big|a_{kl}\big\| a_{kl}\big|$$

$$- \Big(1 + hL\big|a_{ik}\big| - hL\big|a_{kk}\big|\Big) h^2 L^2 \big|a_{jl}\big\| a_{lj}\big|$$

$$- \Big(1 + hL\big|a_{il}\big| - hL\big|a_{ll}\big|\Big) h^2 L^2 \big|a_{jk}\big\| a_{kj}\big|$$

$$h^3 L^3 \Big\{ \big|a_{ij}\big| \Big[\big|a_{jk}\big| \big(\big|a_{kl}\big| - \big|a_{ll}\big| \big) + \big|a_{jl}\big| \big(\big|a_{lk}\big| - \big|a_{kk}\big| \big) + \big|a_{kk}\big\| a_{ll}\big| \Big]$$

$$+ \big|a_{ik}\big| \Big[\big|a_{kl}\big| \big(\big|a_{lj}\big| - \big|a_{jj}\big| \big) + \big|a_{kj}\big| \big(\big|a_{jl}\big| - \big|a_{ll}\big| \big) + \big|a_{jj}\big\| a_{kk}\big| \Big]$$

$$+ \big|a_{il}\big| \Big[\big|a_{lj}\big| \big(\big|a_{jk}\big| - \big|a_{kk}\big| \big) + \big|a_{lk}\big| \big(\big|a_{kj}\big| - \big|a_{jj}\big| \big) + \big|a_{jj}\big\| a_{kk}\big| \Big] \Big\}$$

$$- h^3 L^3 \Big(\big|a_{jk}\big\| a_{kl}\big\| a_{lj}\big| + \big|a_{jl}\big\| a_{kj}\big\| a_{lk}\big| \Big),$$

$$i, j, k, l = 1, 2, 3, 4 \quad i \ne j, \quad j \ne k, \quad k \ne l, \quad l \ne i,$$

and *V* is given by (35).

The first fourteen inequalities in (35) are fulfilled from the assumption (31) and because of $h \le h_0$. The last inequality in (35) also follows from (31), because for $h \le h_0$ we have

$$h < \frac{1}{L\Xi_1 + L^2\Xi_2 + L^3\Xi_3 + L^4\Xi_4},$$

$$1 - hL\Xi_1 - hL^2\Xi_2 - hL^3\Xi_3 - hL^4\Xi_4 > 0.$$

$$h < h_0 < 1, \qquad h^s < h \;\; (s = 2, 3, 4).$$

$$1 - hL\Xi_1 - h^2L^2\Xi_2 - h^3L^3\Xi_3 - h^4L^4\Xi_4 > 0,$$

$h \le h_0$

$$
\begin{aligned}
d(K_{1,k}(h)) \le \frac{1}{V}\Big[& 1 + h_0 L\big(|a_{12}| + |a_{13}| + |a_{14}|\big) \\
& + h_0^2 L^2 \big(|a_{12}||a_{23}| + |a_{12}||a_{24}| + |a_{13}||a_{32}| \\
& \quad + |a_{13}||a_{34}| + |a_{14}||a_{42}| + |a_{14}||a_{43}| \\
& \quad + |a_{22}||a_{33}| + |a_{22}||a_{44}| + |a_{33}||a_{44}|\big) \\
& + h_0^3 L^3 \big(|a_{12}||a_{23}||a_{34}| + |a_{12}||a_{24}||a_{43}| + |a_{12}||a_{33}||a_{44}| \\
& \quad + |a_{13}||a_{22}||a_{44}| + |a_{13}||a_{24}||a_{32}| + |a_{13}||a_{34}||a_{42}| \\
& \quad + |a_{14}||a_{22}||a_{33}| + |a_{14}||a_{23}||a_{42}| + |a_{14}||a_{32}||a_{43}| \\
& \quad + |a_{22}||a_{34}||a_{43}| + |a_{23}||a_{32}||a_{44}| + |a_{24}||a_{33}||a_{42}|\big)\Big] \\
& \cdot L[d(T_k) + d(Y_n(t_k))],
\end{aligned}
$$

$$
\begin{aligned}
d(K_{2,k}(h)) \le \frac{1}{V}\Big[& 1 + h_0 L\big(|a_{21}| + |a_{23}| + |a_{24}|\big) \\
& + h_0^2 L^2 \big(|a_{11}||a_{33}| + |a_{11}||a_{44}| + |a_{13}||a_{21}| \\
& \quad + |a_{14}||a_{21}| + |a_{23}||a_{31}| + |a_{23}||a_{34}| \\
& \quad + |a_{24}||a_{41}| + |a_{24}||a_{43}| + |a_{33}||a_{44}|\big) \\
& + h_0^3 L^3 \big(|a_{11}||a_{23}||a_{44}| + |a_{11}||a_{24}||a_{33}| + |a_{11}||a_{34}||a_{43}| \\
& \quad + |a_{13}||a_{21}||a_{34}| + |a_{13}||a_{24}||a_{41}| + |a_{13}||a_{31}||a_{44}| \\
& \quad + |a_{14}||a_{21}||a_{43}| + |a_{14}||a_{23}||a_{31}| + |a_{14}||a_{33}||a_{41}| \\
& \quad + |a_{21}||a_{33}||a_{44}| + |a_{23}||a_{34}||a_{41}| + |a_{24}||a_{31}||a_{43}|\big)\Big] \\
& \cdot L[d(T_k) + d(Y_n(t_k))],
\end{aligned}
$$

$$
\begin{aligned}
d(K_{3,k}(h)) \le \frac{1}{V}\Big[& 1 + h_0 L\big(|a_{31}| + |a_{32}| + |a_{34}|\big) \\
& + h_0^2 L^2 \big(|a_{11}||a_{22}| + |a_{11}||a_{44}| + |a_{12}||a_{31}|
\end{aligned}
$$

$$+ \left| a_{14} \right| \left| a_{31} \right| + \left| a_{21} \right| \left| a_{32} \right| + \left| a_{22} \right| \left| a_{44} \right|$$

$$+ \left| a_{24} \right| \left| a_{32} \right| + \left| a_{34} \right| \left| a_{41} \right| + \left| a_{34} \right| \left| a_{42} \right| \Big)$$

$$+ h_0^3 L^3 \Big(\left| a_{11} \right| \left| a_{22} \right| \left| a_{34} \right| + \left| a_{11} \right| \left| a_{24} \right| \left| a_{42} \right| + \left| a_{11} \right| \left| a_{32} \right| \left| a_{44} \right|$$

$$+ \left| a_{12} \right| \left| a_{21} \right| \left| a_{44} \right| + \left| a_{12} \right| \left| a_{24} \right| \left| a_{31} \right| + \left| a_{12} \right| \left| a_{34} \right| \left| a_{41} \right|$$

$$+ \left| a_{14} \right| \left| a_{21} \right| \left| a_{32} \right| + \left| a_{14} \right| \left| a_{22} \right| \left| a_{41} \right| + \left| a_{14} \right| \left| a_{31} \right| \left| a_{42} \right|$$

$$+ \left| a_{21} \right| \left| a_{34} \right| \left| a_{42} \right| + \left| a_{22} \right| \left| a_{31} \right| \left| a_{44} \right| + \left| a_{24} \right| \left| a_{32} \right| \left| a_{41} \right| \Big) \Big]$$

$$\cdot L[d(T_k) + d(Y_n(t_k))],$$

$$d(K_{4,k}(h)) \le \frac{1}{\overline{V}} \Big[1 + h_0 L \big(\left| a_{41} \right| + \left| a_{42} \right| + \left| a_{43} \right| \big)$$

$$+ h_0^2 L^2 \Big(\left| a_{11} \right| \left| a_{22} \right| + \left| a_{11} \right| \left| a_{33} \right| + \left| a_{12} \right| \left| a_{41} \right|$$

$$+ \left| a_{13} \right| \left| a_{41} \right| + \left| a_{21} \right| \left| a_{42} \right| + \left| a_{22} \right| \left| a_{33} \right|$$

$$+ \left| a_{23} \right| \left| a_{42} \right| + \left| a_{31} \right| \left| a_{43} \right| + \left| a_{32} \right| \left| a_{43} \right| \Big)$$

$$+ h_0^3 L^3 \Big(\left| a_{11} \right| \left| a_{22} \right| \left| a_{43} \right| + \left| a_{11} \right| \left| a_{23} \right| \left| a_{32} \right| + \left| a_{11} \right| \left| a_{33} \right| \left| a_{42} \right|$$

$$+ \left| a_{12} \right| \left| a_{21} \right| \left| a_{33} \right| + \left| a_{12} \right| \left| a_{23} \right| \left| a_{41} \right| + \left| a_{12} \right| \left| a_{31} \right| \left| a_{43} \right|$$

$$+ \left| a_{13} \right| \left| a_{21} \right| \left| a_{42} \right| + \left| a_{13} \right| \left| a_{22} \right| \left| a_{31} \right| + \left| a_{13} \right| \left| a_{32} \right| \left| a_{41} \right|$$

$$+ \left| a_{21} \right| \left| a_{32} \right| \left| a_{43} \right| + \left| a_{22} \right| \left| a_{33} \right| \left| a_{41} \right| + \left| a_{23} \right| \left| a_{31} \right| \left| a_{42} \right| \Big) \Big]$$

$$\cdot L[d(T_k) + d(Y_n(t_k))],$$

where

$$\overline{V} = \Big(1 - h_0 L \left| a_{11} \right| \Big) \Big(1 - h_0 L \left| a_{22} \right| \Big) \Big(1 - h_0 L \left| a_{33} \right| \Big) \Big(1 - h_0 L \left| a_{44} \right| \Big)$$

$$- h_0^2 L^2 \Big(\left| a_{12} \right| \left| a_{21} \right| + \left| a_{13} \right| \left| a_{31} \right| + \left| a_{14} \right| \left| a_{41} \right| + \left| a_{23} \right| \left| a_{32} \right| + \left| a_{24} \right| \left| a_{42} \right| + \left| a_{34} \right| \left| a_{43} \right| \Big)$$

$$- h_0^3 L^3 \Big(\left| a_{12} \right| \left| a_{23} \right| \left| a_{31} \right| + \left| a_{12} \right| \left| a_{24} \right| \left| a_{41} \right| + \left| a_{13} \right| \left| a_{21} \right| \left| a_{32} \right| + \left| a_{13} \right| \left| a_{34} \right| \left| a_{41} \right|$$

$$+ \left| a_{14} \right| \left| a_{21} \right| \left| a_{42} \right| + \left| a_{14} \right| \left| a_{31} \right| \left| a_{43} \right| + \left| a_{23} \right| \left| a_{34} \right| \left| a_{42} \right| + \left| a_{24} \right| \left| a_{32} \right| \left| a_{43} \right| \Big)$$

$$- h_0^4 L^4 \Big(\left| a_{12} \right| \left| a_{23} \right| \left| a_{34} \right| \left| a_{41} \right| + \left| a_{12} \right| \left| a_{24} \right| \left| a_{31} \right| \left| a_{43} \right| + \left| a_{13} \right| \left| a_{21} \right| \left| a_{34} \right| \left| a_{42} \right|$$

$$+ \left| a_{13} \right| \left| a_{24} \right| \left| a_{32} \right| \left| a_{41} \right| + \left| a_{14} \right| \left| a_{21} \right| \left| a_{32} \right| \left| a_{43} \right| + \left| a_{14} \right| \left| a_{23} \right| \left| a_{31} \right| \left| a_{42} \right| \Big).$$

Using these estimates, from the inequality (33) we obtain

$$d(Y_n(t_{k+1})) \le d(Y_n(t_k))(1 + \nu_4 hL) + \nu_4 hL d(T_k) + [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha] h^{p+1},$$

$$k = 0, 1, \dots, n-1,$$

where

$$\nu_4 = \frac{1}{\overline{V}} \sum_{i=1}^{4} |w_i| \left(1 + \sum_{j=1}^{3} A_{ij} h_0^j L^j\right),$$

and where

$$A_{11} = |a_{12}| + |a_{13}| + |a_{14}|,$$

$$A_{12} = |a_{12}|\left(|a_{23}| + |a_{24}|\right) + |a_{13}|\left(|a_{32}| + |a_{34}|\right) + |a_{14}|\left(|a_{42}| + |a_{43}|\right)$$
$$+ |a_{22}||a_{33}| + |a_{22}||a_{44}| + |a_{33}||a_{44}|,$$

$$A_{13} = |a_{12}|\left(|a_{23}||a_{34}| + |a_{24}||a_{43}| + |a_{33}||a_{44}|\right)$$
$$+ |a_{13}|\left(|a_{22}||a_{44}| + |a_{24}||a_{32}| + |a_{34}||a_{42}|\right)$$
$$+ |a_{14}|\left(|a_{22}||a_{33}| + |a_{23}||a_{42}| + |a_{32}||a_{43}|\right)$$
$$+ |a_{22}||a_{34}||a_{43}| + |a_{23}||a_{32}||a_{44}| + |a_{24}||a_{33}||a_{42}|,$$

$$A_{21} = |a_{21}| + |a_{23}| + |a_{24}|,$$

$$A_{22} = |a_{21}|\left(|a_{13}| + |a_{14}|\right) + |a_{23}|\left(|a_{31}| + |a_{34}|\right) + |a_{24}|\left(|a_{41}| + |a_{43}|\right)$$
$$+ |a_{11}||a_{33}| + |a_{11}||a_{44}| + |a_{33}||a_{44}|,$$

$$A_{23} = |a_{21}|\left(|a_{13}||a_{34}| + |a_{14}||a_{43}| + |a_{33}||a_{44}|\right)$$
$$+ |a_{23}|\left(|a_{11}||a_{44}| + |a_{14}||a_{31}| + |a_{34}||a_{41}|\right)$$
$$+ |a_{24}|\left(|a_{11}||a_{33}| + |a_{13}||a_{41}| + |a_{31}||a_{43}|\right)$$
$$+ |a_{11}||a_{34}||a_{43}| + |a_{13}||a_{31}||a_{44}| + |a_{14}||a_{33}||a_{41}|,$$

$$A_{31} = |a_{31}| + |a_{33}| + |a_{34}|,$$

$$A_{32} = |a_{31}|\left(|a_{12}| + |a_{14}|\right) + |a_{32}|\left(|a_{21}| + |a_{24}|\right) + |a_{34}|\left(|a_{41}| + |a_{42}|\right)$$
$$+ |a_{11}||a_{22}| + |a_{11}||a_{44}| + |a_{22}||a_{44}|,$$

$$A_{33} = |a_{31}|\left(|a_{12}||a_{24}| + |a_{14}||a_{42}| + |a_{22}||a_{44}|\right)$$
$$+ |a_{32}|\left(|a_{11}||a_{44}| + |a_{14}||a_{21}| + |a_{24}||a_{41}|\right)$$
$$+ |a_{34}|\left(|a_{11}||a_{22}| + |a_{12}||a_{41}| + |a_{21}||a_{42}|\right)$$
$$+ |a_{11}||a_{24}||a_{42}| + |a_{12}||a_{21}||a_{44}| + |a_{14}||a_{22}||a_{41}|,$$

$$A_{41} = |a_{41}| + |a_{42}| + |a_{43}|,$$

$$A_{42} = |a_{41}|\left(|a_{12}| + |a_{13}|\right) + |a_{42}|\left(|a_{21}| + |a_{23}|\right) + |a_{43}|\left(|a_{31}| + |a_{32}|\right)$$
$$+ |a_{11}||a_{22}| + |a_{11}||a_{33}| + |a_{22}||a_{33}|,$$

$$A_{43} = \left| a_{41} \right| \left(\left| a_{12} \right| \left| a_{23} \right| + \left| a_{13} \right| \left| a_{32} \right| + \left| a_{22} \right| \left| a_{33} \right| \right)$$
$$+ \left| a_{42} \right| \left(\left| a_{11} \right| \left| a_{33} \right| + \left| a_{13} \right| \left| a_{21} \right| + \left| a_{23} \right| \left| a_{31} \right| \right)$$
$$+ \left| a_{43} \right| \left(\left| a_{11} \right| \left| a_{22} \right| + \left| a_{12} \right| \left| a_{31} \right| + \left| a_{21} \right| \left| a_{32} \right| \right)$$
$$+ \left| a_{11} \right| \left| a_{23} \right| \left| a_{32} \right| + \left| a_{12} \right| \left| a_{21} \right| \left| a_{33} \right| + \left| a_{13} \right| \left| a_{22} \right| \left| a_{31} \right|.$$

Proceeding further, as in the proof of the previous theorem, we get

$$d(Y_n(t_k)) \le R d(Y_n(t_0)) + S \cdot \max_{l=0,1,\dots,k} d(T_l) + Q h^p, \tag{37}$$

$$R = \exp(\nu_4 \eta_4^* L), \quad S \doteq R - 1, \quad Q = \frac{\exp(\nu_4 \eta_4^* L) - 1}{\nu_4 L} [d(\Psi(\Delta_t, \Delta_y)) + 2\alpha].$$

Since $d(T_o) = 0$, the inequality (32) is an obvious consequence of (37).

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